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THE STUDY OF INVARIANT SURFACES AND THEIR BREAK-UP BY THE HAMILTON-JACOBI METHOD*

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Summary

We describe a method to compute invariant tori in phase space for classical non-integrable Hamiltonian systems. Our procedure is to solve the Hamilton-Jacobi equation stated as a system of equations for Fourier coefficients of the generating function. The system is truncated to a finite number of Fourier modes and solved numerically by Newton's method. The resulting canonical transformation serves to reduce greatly the non-integrable part of the Hamiltonian. In examples studied to date the convergence properties of the method are excellent, even near chaotic regions and on the separatrices of isolated broad resonances. We propose a criterion for breakup of invariant surfaces, namely the vanishing of the Jacobian of the canonical transformation to new angle variables. By comparison with results from tracking, we find in an example with two nearly overlapping resonances that this criterion can be implemented with sufficient accuracy to determine critical parameters for the breakup ('transition to chaos') to an accuracy of 5-10%.

The Hamilton-Jacobi Equation

We present results for a system with one degree of freedom having a periodic time-dependent Hamiltonian. The generalization to higher dimensions will be obvious. In angle-action variables the Hamiltonian is

$$H(\phi, J, \theta) = H_0(J) + V(\phi, J, \theta), \tag{1}$$

where θ is the machine azimuth or 'time', and the perturbation V is periodic in θ with period 2π. We seek a canonical transformation $(\phi, J) \mapsto (\psi, K)$ in the form[1]

$$J = K + G_\phi(\phi, K, \theta) \ , \tag{2}$$

$$\psi = \phi + G_K(\phi, K, \theta) \ , \tag{3}$$

such that the new Hamiltonian becomes a function of K alone. Subscripts denote partial derivatives. The Hamilton-Jacobi equation to determine the generator G is the requirement that the new Hamiltonian H indeed depend only on K; namely

$$H_0(K + G_\phi) + V(\phi, K + G_\phi, \theta) + G_\theta = H_1(K) \ . \tag{4}$$

We are interested in periodic solutions of (4) with the Fourier development

$$G(\phi, K, \theta) = \sum_{m,n} g_{mn}(K) e^{i(m\phi - n\theta)} \ . \tag{5}$$

We rearrange (4) by adding and subtracting terms so as to isolate terms linear in G_ϕ and G_θ. We then take the Fourier transform for $m \neq 0$ to cast Eq. (4) in the form

$$g = A(g) \ , \tag{6}$$

where $g = [g_{mn}]$ is a vector of Fourier coefficients and

$$A_{mn}(g) = \frac{i}{(\nu(K)m - n)} \frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} d\phi d\theta e^{-i(m\phi - n\theta)} \times \left[H(\phi, K + G_\phi, \theta) - H_0(K) - \nu(K)G_\phi\right] \ , \ m \neq 0 \ , \tag{7}$$

where $\nu(K) = \partial H_0/\partial K$. To truncate the system (6) for numerical solution we restrict (m,n) to some bounded set B of integers, with $m \neq 0$, and put

$$G_\phi = \sum_{(m,n)\in B} i m g_{mn}(K) e^{i(m\phi - n\theta)} \ . \tag{8}$$

In an iterative solution of (6) the set B is selected so that at iterate $(p+1)$ all $A_{mn}(g^{(p)})$ with $(m,n) \in B$ are greater than some preassigned small number; here $g^{(p)}$ is the p^{th} iterate. Only the amplitudes g_{mn} for $m \neq 0$ are required to calculate G_ϕ; the $m = 0$ amplitude and also the function $H_1(K)$ are determined from (4) *a posteriori*. Once G_ϕ is known, $J(\phi, \theta)$ may be plotted from Eq. (2). The action K is an input parameter.

The equation in the form (6) is suitable for the examples treated below, but not for typical accelerator problems involving short nonlinear lattice elements. For the latter, the Fourier analysis in θ has slow convergence and should be avoided. Retaining the Fourier analysis in ϕ, and using the periodic Green function for the operator $im\nu + \partial/\partial\theta$, we can find an integral equation for the amplitudes $g_m(K;\theta)$. The system can be discretized to provide an equation for the variables $g_m(K;\theta_i)$, $m \geq 1$, where the θ_i are mesh points located only in the nonlinear elements of the lattice. The solution is automatically periodic in θ. An alternative procedure is to treat the equation as a system of differential equations in θ. The equation must be integrated only once around the accelerator with periodicity achieved by iteration, in analogy to nonlinear closed orbit calculations.

