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ON TRANSITION RADIATION*

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ABSTRACT

A treatment of transition radiation between two dielectric media is presented which is based on the exact expressions for the fields of the particle in the two media. Expressions for the spectral distribution of the energy emitted foreward and for that emitted backward are derived. The results are in accord with experimental findings for ultra-relativistic particles. It is indicated how the treatment can be extended to the case of a plate and to that of a wave–guide, as well as to emission by a monopole. The case of the simultaneous emission of transition radiation and Čerenkov radiation is considered and the relationship between them is clarified. It is also found that the Čerenkov wave emitted by the particle in the backward medium will be partially reflected, and partially refracted into the foreward medium, after the particle crosses the boundary between the two media. The linear energy density for the refracted wave is calculated and it is shown that under certain feasible conditions this is amenable to experimental verification.

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1. Introduction

The existence of transition radiation was first suggested by Frank and Ginzburg in 1945 ([1]). They argued that a uniformly moving charged particle must emit radiation when it crosses the boundary between two media. They gave it the name transition radiation and they derived its angular distribution for the case when a charged particle moves from vacuum into an ideal conductor. Their method was to consider the radiation to result essentially from the annihilation of the charge with its image in the conductor.

The calculation of transition radiation for the case of two dielectric media was first carried out by Beck ([2]). His approach was to use the method of images for finding the field of the particle in the two media and then to introduce the transition radiation field in order to satisfy a continuity condition on the field with time.

This same case was subsequently treated by Garibian ([3]) who looked, from the start, for wave solutions in the radiation zone. He then solved a boundary value problem with the fields expanded in plane incoming and outgoing waves. Similar treatments were given also by Ginzburg and Tsytovich ([4]). Garibian's method was later used by Dooher ([5]) to calculate transition radiation from magnetic monopoles. This same case has also been treated more recently by Frank ([6]).

Other aspects of transition radiation have also been considered in the literature. Pafomov ([7]) and Garibian and Chalikian ([8]) treated the case of transition radiation in a plate. The case of emission from a stack of foils was first treated by Garibian ([9]) and has recently been given an exhaustive treatment by Artru, Yodh and Mennessier ([10]). The case of oblique incidence of the particle on the interface between the two media was treated by Pafomov ([11]). Transition radiation in wave–guides was discussed by Barsukov ([12]). X-ray production by transition radiation in a slab was treated by Garibian ([13]), while Alikhanian and Chechin ([14]) have developed an eikonal approximation to treat this case. The relation between transition radiation and Čerenkov radiation has been investigated by Zrelov and Ružička ([15]). Ramsay and McKee ([16]) have recently studied the contribution of transition radiation to x-ray production by protons.

Experimental study of transition radiation has been quite extensive ([4,17,18]). In recent years great experimental interest has been generated by the possibility of using transition radiation as a means for the detection of ultra-relativistic particles.

Frank [19] has always emphasized the feasibility of utilizing transition radiation to this end. It is now envisaged to use transition radiation detectors in energy ranges not covered by other types of detectors or where they may possess higher efficiency [20]. Other uses have also been suggested for transition radiation in the literature. A significant recent suggestion has been made by Chu *et al.,* [21] to employ transition radiation emitters as sources of soft x-rays.

In the present article we undertake a treatment of transition radiation which attempts to complement the derivations which we referred to above. In particular, the method of Beck seems to suffer from the *ad hoc* fashion in which he introduces the radiation field. Furthermore, as we shall see later on, his particle fields do not possess the proper boundary conditions.

The method of Garibian does not suffer from any such limitations. But it is supposed to apply to the radiation zone and, in particular, away from the trajectory of the particle [9,18]. It thus does not hold for the region of small angles about the particle's path. But as Garibian [9] asserts, this region especially in the foreward direction becomes very important in the case of relativistic particles. These are the particles to which most of the experimental effort is now directed [22].

Another aspect of Garibian's method must be noted. His treatment consists in setting up the problem as a stationary scattering problem with incoming particle and free fields at $t = -\infty$ and outgoing such fields at $t = \infty$. This tends to obscure the transitional nature of the phenomenon. But what is of more significance, the resulting expressions for the radiated energy will represent not only transition radiation but also Čerenkov radiation, whenever this latter can take place. Since for the high velocities we are considering, Čerenkov emission is quite certain to occur, this limitation of Garibian's formulas could be significant. Attention has been drawn to this point by Pafomov and Frank [23]

In our treatment we start from Frank's assertion [17] that the theory of transition radiation must be a consequence of Maxwell's equations. With this spirit we base our considerations on the exact solutions for the electromagnetic field of a charged particle moving in a homogeneous medium [24,25]. To obtain transition radiation we make use of the basic idea due to Frank and Ginzburg, which has been emphasized by many authors [26]; namely, field re-adjustment. According to this idea, as the particle crosses from one medium into the other, the fields which it has established in each medium will now have to change. This brings about a re-adjustment of the fields which causes the appearance of transition radiation.

But since in our treatment we use the exact fields, we expect that our solution will give a good description of transition radiation close to the trajectory of the particle. This is the important region at high energies. Our treatment can thus be viewed as complementary to the treatment of Garibian which is supposed to hold at large angles.

We also treat the case when Čerenkov radiation can occur. We find that although transition radiation and Čerenkov radiation could occur simultaneously, they will not show mutual interference. This is because they occur for different frequency ranges. An interesting finding results from our analysis of the Čerenkov wave in one medium after the particle crosses into the other medium. We find that depending on the relative magnitudes of the susceptibilities of the two media, this wave would either suffer total reflection at the boundary or be partially reflected and partially refracted. In our opinion, this result merits experimental verification.

Our treatment also has the advantage that it can be readily generalized to the case of monopoles, dyons, and electric dipoles. This should prove useful for the experimental efforts directed at detecting them. It also constitutes a suitable framework for the treatment of transition radiation in wave-guides. Finally, since the fields we use apply to the case of media which show dispersion as well as dissipation ([25]), the present treatment can be generalized to include the presence of dissipation. However, we will not do this here. Since transition radiation must exist even in the absence of dissipation, we feel that adding this may cloud the issue and not add to our understanding of the phenomenon.

The arrangement of material is as follows. In section 2 we derive the field solutions which hold in the geometry we are using. We use these in section 3 in order to derive the spectrum of the total energy radiated in the foreward direction as well as that radiated in the backward direction. In section 4 we indicate how our method can be extended to other situations. The case when Čerenkov radiation can also be emitted we treat in section 5. We present our conclusions in section 6.

2. Transition Radiation at an Interface

2.1 The Solutions for the Electromagnetic Fields

We consider two semi-infinite homogeneous dispersive media with interface coinciding with the xy–plane. We will refer to the medium below the xy–plane as region 1 and that above it as region 2. They possess electromagnetic susceptibilities which we designate, respectively, as $\epsilon_1(\omega)$, $\mu_1(\omega)$ and $\epsilon_2(\omega)$, $\mu_2(\omega)$, where ω is the frequency of the electromagnetic radiation in the medium. Since, as we mentioned in the introduction, we are neglecting dissipation, the susceptibilities will be real and will satisfy the relations:

$$\begin{aligned} \epsilon_i(\omega) &= \epsilon_i(-\omega) \\ \mu_i(\omega) &= \mu_i(-\omega) \end{aligned} \quad , \quad \mathrm{i} = 1,2 \qquad (2.1)$$

A particle of charge e and velocity v is incident along the z–axis from $-\infty$ so that it passes the origin at $t = 0$. We assume that the particle's velocity is large enough not to suffer appreciable change over the particle's trajectory. However, we will not assume it so large as to give rise to Čerenkov radiation. We will discuss the case when Čerenkov radiation can also be emitted in section 5, below.

The fields of the particle in the two media must fulfill two conditions. They must be solutions of the appropriate Maxwell's equations and they must satisfy the proper boundary conditions. We will meet the first condition by basing our treatment on the exact solutions for the fields of a charged particle in an infinite medium. We will then find the proper combinations of these fields which will satisfy the boundary conditions.

These fields and all relations relevant to them we take from our reference ([25]). However, to avoid excessive repetition we will dispense with any further mention of this source in the sequel. It will then be understood that any relations for the fields which we introduce without demonstration come from this reference.

In order to establish our notation, we start with the case of a particle moving in an infinite homogeneous medium with susceptibilities given by $\epsilon(\omega)$, $\mu(\omega)$. The vector potential of the particle will have the following form:

$$\vec{A}(\vec{x}, t) = \vec{k} \left\{ \int_{K_1^2(\omega) < \frac{1}{\beta^2}} d\omega \; A(\rho, z, t, \omega) \; + c.c. \right\} \qquad (2.2)$$

where $\vec{k}$ is the unit vector along the z–axis, (ρ, z) are the spatial cylindrical coordinates and $K_1^2(\omega)$ is defined in th following equation. The spectral function of the vector potential, namely $A(\rho, z, t, \omega)$ is given as follows:

$$A(\rho, z, t, \omega) \;= \frac{e}{\pi c}\, \mu(\omega)\, e^{i\omega\left(\frac{z}{v} - t\right)}\, K_0\!\left(\frac{\omega\rho}{\gamma' v}\right)$$

$$\gamma' = \frac{1}{\sqrt{1-\beta'^2}} \tag{2.3}$$

$$\beta' = \frac{v}{c \,/\, \sqrt{\epsilon\mu}} \;=\; \frac{v}{c\,'(\omega)}$$

$$K^2(\omega) = \epsilon(\omega)\; \mu(\omega)$$

where K_0 is the modified Bessel function of the third kind of order zero, c is the velocity of light in vacuum, and *c.c.* stands for the complex conjugate. The spectral function for the scalar potential, namely $\varphi(\rho, z, t, \omega)$, is obtained from that for the vector potential via the gauge condition:

$$\vec{\nabla} \cdot \vec{A}(\rho, z, t, \omega) \;\; - i\, \frac{\omega}{(c/\,\epsilon\mu)}\; \varphi(\rho, z, t, \omega) \;\; = 0 \qquad . \tag{2.4}$$

The electric and magnetic fields are obtained from these potentials via their usual definitions

In the present case we have to differentiate between two physically different situations depending on whether t is smaller, or larger, than zero. For $t < 0$ the particle is incident on the boundary from region 1, whereas for $t > 0$ it is going out away from the boundary in region 2. We will designate the fields for these two cases respectively with the superscript i for incident and o for outgoing.

2.2 The Case for $t < 0$

Since the particle is in region 1 now, we must have there an incoming particle field. But since this field will be reflected at the boundary, we must also have there a free reflected field. In region 1, then, the spectral function for the vector potential will have the following form:

$$A_1^{(i)}(\rho, z, t, \omega) \;\; = \frac{e}{\pi c}\, \mu_1(\omega)\, K_0\!\left(\frac{\omega\rho}{\gamma_1' v}\right)\left[e^{i\omega\left(\frac{z}{v} - t\right)} + A_1^{(i)}(\omega)\; e^{-i\omega\left(\frac{z}{v} + t\right)}\right] \tag{2.5}$$

where $A_1^{(i)}(\omega)$ is an arbitrary parameter which will be determined by satisfying the boundary conditions at the interface

In region 2 we can only have a free–field solution. This can be viewed as that part of the particle field in region 1 which is refracted into region 2. It must then correspond to an outgoing field and will be given by:

$$A_2^{(i)}(\rho, z, t, \omega) \quad = \frac{e}{\pi c}\, \mu_2\, A_2^{(i)}(\omega)\, K_0\left(\frac{\omega\rho}{\gamma_1' v}\right) e^{i\omega\left(\frac{\eta z}{v} - t\right)} \tag{2.6}$$

where the arbitrary parameter $A_2^{(i)}(\omega)$ will again be determined by satisfying the boundary conditions at the interface. The parameter η is introduced so as to make the above potential a free–field solution in region 2. That this is not automatically so is dictated by the fact that in order to satisfy the boundary conditions at the interface, we must use γ_1' in place of γ_2' in the argument of the Bessel function.

