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SYNCHROTRON RADIATION IN WEAK ELECTROMAGNETIC FIELDS*

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ABSTRACT

We have calculated the emission rate and spectrum for synchrotron radiation from a relativistic electron in a weak external homogeneous electric field and compared the results with the magnetic synchrotron radiation formulas. The synchrotron radiation from a relativistic electron in a weak electromagnetic field, whether electric or magnetic in nature, can always be described by radiation formulas involving a Lorentz invariant radiation parameter $\Upsilon = |\Pi_\mu F^{\mu\nu} \Pi^\lambda F_{\lambda\nu}|^{1/2} / m_e c F_c$, where Π_μ is the electron mechanical momentum, $F_{\mu\nu}$ is the external field and $F_c \equiv m_e^2 c^3 / e\hbar$.

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The quantum mechanical problem of calculating the synchrotron radiation of photons from relativistic electrons in a homogeneous external magnetic field has been addressed by various authors.[1–3] Sokolov *et al.*[1] utilized the Dirac wave functions of an electron in a constant magnetic field to calculate synchrotron radiation. The transition amplitude for $e^- \rightarrow e^- + \gamma$ was computed by perturbation theory (*i.e.*, to first order in the fine structure constant α) and the power spectrum obtained by squaring the amplitude and summing over final states. Recently Tsai and Yildiz[3] have presented a more efficient method for calculating radiation in external fields based on Schwinger's source theory formulation of quantum field theory.[4] This latter approach, which we use in this paper, eliminates the need for using wave functions by replacing the sum over final states by expectation values obtained directly from the Dirac equation.

The previous results for radiation in external magnetic fields are of course applicable in all Lorentz frames where $H^2 - E^2 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} > 0$ and $\overline{E} \cdot \overline{H} = \frac{1}{4} \widetilde{F}_{\mu\nu} F^{\mu\nu} = 0$.[5] There are, however, physical systems in which the fields are electric in nature ($H^2 - E^2 < 0$) rather than magnetic. An example is synchrotron radiation emitted during the collision of electron-positron beams from a high energy accelerator.[6] The corresponding problem of radiation in a homogeneous electric field has received far less attention, possibly because of the well-known difficulty of the "Klein catastrophe," that is spontaneous pair creation by an electric field.[7] In the weak field limit $|\frac{1}{2} F_{\mu\nu} F^{\mu\nu}|^{1/2} \ll F_c \equiv m_e^2/e$ ($\simeq 4.4 \times 10^{13}$ G $\simeq 1.3 \times 10^{16}$ V/cm) that we consider, pair creation effects are negligible,[8] and we may calculate synchrotron radiation in fields which are either electric or magnetic in the same manner.

In this paper we specifically calculate the emission rate and spectrum for synchrotron radiation from a relativistic electron in a weak external homogeneous electric field following the method used by Tsai and Yildiz[3] to calculate magnetic synchrotron radiation. We find that synchrotron radiation from a relativistic electron in a weak electromagnetic field, whether electric or magnetic in nature, satisfying $\overline{E} \cdot \overline{H} = 0$ can always be described by radiation formulas involving a Lorentz invariant radiation parameter $\Upsilon = \left|\Pi_\mu F^{\mu\nu} \Pi^\lambda F_{\lambda\nu}\right|^{1/2} / m_e F_c$, where Π_μ is the electron mechanical momentum. Classical radiation corresponds to $\Upsilon \ll 1$ in which the classical synchrotron energy is $\omega_c = 3\Upsilon\mathcal{E}/2$, $\mathcal{E}$ being the electron energy, whereas $\Upsilon \gg 1$ corresponds to extreme quantum radiation in which the peak position of the power spectrum approaches the electron energy.

