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COVARIANT CONSTANT CHROMOMAGNETIC FIELDS AND ELIMINATION OF THE ONE LOOP INSTABILITIES*

CURT A. FLORY
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

Calculation of the effective Lagrangian for QCD in the presence of a covariant constant chromomagnetic background field encounters unstable quantum fluctuations at the one loop level. Previous computations simply regularize these unstable modes, which generates a residual imaginary contribution to the effective Lagrangian. Here we show that the one loop unstable modes are completely stabilized as one goes beyond the one loop approximation, and there is no need for an ad hoc regularization prescription. We show that our higher order computation yields an effective Lagrangian whose real part agrees with previous computations, and an imaginary part which is unambiguously zero.

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1. Introduction

Much work has been done on the calculation of the effective Lagrangian for QCD in the presence of a covariant constant chromomagnetic background field.[1] In the one loop approximation the computation is ill-defined as a result of the existence of fluctuations corresponding to unstable excitations of the background field. In order to render the calculation finite the unstable modes must be regularized in the infra-red region, which generates a residual imaginary contribution to the effective Lagrangian.

Recently, Dittrich and Reuter[2] have pointed out the controversy with regard to the sign of the imaginary part. They perform the one loop computation of the effective Lagrangian using Hawking's ξ-function regularization prescription[3] and find that the imaginary part vanishes for this procedure. In their work, and the previous work referenced above, this ambiguity arises from the fact that additional input must be made to the one loop computation to specify how the unstable mode is regularized. In this work we will show that the one loop unstable modes are completely stabilized when one goes beyond the one loop approximation, and thus there is no need for an infra-red regularization prescription. We show that our higher order computation yields an effective Lagrangian whose real part agrees with previous computations[1,2] (as it must to have the proper asymptotically free limit), and an imaginary part which is unambiguously zero.

2. Computation of the Effective Lagrangian

A. The fluctuation eigenmodes

We compute the effective Lagrangian by means of the Euclidean functional integral. The explicit connection is given by[7]

$$Z_E = N \int [DA]\, exp(\textstyle\int d^4x L_E) \equiv N'\, exp\left(\textstyle\int d^4x\, L_E^{eff}\right) \tag{2.1}$$

where N and N′ are normalization constants. The Lagrangian for the pure SU(2) theory is

$$L_E = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \tag{2.2a}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\,\epsilon^{abc} A^b_\mu A^c_\nu \quad . \tag{2.2b}$$

The configuration we will study is a covariant constant field of the form

$$\bar{A}^a_\mu = -\frac{1}{2} \bar{F}_{\mu\nu} x_\nu \,\delta^{a3} \tag{2.3a}$$

where

$$\bar{F}_{12} = B \quad , \bar{F}_{03} = \epsilon B \qquad with \qquad 0 < \epsilon < 1 \quad . \tag{2.3b}$$

The magnitude of the constant chromomagnetic field is B, and we also allow for a parallel chromoelectric field of magnitude ϵB. This is the form of the most general covariant constant field for the 0(4) invariant Euclidean theory, and connection is made with the previous computations as $\epsilon \to 0$.

The gauge fields are parameterized as

$$A^a_\mu(x) = \bar{A}^a_\mu(x) + b^a_\mu(x) \tag{2.4}$$

and the Lagrangian can be expanded in powers of the fluctuations b^a_μ. With this parameterization, and introducing a background gauge fixing term[4] with gauge fixing parameter equal to one, and including the associated Fadeev-Popov term, the Euclidean functional integral becomes

$$Z_E = N \int [Db]\, exp \left\{ \int d^4x \left[-\frac{1}{4} \bar{F}^a_{\mu\nu} \bar{F}^a_{\mu\nu} + \frac{1}{2} b^a_\mu \,\bar{\Theta}^{ac}_{\mu\nu} b^c_\nu \right.\right.$$
$$\left. + g\epsilon^{acd} b^c_\mu b^d_\nu (\bar{D}_\nu)^{ae} b^e_\mu - \frac{g^2}{4}\Big((b^a_\mu b^a_\mu)^2 - (b^a_\mu b^c_\mu)(b^a_\nu b^c_\nu) \Big) \right] \tag{2.5a}$$
$$\left. + \ell n\, (-\bar{D}_\sigma \bar{D}_\sigma) \right\}$$