We use the boundary conditions in their standard form ([26]); namely, that the tangential components of $\vec{E}$ and $\vec{H}$ and the normal components of $\vec{D}$ and $\vec{B}$ must be continuous across the interface. We may also point out that, in our present case, the potential and, by virtue of eq. (2.4), the divergence of the vector potential are continuous across the boundary. However, this condition contains no new information since it is not independent of that imposed on the fields. By applying the boundary conditions to the fields then, we obtain:

$$\begin{aligned} \eta &= \sqrt{1 - \beta_1'^2 + \beta_2'^2} \\ A_1^{(i)} &= \frac{\epsilon_2 - \epsilon_1 \eta}{\epsilon_2 + \epsilon_1 \eta} \\ A_2^{(i)} &= \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1 \eta} \end{aligned} \tag{2.7}$$

It will be seen from this result that the resulting solutions show the proper behaviour when $\epsilon_1 = \epsilon_2$; i.e., when the two media become identical. The reflected field then vanishes and the remaining fields in region 1 and region 2 become identical.

Finally, we must note the following. Since the Bessel functions in either medium have the same argument, all the spectral integrals for the potentials and fields will be restricted to the same domain of frequencies irrespective of the medium; namely,

$$K_1^2(\omega) < \frac{1}{\beta^2} \tag{2.8}$$

This means that the particle activates the same set of frequencies in either medium.

2.2 The Case for t > 0

In this case we have an outgoing particle in region 2. We must then have an outgoing particle field in this region. In region 1, we will have a free incident field, which is required by the way the particle is moving in region 2. But then this field will be reflected at the interface giving rise to a reflected field in region 1. We summarize this situation as follows:

$$A_1^{(o)}(\rho, z, t, \omega) = \frac{e}{\pi c}\,\mu_1 \left[A_1^{(o)}(\omega)\, e^{i\omega\left(\frac{\eta' z}{v} - t\right)} + A_1^{(o)\prime}(\omega)\, e^{-i\omega\left(\frac{\eta' z}{v} - t\right)} \right] K_0\left(\frac{\omega\rho}{\gamma_2' v}\right)$$

$$A_2^{(o)}(\rho, z, t, \omega) = \frac{e}{\pi c}\,\mu_2\, e^{i\omega\left(\frac{z}{v} - t\right)} K_0\left(\frac{\omega\rho}{\gamma_2' v}\right) \tag{2.9}$$

Again we note that the Bessel functions have the same argument in either medium. But this is different from that in the preceding case by having γ_2' replace γ_1'. The parameter η' is chosen now so as to make $A_1^{(o)}$ a free–field solution in region 1. The parameters $A_1^{(o)}(\omega)$ and $A_1^{(o)\prime}(\omega)$ are then chosen so as to satisfy the boundary conditions at the interface. The result is as follows:

$$\begin{aligned} \eta' &= \sqrt{1 + \beta_1'^2 - \beta_2'^2} \\ A_1^{(o)} &= \frac{\epsilon_2\eta' + \epsilon_1}{2\epsilon_2\eta'} \\ A_1^{(o)\prime} &= \frac{\epsilon_2\eta' - \epsilon_1}{2\epsilon_2\eta'} \end{aligned} \tag{2.10}$$

As to the frequency domain of the spectral integrals in this case, it will be given by:

$$K_2^2(\omega) < \frac{1}{\beta^2} \tag{2.11}$$

Comparison of this with eq. (2.8) shows that the set of frequencies activated by the particle for the outgoing case is different from that for the incoming one.

Finally, we are now in a position to explain our comment in the introduction regarding Beek's solution [2]. For the solution corresponding to our outgoing–solution, eq. (2.9), Beek assumes in region 2 a reflected incoming solution in addition to the particle solution. But in our view this choice is physically untenable since it would assume a source for this solution situated at $z = \infty$. Such a source does not exist in the geometry of the problem contemplated.

3. The Spectrum for Transition Radiation

In the preceding section we have seen that for $t < 0$, when the particle is in region 1, it sets up fields in the two media with spectral range given by inequality (2.8). We will represent these fields by $\left(\vec{E}_1^{(i)}, \vec{B}_1^{(i)}\right)$, $\left(\vec{E}_2^{(i)}, \vec{B}_2^{(i)}\right)$, in either medium. When the particle makes the transition from medium 1 into medium 2 at $t = 0$, it will establish a new set of fields in either medium. We represent these by $\left(\vec{E}_1^{(o)}, \vec{B}_1^{(o)}\right)$, $\left(\vec{E}_2^{(o)}, \vec{B}_2^{(o)}\right)$, for medium 1 and medium 2, respectively. What is significant now is that the spectral range for these new fields is different from that of the fields already established in the media. It is given by inequality (2.11) instead of (2.8). But since the only fields which the medium can sustain under the influence of the particle are these new fields, those already established there must undergo a re-arrangement so as to make a transition into the new fields. In particular, the frequencies activated in the medium must now subscribe to inequality (2.11). Those unsustainable degrees of freedom of the fields which are included in (2.8) and not in (2.11) must now disappear from the fields sustained by the particle. Since we are not allowing for the existence of dissipation, the only mode in which they can disappear is in the form of radiation, or more specifically, transition radiation.

If we then form the difference fields in each medium:

$$\left.\begin{aligned} \vec{E}_j &= \vec{E}_j^{(i)} - \vec{E}^j\,(o) \\ \vec{B}_j &= \vec{B}_j^{(i)} - \vec{B}_j^{(o)} \end{aligned}\right. \quad , \quad j = 1,\, 2 \quad . \tag{3.1}$$

then they will be the ones which have to disappear. These would correspond to the fields of the particle and the virtual charge in the treatment of Frank ([17]). From them we must then obtain the energy flux. That these are the fields which we must use is borne out by the fact that they vanish for two identical media. Since our main interest is centered on the derivation of the radiated energy, the sign of the above fields is immaterial.

As to the range of validity of the resulting flux, we expect it to be in, or close to, the foreward direction. This follows from the following argument. We expect the use of the exact form of the fields to be important at high velocities. But by investigating the form of the fields in our case, say in region 2, we find that they are the sum of plane waves propagating parallel to the z-axis with their amplitude modified by a falling exponential coming from the Bessel function $K_0[\omega\rho/(\gamma_1^{\prime} v)]$.

For very high velocities, $\gamma_1' \gg 1$, and so the amplitude will fall off very gradually, which permits some of these virtual waves to materialize into plane waves travelling foreward. This then justifies our assertion that our results should be considered complementary to those obtained by Garibian's scattering solution.

We calculate the energy flux in the standard fashion. For either medium we form the Poynting vector:

$$\vec{S} = \frac{c}{4\pi}\, \vec{E}_j \times \vec{H}_j \qquad , \quad j = 1,\, 2 \tag{3.2}$$

The time integral of this from $t = 0$ *to* ∞ is then the total energy flux radiated in the medium. The details of the calculation are standard and we give an outline of them in Appendix A. We will only present here the results in order to discuss them.

3.1 The Energy Flux in Region 2

We start with this region because it represents the foreward direction which is of main interest in particle detection. The energy flux in this case is given by eq. (A6) in the Appendix. It has the following form:

$$\begin{aligned}
\int_0^\infty \vec{S}_2\, dt = & \frac{2}{\pi}\, \frac{e^2}{(\beta c)^3} \left\{ \vec{\rho}_1 \int_0^\infty d\omega\, \theta\left(\frac{1}{\beta^2} - K_1^2\right) \theta\left(\frac{1}{\beta^2} - K_2^2\right) \frac{\omega^2 \sin\left(\frac{(\eta - 1)\omega^2}{v}\right)}{(\gamma_1'\gamma_2')^2\, (\epsilon_2 + \epsilon_1\eta)} \right. \\
& \times \left[\gamma_2'\, K_0\left(\frac{\omega\rho}{\gamma_1' v}\right) K_1\left(\frac{\omega\rho}{\gamma_2' v}\right) - \gamma_1'\, K_0\left(\frac{\omega\rho}{\gamma_2' v}\right) K_1\left(\frac{\omega\rho}{\gamma_1' v}\right)\right] \\
& + \vec{k} \int_0^\infty d\omega \left[\theta\left(\frac{1}{\beta^2} - K_1^2\right) \frac{2\epsilon_2\eta\omega^2}{\gamma_1'^2(\epsilon_2 + \epsilon_1\eta)^2}\right. \\
& \times K_1^2\left(\frac{\omega\rho}{\gamma_1' v}\right) + \theta\left(\frac{1}{\beta^2} - K_2^2\right) \frac{\omega^2}{2\epsilon_2\gamma_2'^2} \\
& \times K_1^2\left(\frac{\omega\rho}{\gamma_2' v}\right) - \theta\left(\frac{1}{\beta^2} - K_1^2\right) \theta\left(\frac{1}{\beta^2} - K_2^2\right) \frac{(\eta + 1)\omega^2}{\gamma_1'\gamma_2'\, (\epsilon_2 + \epsilon_1\eta)} \\
& \left.\left. \times K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) K_1\left(\frac{\omega\rho}{\gamma_2' v}\right) \cos\left(\frac{(\eta - 1)\omega^2}{v}\right)\right]\right\}
\end{aligned} \tag{3.3}$$

In the above $\vec{\rho}_1$ is the unit radial cylindrical vector and θ is the step function.

The first remark we have about this result is to note that the radial component is of order $1/\gamma$ compared with the foreward component. This means that for very high velocities transition radiation is concentrated in the foreward direction in a cone with angle $\varphi \sim 1/\gamma$. This result has been derived theoretically by many authors(4) and it has also been verified experimentally by many investigators(22)

Second, it will be seen that both components of the energy flux show z-dependence. This is natural considering that we are dealing with exact solutions which retain the short-distance behaviour of the field. At any rate, the important such term for our purposes is that occurring in the foreward component. This term has its maximum value when the two media coincide. It will then cancel the two other terms as it should. However, as the two media start to differ, this term will start to decrease because of the shrinking brought about in its domain of integration by the two different step functions occurring in it. From this we expect that transition radiation will be the stronger the more the properties of the two media differ. This property has been noted by Frank who finds that the maximum intensity occurs between vacuum and a conductor.(19) In what follows we will assume that we are dealing with two quite distinct media.

3.2 Energy Flux in the Foreward Direction

This is given by the following integral,

$$I_2 = \int_{\rho_{min}}^{\infty} \left(\int_0^{\infty} \vec{S}_2 \; dt \right) \cdot \vec{k} \; 2\pi\rho \; d\rho \quad .$$

For ρ_{min} we take the distance shorter than which our macroscopic description of electromagnetic phenomena would break down. A good choice for this is the plasma wave-length of the medium $\lambda\!\!\!{}^{-}_p$ (26). Putting this in the above integral, we then evaluate the resulting integrals using the standard tables(27).