An Integrable Example

We show results from solving (6) - (8) by Newton's method, starting from $g = 0$. Our first example is a locally integrable case in which some of the invariant surfaces may be expressed analytically, namely the 4^{th} order isolated resonance model with

$$H(\phi, J, \theta) = \nu_0 J + \alpha J^2/2 + \epsilon J^2 \cos(4\phi - \theta) \ , \tag{9}$$

where ν_0, α, and ϵ are constants. We have successfully calculated a variety of qualitatively different curves (surfaces of section at fixed θ) for various choices of the parameters. For details about accuracy, rate of convergence, *etc.*, see Ref. 2. The most difficult curves to compute are the separatrices around wide islands. The method works surprisingly well even for

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such curves, as is seen in Figures 1 and 2. Fig. 1 shows separatrices computed in 9 iterations with 31 modes in the set B. The points in Figs. 1 and 2 are plotted in normalized phase space ($\sqrt{J}\cos\phi$ *vs.* $\sqrt{J}\sin\phi$) at $\theta = 0$. The inner separatrix (almost a square) and the outer separatrix (four lobes intersecting at right angles) are from two different calculations for two different values of K. Fig. 2 is a plot of curves from the exact analytic formulas for comparison. Similar results are obtained for nonintegrable Hamiltonians, in a region sufficiently close to a single resonance.

The Two Resonance Model

Our second example is nonintegrable, and contains all the generic phenomena of nonlinear mechanics in $1\frac{1}{2}$ or 2 dimensions. In a restricted region of phase space it should describe the essential features of one dimensional betatron motion in the presence of nonlinearities. The example is the two-resonance model with the Hamiltonian

$$H = \nu_0 J + \frac{1}{2}\alpha J^2 + \epsilon_1 J^{5/2}\cos(5\phi - 3\theta) + \epsilon_2 J^2 \cos(8\phi - 5\theta). \quad (10)$$

For small perturbation strengths ϵ_1, ϵ_2 we compute an invariant curve (section of invariant surface at $\theta = 0$) for a tune equal to the golden mean $\nu_* = (\sqrt{5} - 1)/2$, which is between the two resonances. Here we refer to the exact perturbed tune $\nu_* = dH_1/dK$, not the unperturbed tune $\nu = \nu_0 + \alpha K$. To maintain the perturbed tune at a preassigned value, we include the equation $\nu_* = dH_1/dK$ as a constraint in the iteration (see Ref. 2). Having found an initial curve we then increase ϵ_1, ϵ_2 (arbitrarily taking $\epsilon_1 = 2\epsilon_2$) and look for the transition to chaotic behavior.

We choose $\nu_0 = 0.5$, $\alpha = 0.1$, and consider a sequence of three cases with strengths and resonance widths ΔJ_1, ΔJ_2 as follows:

(*i*) $\epsilon_1 = 2\epsilon_2 = 6 \times 10^{-5}$, $\Delta J_1 = 0.049$, $\Delta J_2 = 0.054$;

(*ii*) $\epsilon_1 = 2\epsilon_2 = 10^{-4}$, $\Delta J_1 = 0.063$, $\Delta J_2 = 0.070$;

(*iii*) $\epsilon_1 = 2\epsilon_2 = 1.25 \times 10^{-4}$, $\Delta J_1 = 0.070$, $\Delta J_2 = 0.078$;

By the Chirikov resonance overlap criterion,[3] the corresponding invariant curves should be close to breakup, since the resonance separation is $J_{r_1} - J_{r_2} = 0.25$.

To identify the transition to chaos as the $\epsilon' s$ are increased, we propose the criterion that the Jacobian of Eq. (3) vanish at some (ϕ, θ):

$$\partial\psi/\partial\phi = 1 + G_{K\phi} = \partial J/\partial K = 0\ . \quad (11)$$

At such a point it is in general impossible to solve uniquely for ϕ as a function of ψ. Since $\partial\psi/\partial\phi = \partial J/\partial K$ the heuristic picture is that two curves, differing infinitesimally in their K values, make contact one with the other.