We apply the method to first calculate the total photon emission rate from an electron in an external homogeneous field, $F_{\mu\nu}$. The starting point is the action contribution associated with the exchange of a virtual photon, $\frac{1}{2}\int (dx)(dx')\, \overline{\psi}(x)\, M(x,x')\, \psi(x')$, where ψ is the electron field. If we represent $M(x,x') = \langle x | M | x' \rangle$, then according to the optical theorem, the total decay rate, $\Gamma(e^- \to e^- + \gamma)$, is related to the imaginary part of the matrix element M by $\Gamma = -(2m/\mathcal{E})\,\mathrm{Im}\, M$, where m is the electron rest mass. From Ref. 3, the matrix element M has the form

$$\begin{aligned} M = &-\frac{\alpha}{4\pi} \int_0^\infty \frac{ds}{s} \int_0^1 du \left(\det \frac{2\,eq\,Fs}{D}\right)^{1/2} e^{-is\Phi} \\ &\times \left[\left(-4 - tr\, A + 2i\sigma A\right) \left(m + \gamma\, \frac{2(1-u)\,eq\,Fs}{D}\, \Pi \right) \right. \\ &\left. + 2\gamma \left(1 + A^T\right) \frac{2(1-u)\,eq\,Fs}{D}\, \Pi \right] + c.t.\,, \end{aligned} \tag{1}$$

where $q = \pm 1$ is the charge sign, $A = \exp\{2u\, eq\, Fs\} - 1$, $D = A + 2\,(1-u)\, eq\, Fs$ (A and D being second rank tensors), $\Phi = u(\Pi^2 + m^2 - eq\,\sigma F) + \Pi[-1/(2\, eq\, Fs)\, \ell n(-D/D^T)]\Pi$ and $\sigma A \equiv \frac{1}{2}\sigma_{\mu\nu} A^{\mu\nu}$. The contact terms have the form $c.t. = -m_c - \varsigma_c(m + \gamma\Pi)$, where

$$m_c = \frac{\alpha}{2\pi}\, m \int_0^\infty \frac{ds}{s} \int_0^1 du (1+u)\, e^{-ism^2u^2} \tag{2}$$

and

$$\begin{aligned} \varsigma_c &= \frac{\alpha}{2\pi} \int_0^\infty \frac{ds}{s} \int_0^1 du(1-u)\, e^{-ism^2u^2} \\ &\quad - i\,\frac{\alpha}{\pi}\, m^2 \int_0^\infty ds \int_0^1 du\; u(1-u^2)\, e^{-ism^2u^2} \; . \end{aligned} \tag{3}$$

We now specialize to the case of radiation in a pure electric field. For an electric field in the z direction, $F_{30} = -F_{03} = E$, the matrix element (1) can be algebraically simplified in a manner analogous to Ref. 3 with the result

$$\begin{aligned} M \;=\; &\frac{\alpha}{2\pi}\, m \int_0^\infty \frac{dx}{x} \int_0^1 du\; \exp\left\{-i\frac{m^2}{eE}\, ux\right\} \\ &\times \left\{ \frac{e^{-is\phi}}{\Delta^{1/2}} \left[1 + e^{2i\varsigma x} \left(1 + (1-u)\,\frac{\gamma\Pi}{m}\right) \right.\right. \\ &\qquad \left. + \frac{1-u}{m} \left(\frac{1-u}{\Delta} + \frac{u}{\Delta}\, \frac{\sinh x}{x}\, e^{i\varsigma x} - e^{2i\varsigma x} \right) \gamma \cdot \Pi_{\parallel} \right] \\ &\qquad \left. -(1+u) - \left(1 + \frac{\gamma\Pi}{m}\right) (1-u)\left[1 - 2im^2 su(1+u)\right] \right\} , \end{aligned} \tag{4}$$

where $\Delta = \det[D/2\,eq\,Fs] = (1-u)^2 + u(1-u)\sinh 2x/x + u^2(\sinh x/x)^2$, $\varsigma = q\sigma^{03}$, $x = eEus$, $\phi = u(1-u)[m^2 - (\gamma\Pi)^2] + (u/x)[\beta - (1-u)x]\,\Pi_{\parallel}^2 - u^2\,eq\,\sigma F$, $\tanh\beta = (1-u)\sinh x/[(1-u)\cosh x + u\sinh x/x]$, $\Pi_{\parallel}^2 = \Pi_0\Pi^0 + \Pi_3\Pi^3$ and $\gamma\cdot\Pi_{\parallel} = \gamma_0\Pi^0 + \gamma_3\Pi^3$.