We first consider the term showing z-dependence. This integrates to the following:

$$4\alpha(\hbar\omega_p) \int_0^{\infty} d\nu \; \theta\!\left(\frac{1}{\beta^2} - K_1^2\right) \theta\!\left(\frac{1}{\beta^2} - K_2^2\right) \gamma_2' \; \frac{(\gamma_1'/\gamma_2')}{[\,1-(\gamma_1'/\gamma_2')^2\,]}$$

$$\times \left\{ \frac{\gamma_2'}{\gamma_1'} \; \nu K_0\!\left(\frac{\gamma_2'}{\gamma_1'}\,\nu\right) K_1(\nu) - \nu \, K_0(\nu) \, K_1\!\left(\frac{\gamma_2'}{\gamma_1'}\,\nu\right) \right\} \cos\!\left[\left(\gamma_2'(\eta-1)\;\frac{z}{\lambda\!\!\!{}^{-}_p}\right)\nu\,\right] .$$

where α is the fine-structure constant, $\hbar$ is Planck's constant, ω_p is the plasma frequency in region 2 and $\nu = [\omega/(\beta\gamma_2'\omega_p)]$. We have just shown that to optimize transition radiation we must consider two quite different media. If $\beta_2' > \beta_1'$ then

$(\eta - 1) \gtrsim 1$ and $[(\eta - 1)(z/\lambda\!\!\!/_p)] \gtrsim 1$. Since $\gamma_2' \gg 1$, and all the terms multiplying the cosine function are smooth functions of ν, this function will oscillate so much for small changes in ν as to make the value of the whole integral zero. If $\beta_2' > \beta_1'$ then $(\eta - 1)$ could become very small. But then for sufficiently high velocities the product $\left(\gamma_2'(\eta - 1)(z/\lambda\!\!\!/_p)\right)$ will become very large and our reasoning again holds. This means that for two quite different media and for particles of very high velocities we can neglect the contribution of the z–dependent term to the energy flux in the foreward direction.

The remaining two terms in I_2 integrate to give us the following final result for the energy flux in the foreward direction:

$$I_2 = \alpha(\hbar\omega_p) \left\{ \int_0^\infty d\nu \left\{ \left[\theta\!\left(\frac{1}{\beta^2} - K_1^2\right) \; \frac{4\epsilon_2\eta}{(\epsilon_2 + \epsilon_1\eta)^2} \; \gamma_1' \right]_{\omega \,=\, \beta\gamma_1'\omega_p\nu} + \left[\theta\!\left(\frac{1}{\beta^2} - K_2^2\right) \frac{\gamma_2'}{\epsilon_2} \right]_{\omega \,=\, \beta\gamma_2'\omega_p\nu} \right\} \right\} f(\nu) \quad , \tag{3.4}$$

where $f(\nu)$ is the following dimensionless function:

$$f(\nu) = 2\left[\nu K_0(\nu)\, K_1(\nu) - \frac{\nu^2}{2} \left(K_1^2(\nu) - K_2^2(\nu) \right) \right] \quad . \tag{3.4a}$$

Since the terms in the curved brackets in eq. (3.4) are functions of ω, we indicate the substitution for each in terms of ν.

The first remark we make about eq. (3.4) is that since at high frequencies the dielectric constant goes to unity, the high frequency behaviour of the spectrum is determined by the function $f(\nu)$. This is a familiar function in physics.
It occurs in the semiclassical treatment of bremsstrahlung.([26]) We plot it in figure 1 on a log–log scale so as to exhibit its high frequency behaviour. We see from this figure that even at $\nu = 2.5$ this function has still the value 0.01. Since this is multiplied by $\gamma_{1,2}'$ we see that even at this frequency its contribution to the spectral integral is still appreciable. Now for substances with densities of order unity([26]) gives $\omega_p \approx 3 \times 10^{16}$ sec^{-1}. For $\gamma \sim 1000$ this would give for the frequency $\omega \approx 7.5 \times 10^{19}$ sec^{-1}. This shows then that the spectrum of transition radiation extends well into the x-ray region for ultra-relativistic velocities of the particle. This is a result which is well–established by experiment.([22])

Next, for the ultra-relativistic particles we are considering we can replace γ_1' and γ_2' by (E/mc^2) where E is the energy of the particle and mc^2 is its rest energy. By taking this factor outside the integral sign, since it is a constant within the context of our assumptions, we show that the total energy radiated is proportional to the energy of the particle. This result was first proved independently by Garibian([28]) and Barsukov([12]). It is amply verified by experiment.([29])

In the high frequency limit where we can take $\epsilon \rightarrow 1$, we can evaluate eq. (3.4) to obtain an estimate for the total radiated energy. The result is as follows:

$$I_2 \approx 2\pi\,\alpha\gamma(\hbar\omega_p) \quad .$$

From eq. (3.4) we obtain for the total number of photons emitted in the foreward direction the following expression:

$$N_2 = \frac{\alpha}{\beta}\left\{ \int_0^\infty d\nu \left\{ \left[\theta\left(\frac{1}{\beta^2} - K_1^2\right) \frac{\epsilon_2 \eta}{\left(\frac{\epsilon_2 + \epsilon_1 \eta}{2}\right)^2} \right]_{\omega \,=\, \beta\gamma_1'\omega_p\nu} + \left[\theta\left(\frac{1}{\beta^2} - K_2^2\right) \frac{1}{\epsilon_2} \right]_{\omega \,=\, \beta\gamma_2'\omega_p\nu} \right\} \right\} g(\nu) \quad , \tag{3.5}$$

where $g(\nu) = f(\nu)/\nu$,

$$g(\nu) = 2\left[K_0(\nu)\, K_1(\nu) - \frac{\nu}{2}\, (\, K_1^2(\nu) - K_2^2(\nu)\,) \right] \quad . \tag{3.6}$$

In figure 1 we give a plot of this function which, as we can see, still extends into the x–ray region.

From our estimate for I_2 we obtain the value $2\pi\alpha$ for the number of x–ray photons produced.

3.3 Energy Flux in the Backward Direction

We give the full expression for the Poynting vector in Appendix A. We will reproduce here only the z–component since we are mainly concerned with it:

$$\int_0^\infty \vec{k}\cdot\vec{S}\;dt = \Bigg\{ \int_0^\infty d\omega\;\frac{\omega^2}{\epsilon_1(\gamma_1'\gamma_2')^2}\;\Bigg[\;\theta\Big(\frac{1}{\beta^2}-K_1^2\Big)\gamma_2'^{\,2}K_1^2\Big(\frac{\omega\rho}{\gamma_1' v}\Big)\;\frac{\epsilon_2-\epsilon_1\eta}{\epsilon_2+\epsilon_1\eta}$$

$$\times\Big\{\cos\Big(\frac{\omega z}{v}\Big)-\frac{\epsilon_2-\epsilon_1\eta}{\epsilon_2+\epsilon_1\eta}\Big\}+\;\theta\Big(\frac{1}{\beta^2}-K_2^2\Big)\gamma_1'^{\,2}K_1^2\Big(\frac{\omega\rho}{\gamma_2' v}\Big)\;\frac{(\epsilon_2\eta')^2-\epsilon_1^2}{4\epsilon_2^2\eta'}$$

$$\times\Big\{\cos\Big(\frac{2\omega\eta' z}{v}\Big)-\frac{\epsilon_2\eta'-\epsilon_1}{\epsilon_2\eta'+\epsilon_1}\Big\}-\;\theta\Big(\frac{1}{\beta^2}-K_1^2\Big)\,\theta\Big(\frac{1}{\beta^2}-K_2^2\Big) \tag{3.7}$$

$$\gamma_1'\;\gamma_2' K_1\Big(\frac{\omega\rho}{\gamma_1' v}\Big)\,K_1\Big(\frac{\omega\rho}{\gamma_2' v}\Big)\Big(\frac{\epsilon_2^2\eta'-\epsilon_1^2\eta}{[\epsilon_2+\epsilon_1\eta]\epsilon_2\eta'}\;\cos\Big\{\frac{(\eta'+1)\omega z}{v}\Big\}$$

$$-\;\Big(\frac{\eta'+1}{2}\Big)\;\frac{[\epsilon_2-\epsilon_1\eta]\,[\epsilon_2\eta'-\epsilon_1]}{[\epsilon_2+\epsilon_1\eta]\,\epsilon_2\eta'}\;\cos\Big\{\frac{(\eta'-1)\omega z}{v}\Big\}\Big)\Bigg]\Bigg\}\qquad.$$

Since it is experimentally established[17] that transition radiation is emitted backwards in region 1, the above expression must be negative. This would indeed be the case except for the presence of the z–dependent term in each of the first two terms. At $z=0$ they cause each term to become positive. However, in this case we must take the third mixed term into account and it will compensate for the z–dependent terms in the first two terms leaving the whole result negative. This will be seen most readily for media close in their properties.

Alternatively, for media of quite different properties, we can apply the argument which we used with eq. (3.3) above. For any finite z the z–dependent terms would oscillate so much as to cause the integrals in which they occur to vanish. This shows that for any finite z the above result is negative. Then by continuity it must be negative at $z=0$ also. The above flux then has the correct sign.

By a similar procedure to that applied to the foreward flux, we find for the total energy emitted backward the following expression:

$$I_2=\frac{1}{2}\,\alpha(\hbar\omega_p)\int_0^\infty d\nu\;\Bigg\{\Bigg[\theta\Big(\frac{1}{\beta^2}-K_1^2\Big)\;\Big(\frac{\epsilon_2-\epsilon_1\eta}{\epsilon_2+\epsilon_1\eta}\Big)^2\;\gamma_1'\Bigg]_{\omega\;=\;\gamma_1'\beta\,\omega_p\,\nu}$$

$$+\;\Bigg[\theta\Big(\frac{1}{\beta^2}-K_2^2\Big)\Big(\frac{\epsilon_2\eta'-\epsilon_1}{2\epsilon_2\eta'}\Big)^2\;\eta'\gamma_2'\Bigg]_{\omega\;=\;\gamma_2'\beta\,\omega_p\,\nu}\quad f(\nu)\Bigg\}\qquad. \tag{3.8}$$

This result shows the same general spectral behaviour as the foreward one. Likewise it is seen to be proportional to the energy of the particle. Nevertheless they differ in two important respects. First, this result is smaller than that for the foreward case. A rough estimate would be to say that backward to foreward emission goes as $(\epsilon_2 - \epsilon_1)^2/\epsilon_2$. Furthermore, since $\epsilon \rightarrow 1$ for high frequencies, the occurrence of the terms in the difference of the dielectric constants above suppresses high frequencies. This would indicate that the spectrum for the backward radiation does not extend to high frequencies. These results agree with the findings of Frank([17]) and of Ginzburg and Tsytovich([4]).

4. Other Applications

4.1 The Case of a Plate

We consider a plate of thickness d and two infinite faces parallel to the xy–plane and with the lower face coinciding with this plane. Region 1 is the medium outside the plate and region 2 is that within the plate.