In Figures 7, 9 and 11 we show the invariant curves in Cartesian plots of $J(\phi, \theta = 0)$ for cases (i), (ii) and (iii) respectively, while in Figures 8, 10, and 12 we give the corresponding plots of $\partial J/\partial K(\phi, \theta = 0)$. The latter quantity allows us to test condition (11), since the minimum values of $\partial J/\partial K$ are quite insensitive to θ. The anticipated zeros of $\partial J/\partial K$ are on the verge of appearance in Fig. 12.

In Figures 3, 4, and 6 we show enlargements of small portions of the invariant curves for cases (i), (ii) and (iii), together with points obtained by tracking from initial conditions on the appropriate curve. An orbit from a single initial condition was followed through N turns in θ, with $N = 4000$, 4000, and 1500 for cases (i), (ii) and (iii) respectively. The good agreement between tracking and computed curves indicated in Figures 3 and 4 is maintained over the full range of ϕ. Chaotic behavior is evident in case (iii), but completely absent in case (ii). In Fig. 5 we show an intermediate case, $\epsilon_1 = 2\epsilon_2 = 1.2 \times 10^{-4}$ tracked for 3000 turns, which is ambiguous. It might represent chaos or merely a high-order island chain not yet filled in. We believe that the scatter of points in Figures 5 and 6 is genuine, since we have checked accuracy in integration of Hamilton's equations by backtracking.

Comparing Figures 12, 5 and 6, one sees that condition (11) is first met at roughly that perturbation strength at which chaotic motion appears in tracking. Actually, the Hamilton-Jacobi results for $\partial J/\partial K$ (but not those for J) are slightly ambiguous for $\epsilon_1 = 2\epsilon_2 > 10^{-4}$, since at such large perturbations we encounter a limitation on the number of modes that can be accommodated while retaining convergence of Newton's method. Thus we cannot say precisely where (11) is first satisfied. A more precise determination of the transition should be possible by using a second canonical transformation or a modification of Newton's method. Assessing present results from tracking and $\partial J/\partial K$ together, we estimate that the curve for the golden mean tune breaks up at $\epsilon_1 = 2\epsilon_2 = (1.2 \pm .05) \times 10^{-4}$.

Aside from comparisons to tracking, a stringent test of the method is to do a second canonical transformation, call it $G^{(1)}$, and see how large $G^{(1)}_\psi$ is in comparison to G_ϕ. This gives a measure of the residual distortion of the invariant surface, a correction to the main distortion of harmonic oscillator behavior obtained at the first step. We have computed $G^{(1)}$ to lowest order, which is adequate for a good estimate. Taking absolute values averaged over angles, we find that $\langle|G^{(1)}_\psi|\rangle/\langle|G_\phi|\rangle$ varies from 2.8×10^{-6} (at $\epsilon_1 = 2\epsilon_2 = 6 \times 10^{-5}$) to 4.1×10^{-3} at $\epsilon_1 = 2\epsilon_2 = 1.2 \times 10^{-4}$. Since this ratio is so small compared to 1, it seems that our surfaces are good approximations to actual KAM surfaces, even near the transition to chaos.

A related approach under study is to construct symplectic maps for tracking by solving the Hamilton-Jacobi equation for a nonperiodic G under the initial condition $G(\phi, K, 0) = 0$. Then the new variables (ψ, K) are interpreted as initial conditions (ϕ_0, J_0), and Equations (2) and (3) provide a map $(\phi_0, J_0) \mapsto (\phi, J)$ expressing time evolution of the system. It should be possible to construct a map for a full turn in θ.

Conclusions

We conclude that the Hamilton-Jacobi method provides a promising alternative to canonical perturbation theory and its modern variants. Unlike perturbation theory its algebraic complexity does not increase as more accuracy is demanded, and the required computer programs are quite simple. The generalization of (11) to higher dimensions, namely the condition $\det(1 + G_{\Phi K}) = 0$, may provide a useful criterion for the transition to chaos in the full 5-dimensional phase space of betatron motion. In principle, successive canonical transformations computed by our method on progressively larger mode sets should provide a KAM algorithm[1] with enhanced convergence, leading to exact KAM tori. We give an extended account of this work in Ref. 2.