with

$$\bar{\Theta}^{ac}_{\mu\nu} = \delta_{\mu\nu}(\bar{D}_\sigma \bar{D}\sigma)^{ac} - 2g\,\epsilon^{adc}\,\bar{F}^d_{\mu\nu} \tag{2.5b}$$

and $\bar{D}^{ac}_\sigma = \delta^{ac}\partial_\sigma - g\,\epsilon^{acb}\bar{A}^b_\sigma$.

The functional integral of Eq. (2.5a) is normally computed in the one loop approximation. This corresponds to retaining terms in the exponential only to second order in the quantum fluctuation b^a_μ. This leaves a simple gaussian integration which is trivially done, yielding the effective Lagrangian by Eq. (2.1). The completion of this program is critically dependent upon the necessary condition that

$$\int d^4x\, b^a_\mu(x) \quad \bar{\Theta}^{ac}_{\mu\nu}\; b^c_\nu(x) \quad < \quad 0 \quad , \tag{2.6}$$

which insures that the gaussian integral is convergent. The fact that Eq. (2.6) is not satisfied for the covariant constant field is the origin of the ambiguity in the aforementioned references. In this work, we will investigate the eigenvalue equation

$$\bar{\Theta}^{ac}_{\mu\nu}\; b^c_\nu \;=\; \lambda b^a_\mu \tag{2.7}$$

and categorize the eigenmodes into those that are stable ($\lambda < 0$) and those that are unstable ($\lambda \geq 0$). This categorization is a gauge invariant procedure since the eigenvalue spectrum is trivially shown to be gauge invariant. The contribution of the eigenmodes which are stable will be computed in the usual one loop approximation and give a well defined result. The one loop unstable eigenmodes will be treated beyond the one loop approximation by retaining the higher order terms in the expanded Lagrangian of Eq. (2.5a). Due to the fact that the negative definite quartic term dominates the functional integral for large fluctuations, this also will give a well defined result, and an unambiguous contribution to the effective Lagrangian. In this way a finite effective Lagrangian will be calculated by a well defined gauge invariant procedure.

The operator $\bar{\Theta}^{ac}_{\mu\nu}$ of the eigenvalue equation, Eq. (2.7), can be written in terms of raising and lowering operators by the following familiar procedure. Define

$$\begin{aligned} a_i &\equiv \partial_i + \frac{gB}{2}x_i \qquad a_i^+ \equiv -\partial_i + \frac{gB}{2}x_i \quad , \quad i=1,2 \\ a_j &\equiv \partial_j + \frac{g\epsilon B}{2}x_j \qquad a_j^+ \equiv -\partial_j + \frac{g\epsilon B}{2}x_j \quad , \quad j=0,3 \end{aligned} \tag{2.8a}$$

and form the linear combinations

$$\begin{aligned} C &= a_0 - ia_3 \qquad D = a_1 - ia_2 \\ C^+ &= a_0^+ + ia_3^+ \qquad D^+ = a_1^+ + ia_2^+ \end{aligned} \tag{2.8b}$$

which satisfy

$$[C, C^+] = 2g\,\epsilon B \quad , \quad [D, D^+] = 2gB \; . \tag{2.8c}$$

Also defining $b^{\pm}_{\mu} \equiv b^1_{\mu} \pm ib^2_{\mu}$, the eigenvalue equation can be rewritten as

$$\left\{ \delta_{\mu\nu}(-C^+C - D^+D - gB - g\epsilon B) \mp 2ig\,F_{\mu\nu} \right\} b^{\pm}_{\nu} = \lambda b^{\pm}_{\mu} \quad . \tag{2.9}$$