The problem now is to satisfy the boundary conditions not only at $z = 0$ but also at $z = d$. By inspecting our solutions we see that they have oscillatory terms of the form $e^{i\omega z/v}$, $e^{i\eta\omega z/v}$, and $e^{i\eta'\omega z/v}$. The only way we can make them all match at $z = d$ is if η and η' were each equal to unity. We must then investigate under what conditions we can have

$$\eta \approx \eta' \approx 1 \quad .$$

This can be the case if we have two similar media. But this is not interesting since then transition radiation will be weak. The other case is the much more interesting case of ultra-relativistic energies. We have seen that in this case the high frequency part of the spectra is enhanced. But at high velocities $\epsilon_2 \approx \epsilon_1 \approx 1$, and the above equality will hold. Hence, for high velocities we can expect to satisfy the boundary conditions for the plate to a high degree of accuracy with the following fields.

4.2 Case 1: $t < 0$ or $t > d/v$

This is the case when the particle is outside of the plate either incident or outgoing. The vector potentials are given as follows:

$$A_1(\vec{x}\,,t) = \frac{2e\beta}{d} \sum_{n=1}^{\infty} \mu_1(\omega_n)\, K_0\left(\frac{\omega_n\rho}{\gamma_1' v}\right) e^{i\omega_n\left(\frac{z}{v} - t\right)}$$

$$A_2(\vec{x}\,,t) = \frac{2e\beta}{d} \sum_{n=1}^{\infty} \Bigg\{ \mu_2(\omega_n)\, K_0\left(\frac{\omega_n\rho}{\gamma_1' v}\right)$$

$$\times \left[\frac{\epsilon_1\eta + \epsilon_2}{2\epsilon_1\eta}\, e^{i\omega_n\left(\frac{\eta z}{v} - t\right)} + \frac{\epsilon_1\eta - \epsilon_2}{2\epsilon_1\eta}\, e^{-i\omega_n\left(\frac{\eta z}{v} - t\right)} \right] \Bigg\} \quad , \tag{4.1}$$

where

$$\omega_n = n \cdot \left(\frac{2\pi v}{d}\right) \quad , \quad n = 1,\, 2,\, 3,\, \ldots \quad . \tag{4.2}$$

3.1 CASE 2: $0 < t < d/v$

This is the case when the particle is within the plate. The vector potentials have the following form:

$$A_1(\vec{x}\,,t) = \frac{2e\beta}{d} \sum_{n=1}^{\infty} \frac{2\epsilon_1\mu_1}{\epsilon_1 + \epsilon_2\eta'}\, K_0\left(\frac{\omega_n\rho}{\gamma_2' v}\right) e^{i\omega_n\left(\frac{\eta' z}{v} - t\right)}$$

$$A_2(\vec{x}\,,t) = \frac{2e\beta}{d} \sum_{n=1}^{\infty} \mu_2\, K_0\left(\frac{\omega\rho}{\gamma_2' v}\right) \tag{4.3}$$

$$\times \left[e^{i\omega_n\left(\frac{z}{v} - t\right)} + \frac{\epsilon_1 - \epsilon_2\eta'}{\epsilon_1 + \epsilon_2\eta'}\, e^{-i\omega_n\left(\frac{z}{v} - t\right)} \right] \quad .$$

Within this approximation we can then give an explanation for the highly coherent interference observed in transition radiation at very high energies.[20]

3.2 TRANSITION RADIATION IN A WAVE–GUIDE

Barsukov[12] has treated the case of transition radiation in a cylindrical wave-guide. He assumes the cylinder to have a perfectly conducting wall and to be filled with two different dielectrics with boundary at the xy–plane. Assuming the radius of the cylinder to be a, the fields for this case will be obtained from the fields which we gave in section 2, via the substitution:

$$K_0\left(\frac{\omega\rho}{\gamma_{1,2}' v}\right) \rightarrow \left[K_0\left(\frac{\omega\rho}{\gamma_{1,2}' v}\right) - \frac{K_0\left(\frac{\omega a}{\gamma_{1,2}' v}\right)}{I_0\left(\frac{\omega a}{\gamma_{1,2}' v}\right)}\, I_0\left(\frac{\omega\rho}{\gamma_{1,2}' v}\right) \right] \quad , \tag{4.4}$$

where I_0 is the modified Bessel function of the first kind of order zero. With the fields which result from this transformation, transition radiation can be calculated by the procedure which we have followed above.

3.3 The Case of Dissipation

In all our considerations so far we have neglected dissipation. But in all media dissipation exists to one degree or the other. In the general case then we must consider $\epsilon(\omega)$ and $\mu(\omega)$ as complex quantities satisfying the usual conditions:

$$\begin{aligned} \epsilon(\omega) &= \epsilon^*(-\omega) \\ \mu(\omega) &= \mu^*(-\omega) \quad , \end{aligned} \tag{4.5}$$

where the asterisk stands for complex conjugation. However, as we have shown elsewhere,[25] the fields now will still retain the same form as we have used so far. However, provision must now be made for the fact that the susceptibilities are complex quantities. In particular the domain of integration over the frequency will now be given through $Re \cdot K^2(\omega) < 1/\beta^2$.

3.4 The Case of a Magnetic Monopole

Dooher[5] has suggested the use of transition radiation in the search for monopoles. Our present results carry over readily to this case via the usual transformation. Expressed in terms of the spectral functions for the fields, this transformation has the following form:

$$\begin{aligned} \vec{E}_{mag}(\rho, z, t, \omega) &= \frac{e'}{e} \; \frac{\vec{B}_{elect}(\rho, z, t, \omega)}{\mu(\omega)} \\ \vec{B}_{mag}(\rho, z, t, \omega) &= -\frac{e'}{e} \; \epsilon(\omega) \; \vec{E}_{elect}(\rho, z, t, \omega) \quad , \end{aligned} \tag{4.6}$$

where e' stands for the magnetic charge. The effect of this transformation on the energy flux is to multiply the spectral function by $e'\epsilon_2/e\mu_2$ for foreward emission and by $e'\epsilon_1/e\mu_1$ for backward emission. Thus the total foreward emitted energy for a monopole would be given by the following expression:

$$\begin{aligned} I_2' = \alpha'(\hbar\omega_p) \int_0^\infty d\nu \; \Bigg\{ & \left[\theta\left(\frac{1}{\beta^2} - K_1^2\right) \; \frac{4\epsilon_2^2 \eta}{\mu_2(\epsilon_2 + \epsilon_1 \eta)^2} \; \gamma_1' \right]_{\omega \, = \, \beta \gamma_1' \omega_p \nu} \\ & + \left[\theta\left(\frac{1}{\beta^2} - K_2^2\right) \frac{\gamma_2'}{\mu_2} \right]_{\omega \, = \, \beta \gamma_2' \omega_p \nu} \Bigg\} f(\nu) \quad . \end{aligned} \tag{4.7}$$

Because of the extra factor of (ϵ_2/μ_2) in the integrand we expect that the spectrum for monopoles would show variation from that for electrons at low frequencies. For very high frequencies there should not be much difference due to the fact that $\epsilon_2 \to 1$.

For dyons the radiation will be the sum of the contributions coming from each type of charge on it independent of the other. This is guaranteed by the transformation (4.6).

In the standard fashion the fields of an electric (magnetic) dipole are obtained from those for an electric (magnetic) charge by operating on them with $(1/e)\,\vec{p}\cdot\vec{\nabla}$ $[(1/e')\,\vec{q}\cdot\vec{\nabla}]$ where $\vec{p}\,(\vec{q})$ is the electric (magnetic) dipole. Consequently, the present method can be readily extended to treat transition radiation from dipoles.

3.5 Čerenkov Radiation and Transition Radiation

Transition radiation is a high velocity phenomenon. But at high enough velocities, Čerenkov radiation will occur. We illustrate this in figure 2, where we give a sketch of the dielectric constant based on the one–oscillator model. The notation is clarified in Appendix B. We see that for $\beta > \beta_{min}$ there will always be Čerenkov emission. For such velocities we must write the field of the particle in the following form:

$$A(\rho, z, t) = \left\{ \frac{e}{\pi c} \int_{K^2 < \frac{1}{\beta^2}} d\omega\, \mu\, K_0\left(\frac{\omega\rho}{\gamma' v}\right) e^{i\omega\left(\frac{z}{v} - t\right)} + \frac{ie}{2c} \int_{K^2 > \frac{1}{\beta^2}} d\omega\, \mu\, H_0^{(1)}\left(\frac{\omega\rho}{\gamma' v}\right) e^{i\omega\left(\frac{z}{v} - t\right)} + c.c. \right\}, \tag{5.1}$$

where $H_0^{(1)}$ is the first Hankel function of order zero. The first term in the above expression is just the field which we have handled so far. It is responsible for transition radiation. The second term gives rise to Čerenkov radiation. It does not enter into the field rearrangement process and so it cannot contribute to transition radiation. In either of the above two terms, γ' is properly defined so as to be real.

Furthermore, due to the different spectral composition of both fields, there can be no interference between Čerenkov radiation and transition radiation. In figure

2 we take a general value for β which gives rise to Čerenkov radiation. Then we see that the Čerenkov spectrum is restricted to the interval (ω_1, ω_2), whereas the transition spectrum covers the rest. In particular, it can extend to quite high frequencies.

We now turn to the treatment of a particle moving from region 1 into region 2 as in section 2 above. If the particle can emit Čerenkov radiation in region 1, then for $t < 0$, only the first term in the field given by eq. (5.1) need satisfy the boundary conditions. The second term trails behind the particle and does not reach the boundary until the particle arrives there. What would be of interest to us is the behaviour of this term after the particle crosses the boundary. There are two cases to distinguish now depending on whether the particle could, or could not, emit Čerenkov radiation in region 2. We will now turn to the treatment of each of these two cases.

3.6 Case 1: $\beta_2' < 1$

The particle now cannot emit any Čerenkov radiation in region 2. However, the Čerenkov signal emitted by the particle for $t < 0$ will now hit the boundary, where it will suffer reflection and refraction. This can be readily deduced from the potentials which now have the following spectral form:

$$A(\rho, z, t, \omega) = \begin{cases} \frac{ie}{c}\,\mu H_0^{(1)}\left(\frac{\omega\rho}{\gamma_1' v}\right)\left[e^{i\omega\left(\frac{z}{v}-t\right)} - \frac{\epsilon_1\eta - \epsilon_2}{\epsilon_1\eta + \epsilon_2}\, e^{-i\omega\left(\frac{z}{v}-t\right)}\right], & \text{for } z < 0 \\ \frac{ie}{c}\,\frac{\epsilon_2\mu_2}{\epsilon_1\eta + \epsilon_2}\, H_0^{(1)}\left(\frac{\omega\rho}{\gamma_1' v}\right) e^{i\omega\left(\frac{\eta z}{v}-t\right)}, & \text{for } z > 0, \end{cases} \tag{5.2}$$

where all the symbols have the same meaning as before. In particular, since $\beta_1' > 1$ now, γ_1' is written as follows:

$$\gamma_1' = \frac{1}{\sqrt{\beta_1'^2 - 1}} \quad .$$

Our main interest is in the refracted wave in region 2. This will have different forms depending on the sign of η^2. From its definition, eq. (2.7), we have:

$$\eta^2 = 1 - (\beta_1'^2 - \beta_2'^2) \quad .$$

Since $\beta_1'^2 > 1$, the value of η^2 could be positive, negative or zero, depending on the relative magnitudes of $\beta_1'^2$ and $\beta_2'^2$. In figure 3 we indicate the three ranges of values of $\beta_1'^2$ and $\beta_2'^2$ which correspond to these three possibilities for the values of η^2. These ranges all lie to the right of the line $\beta_1'^2 = 1$ and below the line $\beta_2'^2 = 1$. We will now turn to the discussion of the form of the refracted wave for these three possibilities.

a) $\underline{\eta^2 > 0}$ In region 1, there will now be the original incident wave plus a reflected Čerenkov wave with apex at $z = -vt$. In region 2, the refracted wave will be represented by dispersed Čerenkov waves with vertices at $z(\omega) = vt/\eta(\omega)$. These waves are polarized along $\vec{\ell}_2(\omega)$ and their wave-fronts propagate with velocity $c_2'(\omega)$ along $\vec{n}_2(\omega)$, where:

$$
\begin{aligned}
\vec{\ell}_2(\omega) &= \frac{\eta\gamma_1' \,\vec{\rho}_1 - \vec{k}}{\beta_2'\gamma_1'} \quad , \\
\vec{n}_2(\omega) &= \frac{\vec{\rho}_1 + \eta\gamma_1' \,\vec{k}}{\beta_2'\gamma_1'} \quad .
\end{aligned}
\tag{5.3}
$$

We give the corresponding vectors for the other two waves as well as the fields for all three waves in Appendix C, and we give a schematic representation of these waves in figure 4.