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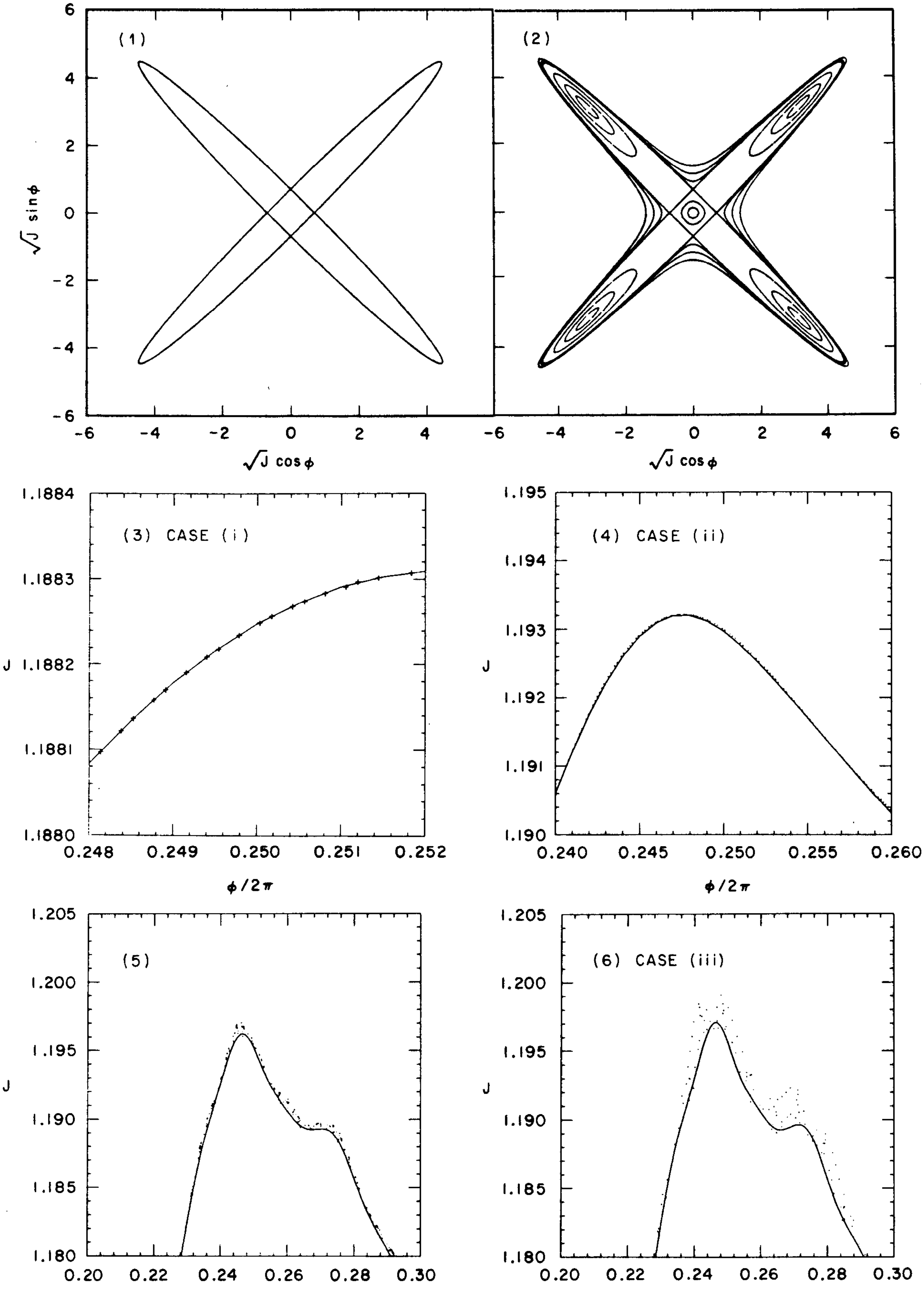