The eigenvalue spectrum is immediately obtained using the commutation relations of Eq. (2.8c) and defining $b^+_{0+i3} \equiv b^+_0 + ib^+_3$, etc.:

$$\begin{aligned} b^{\pm}_{0\pm i3} &: \lambda = -2ng\epsilon B - 2mgB + g\epsilon B - gB \\ b^{\pm}_{0\mp i3} &: \lambda = -2ng\epsilon B - 2mgB - 3g\epsilon B - gB \\ b^{\pm}_{1\pm i2} &: \lambda = -2ng\epsilon B - 2mgB - g\epsilon B + gB \\ b^{\pm}_{1\mp i2} &: \lambda = -2ng\epsilon B - 2mgB - g\epsilon B - 3gB \; , \end{aligned} \tag{2.10}$$

where (m,n $= 0,1,2 \ldots$). It is now apparent which of the modes are stable, and which are unstable to one loop. All modes are stable ($\lambda < 0$) except for b^+_{1+i2} and b^-_{1-i2} with m=0 and $0 \leq n \leq \frac{1}{2\epsilon} - \frac{1}{2}$. The contribution to the functional integral for the stable and unstable modes will now be treated in turn.

B. One loop stable modes

The contribution of the one loop stable (OLS) modes to the effective Lagrangian is determined by evaluating the functional integral of Eq. (2.5a), retaining terms only to second order in the fluctuation field. This yields

$$Z_E^{(OLS)} \sim \int [\mathcal{D}b]^{(OLS)} \, exp\left\{\frac{1}{2} \int d^4x \; b_\mu^a(x) \; \bar{\Theta}_{\mu\nu}^{ac} \; b_\nu^c(x)\right\} \quad , \tag{2.11}$$

where the functional integral is over all the OLS modes. When the fluctuation fields are expanded in the eigenstates of the operator $\bar{\Theta}_{\mu\nu}^{ac}$, the functional integral becomes a product of integrals over each of the eigenmodes. For example, the contribution from the set of eigenmodes corresponding to the color and spin state b_{0+i3}^{+} is

$$\begin{aligned} Z_E^{(OLS)}(b_{0+i3}^{+}) \sim \Big[\int \prod_{n,m} db_{nm}^{(OLS)} \; exp\Big\{ \frac{1}{2}(-2ng\epsilon B - 2mgB + g\epsilon B - gB) \\ \times \int d^4x \; b_\mu^a(x) \, b_\mu^a(x)\Big\}\Big]^c . \end{aligned} \tag{2.12}$$

The power c reflects the degeneracy of each of the (n,m) eigenstates originating in the arbitrariness in the choice of the coordinate origin for the fluctuation. This constant is calculated in the appendix of reference 5, and is $\epsilon(gB/_{2\pi})^2 \int d^4x$. The gaussian integral of Eq. (2.12) yields

$$Z^{(OLS)} \, (b_{0+i3}^{+}) \sim \Big[\prod_{n,m} (2ng\epsilon B \; + \; 2mgB - g\epsilon B + gB)^{-1/2}\Big]^c , \tag{2.13}$$

which gives a contribution to the effective Lagrangian of

$$L_E^{eff}(b_{0+i3}^{+}) = -\frac{\epsilon}{2}\left(\frac{gB}{2\pi}\right)^2 \sum_{n,m} \ell n(2ng\epsilon B + 2mgB - g\epsilon B + gB) \quad . \tag{2.14}$$

Using the identity

$$\ell na = -\int_o^\infty \frac{ds}{s} \, e^{-as} \quad , \tag{2.15}$$

and doing the sum over m and n gives for the contribution of the modes from the color and spin state b_{0+i3}^{+}

$$L_E^{eff}(b_{0+i3}^{+}) = \frac{\epsilon g^2 B^2}{16\pi^2} \int_o^\infty \frac{ds}{s} \, \frac{e^{2g\epsilon Bs}}{sinh(g\epsilon Bs) sinh(gBs)} \quad . \tag{2.16}$$