For the refracted wave, the quantity of significance from the experimental point of view is the energy carried by the wave per unit length of the z–axis. To this end we use the fields given in Appendix C in order to calculate the time-integral of the Poynting flux for this wave. The result is as follows:

$$
\int_0^\infty \vec{S}_2(\rho, z, t, \omega)\, dt = \frac{2}{\pi\rho}\frac{e^2}{c^2} \int\limits_{K_1^2 > \frac{1}{\beta^2}} d\omega \, (\beta_2'\gamma_1' \,\vec{n}_2) \, \frac{\epsilon_1\epsilon_2}{(\epsilon_1\eta + \epsilon_2)^2} \, \mu_1\omega \left(1 - \frac{1}{\beta_1'^2}\right)
\tag{5.4}
$$

We give the corresponding result for the incident and the reflected waves in Appendix C. It will be seen from them that the conservation of energy is satisfied.

In calculating the radiated energy density from eq. (5.4), we must be careful to use the same normalization in region 2 as in region 1. To this effect we recall that in the usual case of a particle emitting Čerenkov radiation, the energy radiated is that expended by the particle against the reaction force of the medium. By examining eq. (5.2), we see that the refracted wave can be construed as the sum of Čerenkov waves from charges $e[\epsilon_2\mu_2/(\epsilon_1\eta + \epsilon_2)]$ moving with velocity $v/(\eta(\omega)$. Now in time dt a particle moving with velocity v will have the energy dissipated by it spread over a distance $dz = vdt$. The above virtual charge will have it spread over $dz' = (v/\eta)dt = dz/\eta$. Hence, if we want to calculate the energy spread over dz we must use not $\vec{S}(\omega)$ but $\eta(\omega)\vec{S}(\omega)$, where $\vec{S}$ refers to the integrand in eq. (5.4). With this, we have for the radiated energy density in region 2, the following expression:

$$\frac{dU_2}{dz} = \alpha \frac{u_0}{\chi_p} \int_{K_1^2 > \frac{1}{\beta^2}} d\xi \, \frac{\eta\epsilon_1\epsilon_2}{\left(\frac{\epsilon_1\eta + \epsilon_2}{2}\right)^2} \, \mu_1\xi\left(1 - \frac{1}{\beta_2'^2}\right) \quad , \tag{5.5}$$

where,

$$\begin{aligned} \xi &= \frac{\omega}{\omega_p} \\ u_0 &= \frac{\hbar\,\omega_p}{\chi_p} \quad . \end{aligned} \tag{5.6}$$

In the above, ω_p is the plasma frequency in region 1, and u_0 is the natural unit to use for the energy density in the present problem. Again, by comparing the above result with the corresponding results for the incident and the reflected waves, which we give in Appendix C, we see that energy conservation is satisfied.

To give an idea about the magnitude of this result we estimate it for the case of two purely dielectric media which are not too distinct from each other. We then consider the following choice for the relevant quantities:

$$\begin{aligned} \mu_1 = \mu_2 &= 1 \\ \beta_1'^2 &= 1 + \delta \\ \beta_2'^2 &= 1 - \delta \\ 0 < \delta &\ll 1 \quad . \end{aligned} \tag{5.7}$$

This choice corresponds to a point close to the upper left-hand corner in range 1 in figure 3. To lowest nonvanishing order in powers of δ, eq. (5.5) acquires the following form:

$$\frac{dU_2}{dz} = \alpha \frac{u_0}{\chi_p} \int_{K_1^2 > \frac{1}{\beta^2}} d\xi \left(1 - \frac{\delta^2}{4}\right) \mu_1\xi\left(1 - \frac{1}{\beta_1'^2}\right) \quad , \tag{5.8}$$

Comparing this with eq. (A3.3) we see that it differs by terms of order δ^2 from the energy radiated in the form of Čerenkov radiation by a moving charge in region 1.

Hence, under the present assumptions, it should be quite feasible to detect the refracted wave in region 2.

b) $\underline{\eta^2 = 0}$ By inspection of figure 4, we see that in the present case, the vertex of the refracted wave receeds to infinity and the conical wave front turns into a cylindrical one. However, either by calculating the time integral of the Poynting flux for this wave from its fields, or by the use of eq. (5.5), we show that the wave carries no energy. For media represented by the straight segment in Fig. 3, the incident wave in region 1 will be totally reflected. This is reminiscent of the case of total reflection of light within the denser of two media in optics.

c) $\underline{\eta^2 < 0}$ In this case the parameter η will be pure imaginary: $\eta = i|\eta|$. This affects the refracted wave in two ways: (1) this will now be a purely cylindrical wave spreading away from the z–axis, and (2) it will suffer damping along this axis. We obtain for the time-integral of the Poynting flux the following expression:

$$\int_0^\infty \vec{S}_2(\rho, z, i, \omega)\, dt = \vec{\rho}_1 \frac{2}{\pi\rho}\frac{e^2}{c^2} \int_{K_1^2 > \frac{1}{\beta^2}} d\omega\, e^{-2\frac{\omega|\eta|z}{\beta c}} \frac{\epsilon_1\epsilon_2}{(\epsilon_1^2|\eta| + \epsilon_2^2)} \mu_1\omega\left(1 - \frac{1}{\beta_1'^2}\right) \tag{5.9}$$

In order to obtain the energy density from this, we cannot now make use of the argument which we used in case (a), above. However, since our result must vanish with vanishing η, we must again introduce a factor of $|\eta|$ just as we did in that case. The result then is as follows:

$$\frac{dU_2}{dz} = \alpha\, u_0 \int_{K_1^2 > \frac{1}{\beta^2}} d\xi\, e^{-2\xi\frac{|\eta|z}{\beta \lambda\!\!\!^{-}_p}} \frac{4|\eta|\epsilon_1\epsilon_2}{(\epsilon_1^2|\eta|^2 + \epsilon_2^2)} \mu_1\xi\left(1 - \frac{1}{\beta_1'^2}\right) \quad . \tag{5.10}$$

This gives the following result for the total energy which is transmitted from region 1 into region 2:

$$U_2 = \alpha\beta(\hbar\omega_p) \int_{K_1^2 > \frac{1}{\beta^2}} d\xi \frac{2\epsilon_1\epsilon_2}{(\epsilon_1^2|\eta|^2 + \epsilon_2^2)} \mu_1\left(1 - \frac{1}{\beta_1'^2}\right) \quad . \tag{5.11}$$

From this result we see that the more the two media differ from each other, the less energy is transferred from region 1 into region 2.

The result (5.10) will prove to be more difficult to measure than result (5.5). This is not only due to the presense of the damping factor, but also because of the two factors of $|\eta|$: one in the exponent and the other in the integrand. These two factors act in opposite senses. If we choose two media which are not too dissimilar, then $|\eta|$ will be very small. This decreases the exponent which is a desirable result. However, it also decreases the value of the integrand. The other choice of two quite dissimilar media would have the opposite effect. The net result is that the integrand will always be multiplied by a small factor in comparison with that resulting from eq. (5.5).

We illustrate this with the case of two purely dielectric media which are not too dissimilar from each other. This would mean that their susceptibilities would lie close to the upper left-hand corner in range 2 of figure 3. We can then make the following choice for the relevant parameters:

$$\begin{aligned} |\eta| &= \delta \\ \beta_1'^2 &= 2 + \frac{\delta^2}{2} \quad , \\ \beta_2'^2 &= 1 - \frac{\delta^2}{2} \quad , \end{aligned} \tag{5.12}$$

where, $0 < \delta \ll 1$. To lowest order in δ, eq. (5.10) becomes:

$$\frac{dU_2}{dz} = \alpha\, u_0 \int_{K_1^2 > \frac{1}{\beta^2}} d\xi \;\cdot 4\delta \cdot \mu_1 \xi \left(1 - \frac{1}{\beta_1'^2}\right) \quad . \tag{5.13}$$

This result is of order δ compared to the energy radiated by the incident particle in region 1.

3.7 The Causality Condition

In deriving the fields to be used for the present case from those which we give in Appendix C, we must take care not to violate the causality principle. The reason for this is that for η real and positive the fields satisfy causality automatically so long as we do not extend them beyond the Čerenkov cone. But in the present case the wave fronts are cylinders which extend from $z = 0$ to infinity. Hence, in deriving the fields for this case from those given in Appendix C, we must multiply the spectral integrand by the step function $\theta(z - c_2'(\omega)t)$. This will guarantee that causality is properly taken care of. We have used these causal fields in deriving eq. (5.9).

3.8 Case 2: $\beta_2' > 1$

In this case the particle will be able to emit Čerenkov radiation in region 2 as well. For $t < 0$ there will only be a Čerenkov wave in region 1. After the particle crosses the boundary between the two regions, this wave will suffer reflection and refraction in the manner which we have discussed in the preceding case. As to the signal which the particle starts to radiate in region 2, this will travel as a normal Čerenkov wave characteristic of this region. It will be polarized along $\vec{\ell}_2'$ and will propagate along $\vec{n}_2'$, where these orthonormal vectors are given as follows:

$$\begin{aligned} \vec{\ell}_2' &= \frac{\gamma_2'\,\vec{\rho}_1 - \vec{k}}{\beta_2'\gamma_2'} \quad . \\ \vec{n}_2' &= \frac{\vec{\rho}_1 + \gamma_2'\,\vec{k}}{\beta_2'\gamma_2'} \end{aligned} \qquad (5.14)$$

Causality restricts this wave to lie entirely within the foreward cone defined by $\vec{n}_2'$ at the origin. Since the energy travels in the foreward direction, the fields from this wave which are refracted backwards at the boundary can transmit no energy into region 1 and so need not be considered by us.