To further simplify Eq. (4), we will approximate M (accurate to order α) by its expectation value taken between fields obeying the Dirac equation $(m+\gamma\Pi)\psi = 0$ assuming that $E \ll F_c$ so that spontaneous pair creation is negligible. Assuming without loss of generality that $\Pi_1 = 0$ and $\Pi_2 = p_\perp =$ constant, the Dirac wavefunction ψ (where $\int \bar{\psi}\psi(dx) = 1$) is a simultaneous eigenstate of the Hamiltonian H, momentum Π_2 and $\gamma_2\varsigma$:

$$\gamma_2\left(m + \gamma\cdot\Pi_{\parallel}\right)\psi \;=\; \Pi_2\psi \;=\; p_\perp\psi \;, \tag{5}$$

$$\gamma_2\varsigma\psi \;=\; \varsigma'\psi \quad , \qquad \varsigma' \;=\; \pm 1 \;. \tag{6}$$

We decompose the eigenfunctions ψ into the two subspaces of $i\gamma_2$ using the projection operators $P_\pm = \frac{1}{2}\,(1 \pm i\gamma_2)$ such that $P_\pm\psi = \psi_\pm$. In each subspace $\psi_\pm$ is an eigenfunction of $i\gamma_2$ and $-i\varsigma$: $i\gamma_2\psi_\pm = \pm\psi_\pm$, $-i\varsigma\psi_\pm = \pm\varsigma'\psi_\pm$. Applying $P_\pm$ to Eq. (5) yields $\gamma\cdot\Pi_{\parallel}\psi_\mp = (\pm i p_\perp - m)\psi_\pm$ or equivalently

$$\Pi_{\parallel}^2\psi_\pm \;=\; -\left(p_\perp^2 + m^2 + \varsigma eE\right)\psi_\pm \;. \tag{7}$$

The following expectation values are then easily derived,

$$\begin{aligned} \langle\gamma_2\rangle \;&=\; \frac{p_\perp}{m} \quad , \qquad \langle\varsigma\rangle = -\langle\gamma_2\rangle\;\varsigma' \;, \\ \left\langle\gamma\cdot\Pi_{\parallel}\right\rangle \;&=\; -\frac{p_\perp^2 + m^2}{m} \quad , \qquad \left\langle\varsigma\gamma\cdot\Pi_{\parallel}\right\rangle \;= 0 \;. \end{aligned} \tag{8}$$

Using Eqs. (7) and (8), the matrix element M can be written approximately as

$$
\begin{aligned}
M \; = \; & \frac{\alpha}{2\pi}\, m \int_0^\infty \frac{dx}{x} \int_0^1 du \, \exp\left\{-i\, \frac{m^2}{eE}\, ux\right\} \\
& \times \left\{ \Delta^{-1/2} \exp\left\{ i[\beta-(1-u)x]\, \frac{p_\perp^2+m^2}{eE} \right\} \right. \\
& \times \left[e^{i\varsigma(\beta-x)} \left(1+u\, e^{2i\varsigma x}\right) - \frac{p_\perp^2+m^2}{m^2}\,(1-u) \right. \\
& \left.\left. \times \left(\frac{1-u}{\Delta}\, \cosh(\beta-x) + \frac{u}{\Delta}\, \frac{\sinh x}{x}\, \cosh\beta - \cosh(\beta+x) \right) \right] - (1+u) \right\} .
\end{aligned}
\tag{9}
$$

This form of M in an electric field is very similar to that in a magnetic field [*cf.*, Eq. (56) of Ref. 3]. Indeed one can make an immediate duality transformation to obtain it from the magnetic case: $H \to iE$, $x \to ix = ieEus$, $\beta \to i\beta$, $\varsigma \to i\varsigma = iq\sigma^{03}$ and $\mathcal{E}^2 - m^2 \to -p_\perp^2 - m^2$. We now specialize to synchrotron radiation in a weak electric field which is defined as the high transverse momentum $(p_\perp \gg m)$–weak field $(eE/m^2 \ll 1)$ limit[9] of Eq. (9). In this limit the total decay rate can be written down by inspection of the magnetic case in Ref. 3,

$$
\begin{aligned}
\Gamma \; = \; & \frac{\alpha}{\sqrt{3}\,\pi}\, \frac{m^2}{\mathcal{E}} \int_0^1 du \left[(1+u) \int_\xi^\infty K_{5/3}\,(\eta)\, d\eta \right. \\
& \left. + \frac{2}{3}\, u(3u-2)(1-u)^{-1}\, K_{2/3}\,(\xi) + \varsigma' u\, K_{1/3}\,(\xi) \right] ,
\end{aligned}
\tag{10}
$$

where $K_\nu(\eta)$ is the modified Besel function of the second kind, $\xi = 2u/3\Upsilon(1-u)$, $\varsigma' = \langle \gamma_2 \varsigma \rangle = \pm 1$ and $\Upsilon = (p_\perp/m)(eE/m^2)$.