Performing the same procedure of Eqs. (2.12) - (2.16) for all of the OLS eigenmodes plus the ghost terms yields the contribution to the effective Lagrangian

$$L_E^{eff}(OLS) = \frac{\epsilon g^2 B^2}{8\pi^2}\int_o^\infty \frac{ds}{s}\Big(2\frac{sinh(g\epsilon Bs)}{sinh(gBs)} + 2\frac{sinh(gBs)}{sinh(g\epsilon Bs)} + \frac{1}{sinh(g\epsilon Bs) sinh(gBs)}\Big)$$

$$- \frac{\epsilon g^2 B^2}{4\pi^2}\int_o^\infty \frac{ds}{s} \; e^{sgB(1-\epsilon)} \; \frac{(1-e^{-gBs(1+\epsilon)})}{(1-e^{-2g\epsilon Bs})} \quad . \tag{2.17}$$

Note that as expected the contribution is finite as s $\to \infty$, and has the usual UV (s $\to$ 0) singularities that will be removed by the ordinary UV renormalization procedure in Section D.

C. One loop unstable modes

The contribution of the one loop unstable (OLU) modes to the effective Lagrangian is determined by evaluating the functional integral of Eq. (2.5a), retaining all orders in the fluctuation field. Since the OLU modes have the color and spin states $b^{\pm}_{1\pm i2}$, it is easily seen that the cubic term vanishes, giving the functional integral for the OLU modes

$$Z_E^{(OLU)} \sim \Big[\int \prod_{n=0}^{N} db_n^{(OLU)} \; exp\Big\{\frac{1}{2}(-2ng\epsilon B \; - \; g\epsilon B \; + \; gB) \int d^4x \; b^a_\mu(x) b^a_\mu(x)$$

$$- \frac{g^2}{4}\int \; d^4x \; [(b^a_\mu b^a_\mu)^2 - (b^a_\mu b^c_\mu)(b^a_\nu b^c_\nu)] \Big\}\Big]^c \quad , \tag{2.18}$$

where N $= \frac{1}{2\epsilon} - \frac{1}{2}$.

The nth OLU mode is given by $(C^+)^n \; \phi^a_\mu(x)$, where $\phi^a_\mu(x)$ is the lowest eigenmode and satisfies $C\phi^a_\mu(x) = D\phi^a_\mu(x) = 0$. Using the expressions for C and D from Eq. (2.8), it is easily shown that the nth OLU mode is given by

$$b^{\pm}_{1\pm i2}(n) \; = \; \Big(\frac{\epsilon g^2 B^2}{4\pi^2}\Big)^{\frac{1}{2}} \phi^{\pm}_{1\pm i2}(C^+)^n \; exp\Big\{\frac{-gB(x_1^2 + x_2^2)}{4} - \frac{g\epsilon B(x_0^2 + x_3^2)}{4}\Big\} \tag{2.19}$$

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with the arbitrary normalization chosen such that $\int d^4x\, b^a_\mu\, b^a_\mu = 1$, and $\phi^{\pm}_{1\pm i2}$ is the constant color/spin vector for the OLU modes. Equation (2.19) can be used to simplify Eq. (2.18) to an integral over the strength of the OLU eigenmode fluctuation:

$$Z_E^{(OLU)} \sim \left[\prod_{n=0}^{N} \int_o^\infty d\xi \; exp \left\{ \frac{1}{2}(-2ng\epsilon B - g\epsilon B + gB)\xi^2 - \left(\frac{\epsilon g^4 B^2}{2\pi^2}\right)\xi^4 \right\} \right]^{2c} . \tag{2.20}$$