As to the refracted wave, its behaviour is similar to that which we discussed for the corresponding wave in case 1, above. The allowed ranges of $\beta_1'^2$ and $\beta_2'^2$ are indicated in figure 5. They all lie to the right of the line $\beta_1'^2 = 1$ and above the line $\beta_2'^2 = 1$. In particular, the range of values which allows a nonattenuated wave is quite extensive now. It splits into a domain with $\eta^2 > 1$ and another with $\eta^2 < 1$ as we show on the same figure. Thus the refracted wave and the signal from the particle each has its apex travelling at a different velocity from the other. For this case, then, we expect to see two Čerenkov signals which travel one behind the other. These will differ in the magnitude of their intensity as well as in their spectral distribution. In figure 6 we give a schematic representation of all four waves for the case when the refracted wave precedes the particle. As to the fields and the radiated energy density, they can all be deduced from the results which we have presented in case 1, above.
There is no need for us to reproduce them here.

4. Conclusions

Our treatment of transition radiation based on the idea of field rearrangement suggested by Frank is confirmed by the comparisons which we have made with both theoretical and experimental results. We have seen that our results agree with all findings for highly energetic particles. This prompts us to suggest that experimental efforts be undertaken to verify the spectral distributions which we obtain for the energy emitted in either the foreward or the backward direction.

We would also like to draw attention to the case of the refracted Čerenkov wave in the foreward medium. When the particle does not emit Čerenkov radiation there, this may be mistaken for part of transition radiation. The same may also happen with the reflected Čerenkov wave in the backward medium. At any rate, we feel that an experimental effort needs to be made in order to verify our results for the refracted wave in the foreward medium.

Finally, we point out that the treatment which we have presented above holds for media which show dispersion (and also dissipation) in both ϵ and μ. This should make it applicable to a wide range of media.

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APPENDIX A

A.1 The Explicit Expressions for the Fields

CASE 1: $t < 0$

We express the potentials and the fields as spectral integrals over the shown domain of frequencies which we illustrate for the electric field in region 1:

$$E_1^{(i)}(\rho, z, t, \omega) = \left\{ \int_{K_1^2(\omega) < \frac{1}{\beta^2}} d\omega \; E_1^{(i)}(\rho, z, t, \omega) + c.c. \right\} \quad . \tag{A1}$$

With this notation the respective spectral functions for the fields have the following form:

$$E_1^{(i)}(\rho, z, t, \omega) = \frac{e}{\pi(\beta c)^2} \frac{\omega}{\epsilon_1 \gamma_1'^2} \left\{ -i\vec{k}\, K_0\left(\frac{\omega\rho}{\gamma_1' v}\right) \left[e^{i\omega\left(\frac{z}{v} - t\right)} + \frac{\epsilon_2 - \epsilon_1\eta}{\epsilon_2 + \epsilon_1\eta} \right.\right.$$

$$\left.\left. e^{-i\omega\left(\frac{z}{v} + t\right)} \right] + \vec{\rho}_1\, \gamma_1' K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) \left[e^{i\omega\left(\frac{z}{v} - t\right)} - \frac{\epsilon_2 - \epsilon_1\eta}{\epsilon_2 + \epsilon_1\eta}\, e^{-i\omega\left(\frac{z}{v} + t\right)} \right] \right\} \quad ,$$

$$B_1^{(i)}(\rho, z, t, \omega) = \vec{\varphi}_1 \frac{\beta e}{\pi(\beta c)^2} \frac{\mu_1 \omega}{\gamma_1'} K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) \left(e^{i\omega\left(\frac{z}{v} - t\right)} + \frac{\epsilon_2 - \epsilon_1\eta}{\epsilon_2 + \epsilon_1\eta}\, e^{-i\omega\left(\frac{z}{v} + t\right)} \right) \quad ,$$

$$E_2^{(i)}(\rho, z, t, \omega) = \frac{e}{\pi(\beta c)^2} \frac{\omega}{\gamma_1'^2} \frac{2}{(\epsilon_2 + \epsilon_1\eta)}$$

$$\left[-i\vec{k}\, K_0\left(\frac{\omega\rho}{\gamma_1' v}\right) + \vec{\rho}_1\, \gamma_1' \eta K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) \right] e^{i\omega\left(\frac{\eta z}{v} - t\right)} ,$$

$$B_2^{(i)}(\rho, z, t, \omega) = \vec{\varphi}_1 \frac{\beta e}{\pi(\beta c)^2} \frac{\omega}{\gamma_1'} \frac{2\epsilon_2\mu_2}{\epsilon_2 + \epsilon_1\eta} K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) e^{i\omega\left(\frac{\eta z}{v} - t\right)} \quad . \tag{A2}$$

CASE 2: $t > 0$

In this case the potentials and fields are still defined as in (A1) except that now the domain of integration over frequency is given by:

$$K_2^2(\omega) < \frac{1}{\beta^2} \quad . \tag{A3}$$

The spectral functions for the fields have the following form:

$$E_1^{(o)}(\rho,z,t,\omega) = \frac{e}{\pi(\beta c)^2}\,\frac{\omega}{\gamma_2'^2}\,\frac{1}{2\epsilon_1\epsilon_2\eta'}$$

$$\times\left\{-i\vec{k}\,K_0\!\left(\frac{\omega\rho}{\gamma_2' v}\right)\left[(\epsilon_2\eta'+\epsilon_1)\,e^{i\omega\left(\frac{\eta' z}{v}-t\right)} + (\epsilon_2\eta-\epsilon_1)\,e^{-i\omega\left(\frac{\eta' z}{v}+t\right)}\right]\right.$$

$$\left.+\ \vec{\rho}_1\,\gamma_1'\eta' K_1\!\left(\frac{\omega\rho}{\gamma_2' v}\right)\left[(\epsilon_2\eta'+\epsilon_1)\,e^{i\omega\left(\frac{\eta' z}{v}-t\right)} - (\epsilon_2\eta'-\epsilon_1)\,e^{-i\omega\left(\frac{\eta' z}{v}+t\right)}\right]\right\},$$

$$B_1^{(o)}(\rho,z,t,\omega) = \vec{\phi}_1\frac{\beta e}{\pi(\beta c)^2}\,\frac{\omega}{\gamma_2'}\,\frac{\mu_1}{2\epsilon_2\eta'}\,K_1\!\left(\frac{\omega\rho}{\gamma_2' v}\right)$$

$$\times\left((\epsilon_2\eta'+\epsilon_1)\,e^{i\omega\left(\frac{\eta' z}{v}-t\right)} + (\epsilon_2\eta'-\epsilon_1)\,e^{-i\omega\left(\frac{\eta' z}{v}+t\right)}\right),$$

$$E_2^{(o)}(\rho,z,t,\omega) = \frac{e}{\pi(\beta c)^2}\,\frac{\omega}{\epsilon_2\gamma_2'^2}\left[-i\vec{k}\,K_0\!\left(\frac{\omega\rho}{\gamma_2' v}\right) + \vec{\rho}_1\,\gamma_2' K_1\!\left(\frac{\omega\rho}{\gamma_2' v}\right)\right]e^{i\omega\left(\frac{z}{v}-t\right)},$$

$$B_2^{(o)}(\rho,z,t,\omega) = \vec{\phi}_1\frac{\beta e}{\pi(\beta c)^2}\,\frac{\omega}{\gamma_2'}\,\mu_2\,K_1\!\left(\frac{\omega\rho}{\gamma_2' v}\right)e^{i\omega\left(\frac{z}{v}-t\right)}. \tag{A4}$$

A.2 The Energy Flux

We now form the difference fields from those given above as indicated in the text. Since we are integrating over time for $t = 0$ to ∞, we must use the step function in order to convert this to an integral from $-\infty$ to ∞. To this end we must also convert the integral over frequency to the limits $-\infty$ to ∞. We write the fields as follows:

$$\vec{E}_2 = \frac{e}{\pi(\beta c)^2} \quad (\vec{k}\, E_\ell + \vec{\rho}_1\, E_t)$$

$$\vec{H}_2 = \vec{\varphi}_1 \; \frac{\beta e}{\pi(\beta c)^2}\; H_2 \qquad .$$

From E_ℓ, E_t, H_2, we define three functions F_2, G_2, L_2, respectively such that each satisfies the relation $F_2^*(\omega) = F_2(-\omega)$. Thus from

$$E_\ell = -i\omega\Bigg[\theta\Big(K_1^2 - \frac{1}{\beta^2}\Big)\; \frac{2}{(\epsilon_2+\epsilon_1\eta)\gamma_1'^2}\; K_1\Big(\frac{\omega\rho}{\gamma_1' v}\Big)\; e^{i\omega\eta z/v}$$
$$- \;\theta\Big(K_2^2 - \frac{1}{\beta^2}\Big)\; \frac{1}{\epsilon_2\gamma_2'^2}\, K_0\Big(\frac{\omega\rho}{\gamma_2' v}\Big)\; e^{i\omega z/v}\Bigg]$$

we define:

$$F_2 = -i\omega\Bigg[\theta\Big(K_1^2 - \frac{1}{\beta^2}\Big)\; \frac{2}{(\epsilon_2+\epsilon_1\eta)\gamma_1'^2}\; K_1\Big(\Big|\frac{\omega\rho}{\gamma_1' v}\Big|\Big)\; e^{i\omega\eta z/v}$$
$$+ \;\theta\Big(K_2^2 - \frac{1}{\beta^2}\Big)\; \frac{1}{\epsilon_2\gamma_2'^2}\; K_0\Big(\Big|\frac{\omega\rho}{\gamma_2' v}\Big|\Big)\; e^{i\omega z/v}\Bigg] \qquad . \tag{A5}$$

With the above definitions the Poynting flux takes on the following form:

$$\vec{S} = \beta c\left(\frac{e}{2\pi(\beta c)^2}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\; d\omega'\left[\vec{k}\; G_2 - \vec{\rho}_1\; F_2\right] L_2(\omega')\; e^{-i(\omega+\omega')t} \qquad .$$

By use of the integral representation for the step function in time, we obtain the following result for the time integral:

$$\int_0^\infty \vec{S}\; dt = \beta c\left(\frac{e}{2\pi(\beta c)^2}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\omega\; d\omega' \;\; \frac{[\vec{k}\; G_2(\omega) - \vec{\rho}_1\; F_2(\omega)]\; L_2(\omega')}{i(\omega+\omega'-i\epsilon)} \qquad .$$

We integrate this in the complex ω'–plane and then express the result in the components of the field to obtain:

$$\int_0^\infty \vec{S}\ dt = \frac{e^2}{2\pi(\beta c)^3}\left[\int_0^\infty d\omega\left(\vec{k}\ E_t - \vec{\rho}_1\ E_\ell\right) H^* + \text{c.c.}\right] \quad .$$