For the case of synchrotron radiation from a relativistic electron in a weak homogeneous magnetic field H along the z axis, the radiation parameter $\Upsilon = (\mathcal{E}/m)(eH/m^2)$, where $\Pi_3 = 0$ without loss of generality. The square of the perpendicular mechanical momentum $\Pi_\perp^2 = \Pi_1\Pi^1 + \Pi_2\Pi^2$ obeys the relation $\langle\Pi_\perp^2\rangle = \mathcal{E}^2 - m^2 + (\mathcal{E}/m)eH\varsigma' \simeq \mathcal{E}^2$ (where $\varsigma' = \left\langle q\gamma^0\sigma_3\right\rangle = \pm 1$) which is to be compared with the square of the parallel mechanical momentum in an electric field from Eq. (7), $\langle\Pi_\parallel^2\rangle \simeq -p_\perp^2$. Within the relativistic approximations made, the radiation parameter in both the electric and magnetic cases can be written in the Lorentz invariant form

$$\Upsilon = \frac{|\Pi_\mu F^{\mu\nu}\, \Pi^\lambda F_{\lambda\nu}|^{1/2}}{mF_c} \,. \tag{11}$$

The expectation value ς' in both cases can be written as $q\epsilon_{\alpha\beta\mu\nu}\,\Pi^\alpha S^\beta F^{\mu\nu} / mF_c\Upsilon$, where S^β is the electron four-spin. With these identifications, Eq. (10) is valid in all weak homogeneous electromagnetic fields satisfying $\overline{E}\cdot\overline{H} = 0$.

The synchrotron power spectrum in an electric field, $P(\omega)$, where ω is the photon frequency, can be obtained by a simple modification of the method used to calculate the decay rate. By inserting a unit factor $1 = \int_{-\infty}^{\infty} d\omega\,\delta(\omega - k^0) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} (d\tau/2\pi)\exp\left\{i(\omega - k^0)\tau\right\}$ into the matrix element M, the spectrum $P(\omega)$ is identified from the ω–integrand. The procedure is essentially identical to that for a magnetic field with the result

$$
\begin{aligned}
P(\omega) \;=\; & -\frac{2m}{\mathcal{E}}\,\omega\,\mathrm{Im}\left[\frac{\alpha}{2\pi}\,m\int_0^\infty \frac{ds}{s}\int_0^1 du\; e^{-ism^2u^2}\right.\\
& \times\left\{\Delta^{-1/2}\;\exp\left\{i[\beta-(1-u)x]\;\frac{p_\perp^2+m^2}{eE}\right\}\right.\\
& \times\left[e^{i\varsigma(\beta-x)}\left(1+u\,e^{2i\varsigma x}\right)-\frac{p_\perp^2+m^2}{m^2}\,(1-u)\right.\\
& \times\left(\frac{1-u}{\Delta}\,\cosh(\beta-x)+\frac{u}{\Delta}\,\frac{\sinh x}{x}\,\cosh\beta-\cosh(\beta+x)\right)\\
& \left.\left.+\frac{i}{2ms\Pi^0}\,e^{i\varsigma(\beta+x)}\,\frac{d}{du}\,\gamma^0\right]-\left[1+u+\frac{i}{2ms\Pi^0}\,\frac{d}{du}\,\gamma^0\right]\right\}\\
& \left.\times\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\,\exp\left\{i(\omega-u\Pi^0)\tau-i\,\frac{\tau^2}{4s}\right\}\right]\;.
\end{aligned}
\tag{12}
$$