The factor of two in the exponent comes from the two color/spin states. Note that this integral is convergent only because of the inclusion of the quartic term in the quantum fluctuations. The B-dependence of this integral can be scaled out, giving

$$Z_E^{(OLU)} \sim \left[\prod_{n=0}^{N} \left(\frac{1}{B}\right)^{\frac{1}{2}} \int_o^\infty d\xi \; exp \left\{ \frac{1}{2}(-2ng\epsilon - g\epsilon + g)\xi^2 - \left(\frac{\epsilon g^4}{2\pi^2}\right)\xi^4 \right\} \right]^{2c} . \tag{2.21}$$

The factor coming from the finite ξ-integration is B-independent and does not show up in the renormalized effective Lagrangian. This gives the contribution of the OLU modes to the effective Lagrangian,

$$L_E^{eff}(OLU) = - \sum_{n=0}^{N} \frac{\epsilon g^2 B^2}{4\pi^2} \; \ell n B \; = -\frac{\epsilon g^2 B^2}{8\pi^2} \left(\frac{1}{\epsilon} + 1\right) \ell n B \quad . \tag{2.22}$$

D. Renormalized effective Lagrangian

The total unrenormalized effective Lagrangian is the sum of L_E^{eff}(OLS) and L_E^{eff}(OLU), Eqs. (2.17) and (2.22). It has the usual UV divergences at s $\rightarrow$ 0, and can be renormalized by the standard Coleman-Weinberg conditions [6]

$$L|_{\mathcal{F}=0} = 0 \quad , \quad \frac{\partial L}{\partial \mathcal{F}}|_{g^2\mathcal{F}=\mu^2/2} = -1 \tag{2.23}$$

where $\mathcal{F} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} = \frac{1}{2}(1+\epsilon^2)B^2$ for the field configuration under study. Applying Eq. (2.23) to the total effective Lagrangian computed above yields

$$L_E^{eff} = -\frac{1}{2}(1+\epsilon^2)B^2 - \frac{11 g^2 B^2 (1+\epsilon^2)}{48\pi^2} \left\{ \ell n\left(\frac{g(1+\epsilon^2)^{\frac{1}{2}} B}{\mu}\right) - \frac{1}{2} \right\} \quad . \tag{2.24}$$

3. Discussion

It has been shown that the fluctuations about a covariant constant field configuration can be separated into gauge invariant classes that are either stable or unstable at the one loop level. The contribution to the effective Lagrangian of the one loop stable modes was computed in the usual gaussian approximation. The contribution of the one loop unstable fluctuations was computed by retaining all orders in these OLU modes, yielding a finite well defined result. The obtained effective Lagrangian has the form for a constant chromomagnetic field ($\epsilon \to 0$)

$$L_E^{eff} = -\frac{B^2}{2} - \frac{11g^2B^2}{48\pi^2}\left\{ \ell n\left(\frac{gB}{\mu}\right) - \frac{1}{2}\right\} \tag{3.1}$$

which agrees with the real part of L^{eff} of previous calculations, and unambiguously determines the imaginary part to be zero, in agreement with the regularization procedure of Dittrich and Reuter.[2] (It should be noted that a vanishing imaginary part is what should be expected since the Lagrangian in the exponent of the functional integrand, Eq. (2.1), is a negative definite quantity for all large and real fluctuations, and thus is a well defined and real quantity.) The form of L_E^{eff} indicates that the uniform chromomagnetic vacuum field has a stable minimum at $B = \frac{\mu}{g}\, exp\left(-\frac{24\pi^2}{11g^2}\right)$ in this approximation scheme.

It is also important to realize that although the two sets of modes (which are gauge invariant sets) were treated to different orders, the results obtained are gauge invariant. The approximation is effectively that all modes are calculated to one loop, with an additional "all orders" contribution from the one loop unstable modes, neglecting cross terms between the different gauge invariant subsets. All of the modes could have been treated in the same way, i.e. to quartic order in the fluctuations, giving the same result as was obtained above, but once again with the restriction that the cross terms between gauge invariant subsets would be necessarily neglected for computational feasability.

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