From this we obtain the final result

$$\begin{aligned}
\int_0^\infty \vec{S}\ dt = \frac{2e^2}{\pi(\beta c)^3}\Bigg\{ & \vec{\rho}_1 \int_0^\infty d\omega\ \theta\left(K_1^2 - \frac{1}{\beta^2}\right)\theta\left(K_2^2 - \frac{1}{\beta^2}\right) \frac{\omega^2 sin\left(\frac{(\eta-1)\omega z}{v}\right)}{(\gamma_1'\gamma_2')^2(\epsilon_2+\epsilon_1\eta)} \\
& \times\left[\gamma_2'\ K_0\left(\frac{\omega\rho}{\gamma_1' v}\right) K_1\left(\frac{\omega\rho}{\gamma_2' v}\right) - \gamma_1'\ K_0\left(\frac{\omega\rho}{\gamma_2' v}\right) K_1\left(\frac{\omega\rho}{\gamma_1' v}\right)\right] \\
& + \vec{k}\left[\int_0^\infty d\omega\ \theta\left(K_1^2 - \frac{1}{\beta^2}\right)\frac{2\epsilon_2\eta\omega^2}{\gamma_1'^2(\epsilon_2+\epsilon_1\eta)^2}\ K_1^2\left(\frac{\omega\rho}{\gamma_1' v}\right)\right. \\
& + \int_0^\infty d\omega\ \theta\left(K_2^2 - \frac{1}{\beta^2}\right)\frac{\omega^2}{2\epsilon_2\gamma_2'^2}\ K_1^2\left(\frac{\omega\rho}{\gamma_2' v}\right) \\
& - \int_0^\infty d\omega\ \theta\left(K_1^2 - \frac{1}{\beta^2}\right)\theta\left(K_2^2 - \frac{1}{\beta^2}\right)\frac{(\eta+1)\,\omega^2}{\gamma_1'\gamma_2'(\epsilon_2+\epsilon_1\eta)} \\
& \left.\times\ K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) K_1\left(\frac{\omega\rho}{\gamma_2' v}\right) cos\left(\frac{(\eta-1)\,\omega z}{v}\right)\right]\Bigg\} \quad .
\end{aligned} \tag{A6}$$

For the case of the energy radiated in region 1 we follow the same procedure and we obtain the following result:

$$\int_0^\infty \vec{S}\, dt = \frac{1}{\pi}\, \frac{e^2}{(\beta c)^3} \left\{ \vec{k} \int_0^\infty d\omega \; \frac{\omega^2}{\epsilon_1 \gamma_1' \gamma_2'} \left[\theta\left(K_1^2 - \frac{1}{\beta^2}\right) \gamma_2'\, K_1^2\left(\frac{\omega\rho}{\gamma_1' v}\right) \frac{\epsilon_2 - \epsilon_1\eta}{\epsilon_2 + \epsilon_1\eta} \right.\right.$$

$$\times \left(\cos\left(\frac{\omega z}{v}\right) - \frac{\epsilon_2 - \epsilon_1\eta}{\epsilon_2 + \epsilon_1\eta} \right) + \theta\left(K_2^2 - \frac{1}{\beta^2}\right) \gamma_1'^2\, K_1^2\left(\frac{\omega\rho}{\gamma_2' v}\right) \frac{(\epsilon_2\eta')^2 - \epsilon_1^2}{4\epsilon_2^2\eta'}$$

$$\times \left(\cos\left(\frac{2\omega\eta' z}{v}\right) - \frac{\epsilon_2\eta' - \epsilon_1}{\epsilon_2\eta' + \epsilon_1} \right) - \theta\left(K_1^2 - \frac{1}{\beta^2}\right) \theta\left(K_2^2 - \frac{1}{\beta^2}\right)$$

$$\times \gamma_1' \gamma_2' K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) K_1\left(\frac{\omega\rho}{\gamma_2' v}\right) \left(\frac{\epsilon_2^2\eta' - \epsilon_1^2\eta}{(\epsilon_2 + \epsilon_1\eta)\epsilon_2\eta'} \cos\left(\frac{(\eta'+1)\omega z}{v}\right) \right.$$

$$\left.\left. - \frac{\eta'+1}{2}\; \frac{(\epsilon_2 - \epsilon_1\eta)(\epsilon_2\eta' - \epsilon_1)}{(\epsilon_2 + \epsilon_1\eta)\,\epsilon_2\eta'}\; \cos\left(\frac{(\eta'-1)\omega z}{v}\right)\right)\right]$$

$$- \vec{\rho}_1 \int_0^\infty d\omega \; \frac{\omega^2}{\epsilon_1(\gamma_1'\gamma_2')^3} \left[\theta\left(K_1^2 - \frac{1}{\beta^2}\right) \frac{\epsilon_2 - \epsilon_1\eta}{\epsilon_2 + \epsilon_1\eta}\; \gamma_2'\, K_0\left(\frac{\omega\rho}{\gamma_1' v}\right) K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) sin\left(\frac{2\omega z}{v}\right) \right.$$

$$+ \theta\left(K_2^2 - \frac{1}{\beta^2}\right) \frac{(\epsilon_2\eta')^2 - \epsilon_1^2}{(2\epsilon_2\eta')^2} \gamma_1'^2\, K_0\left(\frac{\omega\rho}{\gamma_2' v}\right) K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) cos\left(\frac{2\omega\eta' z}{v}\right)$$

$$- \theta\left(K_1^2 - \frac{1}{\beta^2}\right) \theta\left(K_2^2 - \frac{1}{\beta^2}\right) \gamma_1' \gamma_2' \left(\frac{\epsilon_2\eta' + \epsilon_1}{2\epsilon_2\eta'} \left\{ \frac{\epsilon_2 - \epsilon_1\eta}{\epsilon_2 + \epsilon_1\eta} \right.\right.$$

$$K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) K_0\left(\frac{\omega\rho}{\gamma_2' v}\right) + \frac{\epsilon_2\eta' - \epsilon_1}{\epsilon_2\eta' + \epsilon_1}$$

$$\left. K_0\left(\frac{\omega\rho}{\gamma_1' v}\right) K_1\left(\frac{\omega\rho}{\gamma_2' v}\right) \right\} sin\left(\frac{(\eta'+1)\omega z}{v}\right) \Bigg) + \left(\frac{\epsilon_2\eta' - \epsilon_1}{2\epsilon_2\eta'} \left\{ \frac{\epsilon_2\eta' - \epsilon_1}{2\epsilon_2\eta'} \right.\right.$$

$$K_0\left(\frac{\omega\rho}{\gamma_1' v}\right) K_1\left(\frac{\omega\rho}{\gamma_2' v}\right) - \frac{\epsilon_2 - \epsilon_1\eta'}{\epsilon_2 + \epsilon_1\eta'}$$

$$\left.\left.\left. K_1\left(\frac{\omega\rho}{\gamma_1' v}\right) K_0\left(\frac{\omega\rho}{\gamma_2' v}\right) \right\} sin\left(\frac{(\eta'-1)\omega z}{v}\right)\right)\right]\right\} \quad . \tag{A7}$$

APPENDIX B

B.1 The Dielectric Constant

We make use of the standard expression for the dielectric constant given by the single–oscillator model. We write this as follows:

$$\epsilon\left(\frac{\omega}{\omega_0}\right) = 1 + \frac{(\epsilon_0 - 1)}{1 - \left(\frac{\omega}{\omega_0}\right)^2 - 2i\xi\left(\frac{\omega}{\omega_0}\right)^2} \quad , \tag{B1}$$

where ω_0 is the natural frequency of the oscillator, ϵ_0 is the static dielectric constant and $\xi = (p/\omega_0)$, $2p$ being the coefficient of the dissipative force. The function which we plot in figure 2 is given by:

$$Re\ \epsilon\left(\frac{\omega}{\omega_0}\right) = 1 + (\epsilon_0 - 1)\ \frac{1 - \left(\frac{\omega}{\omega_0}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left(2\xi\,\frac{\omega}{\omega_0}\right)^2} \quad . \tag{B2}$$

APPENDIX C

Results for the Čerenkov Waves in both Media

The spectral functions for the asymptotic fields of the incident and the reflected waves, respectively, in region 1 are given as follows:

$$\vec{E}_1(\rho, z, t, \omega) = \frac{e}{\sqrt{2\pi(\beta c)^3 \rho}} \; \vec{\ell}_1(\omega) \, \frac{\beta_1'}{\epsilon_1} \sqrt{\frac{\omega}{\gamma_1'}} \; e^{i\left[\omega\left(\frac{\vec{n}_1 \cdot \vec{r}}{c_1'} - t\right) - \frac{\pi}{4}\right]}$$

$$\vec{B}_1(\rho, z, t, \omega) = \vec{\varphi}_1 \frac{\beta e}{\sqrt{\pi(\beta c)^3 \rho}} \; \mu_1 \sqrt{\frac{\omega}{\gamma_1'}} \; e^{i\left[\omega\left(\frac{\vec{n}_1 \cdot \vec{r}}{c_1'} - t\right) - \frac{\pi}{4}\right]}$$

$$\vec{E}_1'(\rho, z, t, \omega) = \frac{e}{\sqrt{\pi(\beta c)^3 \rho}} \; \vec{\ell}_1'(\omega) \frac{\epsilon_1 \eta - \epsilon_2}{\epsilon_1 \eta + \epsilon_2} \, \frac{\beta_1'}{\epsilon_1} \sqrt{\frac{\omega}{\gamma_1'}} \; e^{i\left[\omega\left(\frac{\vec{n}_1' \cdot \vec{r}}{c_1'} - t\right) - \frac{\pi}{4}\right]}$$

$$\vec{B}_1'(\rho, z, t, \omega) = -\vec{\varphi}_1 \frac{\beta e}{\sqrt{\pi(\beta c)^3 \rho}} \frac{\epsilon_1 \eta - \epsilon_2}{\epsilon_1 \eta + \epsilon_2} \; \mu_1 \sqrt{\frac{\omega}{\gamma_1'}} \; e^{i\left[\omega\left(\frac{\vec{n}_1' \cdot \vec{r}}{c_1'} - t\right) - \frac{\pi}{4}\right]} \tag{C1}$$

The two sets of orthonormal vectors occurring above are defined as follows:

$$\begin{aligned} \vec{\ell}_1 &= \frac{\gamma_1' \, \vec{\rho}_1 - \vec{k}}{\beta_1' \gamma_1'} \,, & \vec{n}_1 &= \frac{\vec{\rho}_1 + \gamma_1' \, \vec{k}}{\beta_1' \gamma_1'} \\ \vec{\ell}_1' &= \frac{\gamma_1' \, \vec{\rho}_1 + \vec{k}}{\beta_1' \gamma_1'} \,, & \vec{n}_1' &= \frac{\vec{\rho}_1 - \gamma_1' \, \vec{k}}{\beta_1' \gamma_1'} \end{aligned} \quad . \tag{C2}$$

The respective time-integrals for the Poynting flux of these waves are given by the following expressions:

$$\begin{aligned} \int_0^\infty \vec{S}_1(\rho, z, t, \omega) \, dt &= \frac{1}{2\pi\rho} \frac{e^2}{(\beta c)^2} \int\limits_{K_1^2 < \frac{1}{\beta^2}} d\omega \; \vec{n}_1 \, \frac{\beta_1' \omega}{\epsilon_1 \gamma_1'} \,, \\ \int_0^\infty \vec{S}_1'(\rho, z, t, \omega) \, dt &= \frac{1}{2\pi\rho} \frac{e^2}{(\beta c)^2} \int\limits_{K_1^2 > \frac{1}{\beta^2}} d\omega \; \vec{n}_1' \, \frac{\beta_1' \omega}{\epsilon_1 \gamma_1'} \,. \end{aligned} \tag{C3}$$

The interference terms between the incident and the reflected waves all vanish.