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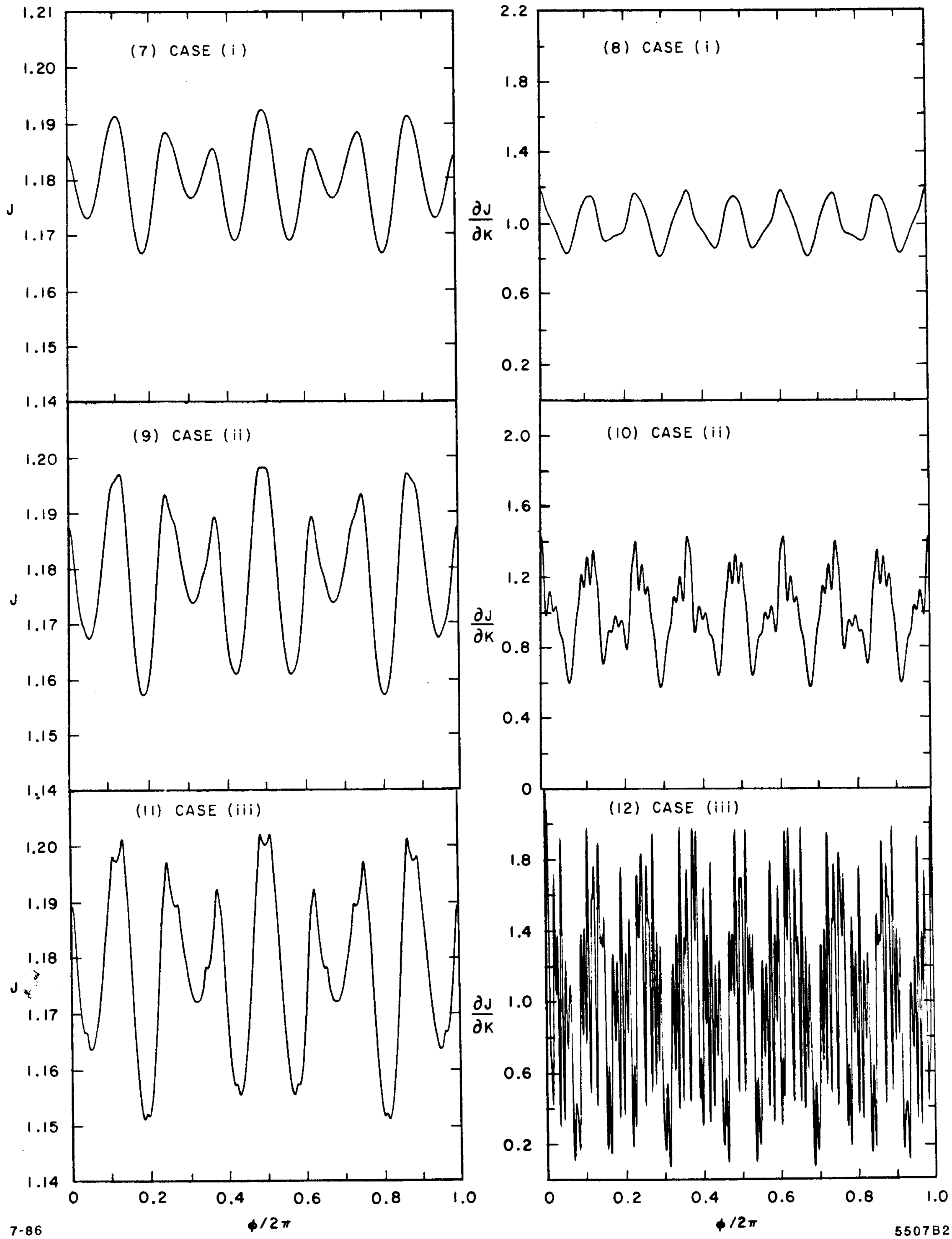
Figures 1-2: Invariant curves for the 4^{th} order single resonance model, by Newton solution of the Hamilton-Jacobi equation and analytic formula, respectively.
Figures 3-6: Small segments of invariant curves compared to points from tracking, for increasing resonance widths.



Figures 7,9,11: Invariant KAM curves $J(\phi, \theta = 0)$.
Figures 8,10,12: The Jacobians $\partial\psi/\partial\phi = \partial J/\partial K$ at $\theta = 0$.
Cases (i) - (iii) are for increasing resonance widths.