We now specialize to the high transverse momentum-weak field limit. In this case $\Pi^0 = \mathcal{E} - V \simeq \mathcal{E}$ and $\langle i\varsigma\gamma^0\rangle = q\,\langle\gamma^3\rangle = q\,\langle\Pi^3\rangle/m \ll q\mathcal{E}/m$ except at asymptotically large distances from the origin $[V(z=0)\equiv 0]$ where tunneling solutions of the Dirac equation (pair creation) are involved. In both electric and magnetic fields $\langle\gamma^0\rangle = \mathcal{E}/m$ so Eq. (12) can be reduced to a form similar to the spectrum in a magnetic field [*cf.*, Eq. (144) in Ref. 3]. The final form of the synchrotron spectrum in an electric field is identical to that in a magnetic field,

$$
\begin{aligned}
P(\omega) \;=\; & \frac{\alpha}{\sqrt{3}\,\pi}\,\frac{m^2}{\mathcal{E}}\,\frac{\omega}{\mathcal{E}}\\
& \times\left[\int_{\xi'}^{\infty} K_{5/3}\,(\eta)\,d\eta+\left(\frac{\omega}{\mathcal{E}}\right)^2\left(1-\frac{\omega}{\mathcal{E}}\right)^{-1}\,K_{2/3}(\xi')+\varsigma'\,\frac{\omega}{\mathcal{E}}\,K_{1/3}(\xi')\right],
\end{aligned}
\tag{13}
$$

where $\xi' = 2(\omega/\mathcal{E})/3\Upsilon(1-\omega/\mathcal{E})$. With the previous identifications of the Lorentz invariant expressions for Υ and ς', Eq. (13) is valid in all weak homogeneous electromagnetic fields satisfying $\overline{E}\cdot\overline{H}=0$.

The photon power spectrum $P(\omega)$ is related to the photon number spectrum $N(\omega)$ by $P(\omega)=\omega N(\omega)$. The total synchrotron power and photon emission rate from an unpolarized relativistic electron can be written in terms of functions of the radiation parameter Υ. The total radiated power from an electron is

$$P = \int_0^{\mathcal{E}} P(\omega)\,d\omega = \frac{2}{3}\alpha m^2 g(\Upsilon)\,, \tag{14}$$

where

$$g(\Upsilon) \simeq \begin{cases} \Upsilon^2, & \Upsilon \ll 1\,, \\ \frac{16}{27}\,3^{-1/3}\,\Gamma(\frac{2}{3})\,\Upsilon^{2/3}, & \Upsilon \gg 1\,. \end{cases} \tag{15}$$

The total emission rate is

$$\Gamma = \int_0^{\mathcal{E}} N(\omega)\,d\omega = \frac{5\alpha}{2\sqrt{3}}\,\frac{m^2}{\mathcal{E}}\,h(\Upsilon)\,, \tag{16}$$

where

$$h(\Upsilon) \simeq \begin{cases} \Upsilon, & \Upsilon \ll 1\,, \\ \frac{14}{15}\,3^{1/6}\,\Gamma(\frac{5}{3})\,\Upsilon^{2/3}, & \Upsilon \gg 1\,. \end{cases} \tag{17}$$

For intermediate values of Υ, there are no simple analytic forms for the functions $g(\Upsilon)$ and $h(\Upsilon)$. Table I contains representative values of these functions in the range $10^{-3} \leq \Upsilon \leq 10^3$.

As a final note we wish to caution that our analytic results for synchrotron radiation in weak electromagnetic fields are in fact physically valid only for $\Upsilon \lesssim 10^5$ before vacuum polarization effects become important. Synchrotron radiation changes to a new synergetic synchrotron–Čerenkov radiation for $\Upsilon > 10^5$ when the vacuum, modified by the external electromagnetic fields, acts like a dielectric medium.[10]

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TABLE I

Representative values of the functions $g(\Upsilon)$ and $h(\Upsilon)$ in the range $10^{-3} \leq \Upsilon \leq 10^{3}$.

Υ	$g(\Upsilon)$	$h(\Upsilon)$
10^{-3}	9.94×10^{-7}	9.99×10^{-4}
10^{-2}	9.45×10^{-5}	9.91×10^{-3}
10^{-1}	6.55×10^{-3}	9.30×10^{-2}
1	1.82×10^{-1}	7.16×10^{-1}
10	1.84	4.24
10^{2}	1.11×10^{1}	2.13×10^{1}
10^{3}	5.56×10^{1}	1.01×10^{2}