The spectral functions for the asymptotic fields of the Čerenkov wave refracted into region 2 have the following forms:

$$\vec{E}_2(\rho, z, t, \omega) = \frac{e}{\sqrt{2\pi(\beta c)^3 \rho}} \ \vec{\ell}_2 \ \frac{\epsilon_2}{\epsilon_1 \eta + \epsilon_2} \ \frac{\beta_2'}{\epsilon_2} \sqrt{\frac{\omega}{\gamma_1'}} \ e^{i\left[\omega\left(\frac{\vec{n}_2 \cdot \vec{r}}{c_2'} - t\right) - \frac{\pi}{4}\right]} \quad ,$$

$$\vec{B}_2(\rho, z, t, \omega) = \vec{\varphi}_1 \frac{\beta e}{\sqrt{2\pi(\beta c)^3 \rho}} \frac{\epsilon_2}{\epsilon_1 \eta + \epsilon_2} \ \mu_2 \sqrt{\frac{\omega}{\gamma_1'}} \ e^{i\left[\omega\left(\frac{\vec{n}_2 \cdot \vec{r}}{c_2'} - t\right) \frac{\pi}{4}\right]} \quad . \tag{C4}$$

where $\vec{\ell}_2, \vec{n}_2$ are given by eq. (5.3).

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Figure Captions

Figure 1. A plot of the dimensionless spectral functions $f(\nu)$ and $g(\nu)$.

Figure 2. A schematic plot of $Re \cdot \epsilon$ as a function of $\left(\omega/\omega_0\right)$.

Figure 3. The ranges of $\beta_1'^2$, $\beta_2'^2$ corresponding to the three possible ranges of values of η^2 for the case $\beta_1'^2 > 1$, $\beta_2'^2 < 1$.

Figure 4. Schematic representation of the incident and the reflected Čerenkov waves in region 1 and of the refracted Čerenkov wave in region 2.

Figure 5. The ranges of $\beta_1'^2$, $\beta_2'^2$ corresponding to the three possible ranges of values of η^2 for the case $\beta_1'^2 > 1$, $\beta_2'^2 > 1$.

Figure 6. Schematic representation of the incident, the reflected and the refracted Čerenkov waves and of the Čerenkov wave emitted by the particle in region 1.

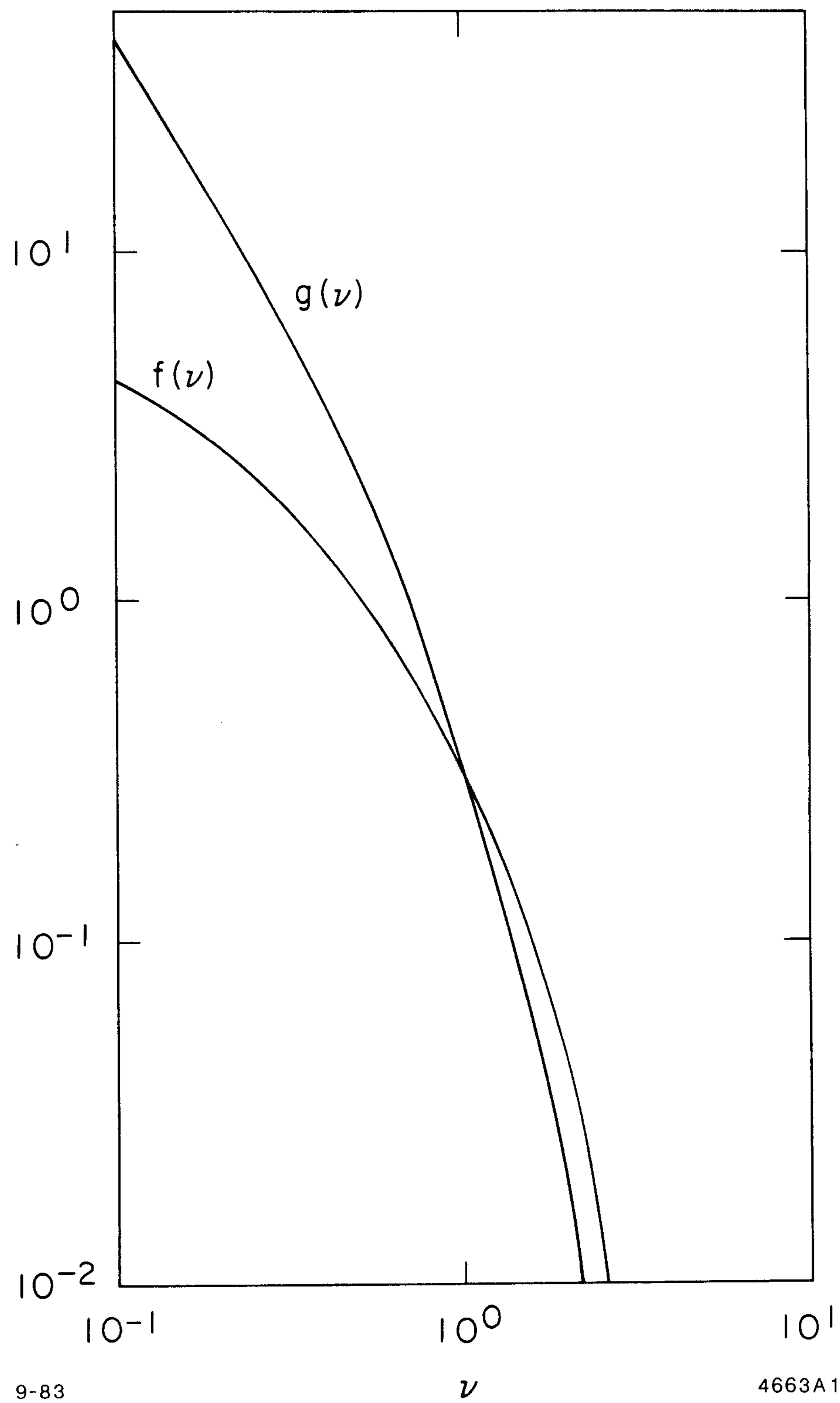
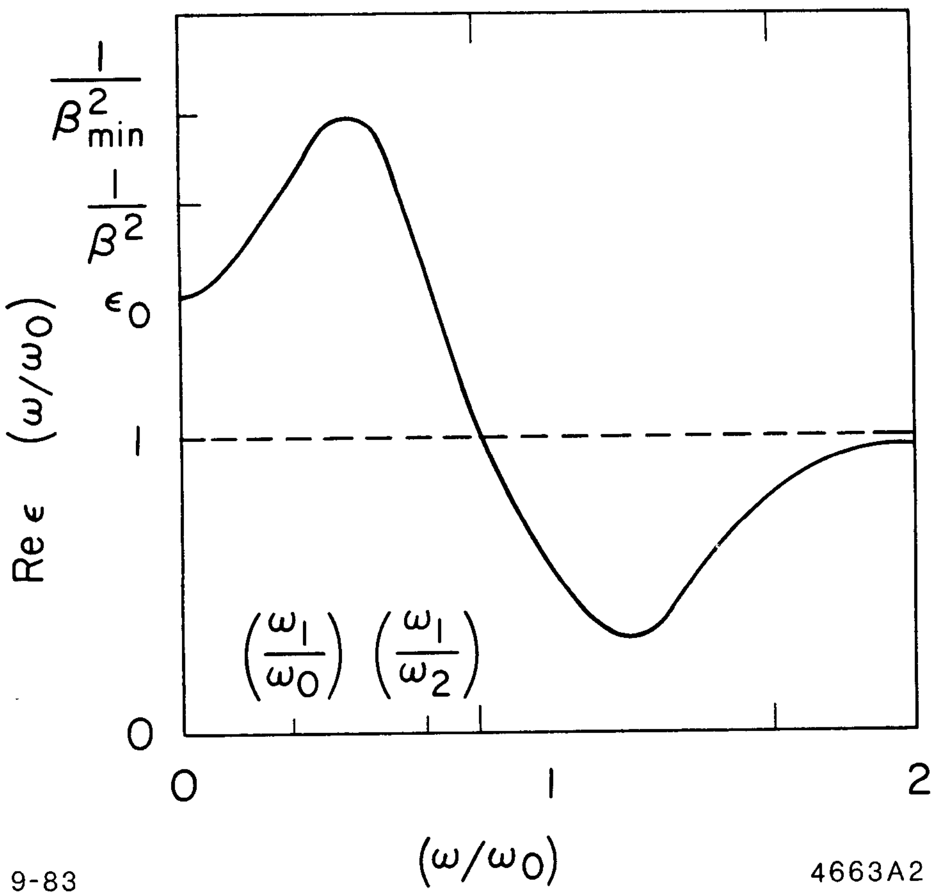


Fig. 1



$\frac{1}{\beta^2_{min}}$
$\frac{1}{\beta^2}$
ϵ_0
Re ϵ (ω/ω_0)
1
0
$\left(\frac{\omega_1}{\omega_0}\right)$ $\left(\frac{\omega_1}{\omega_2}\right)$
0
1
2
(ω/ω_0)
9-83
4663A2

Fig. 2

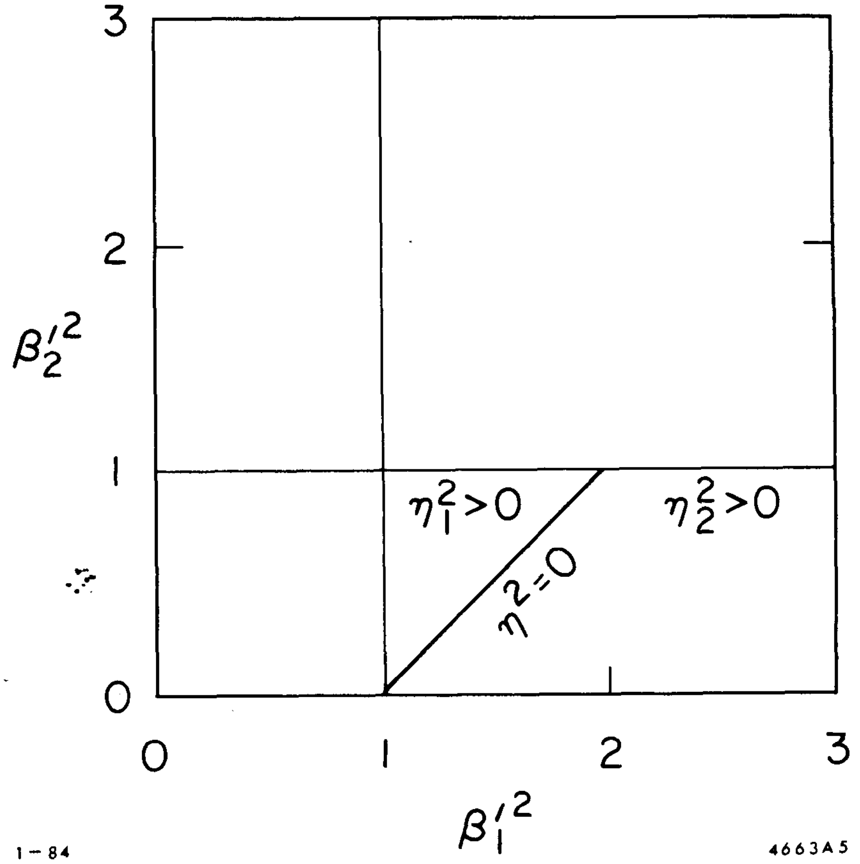


Fig. 3

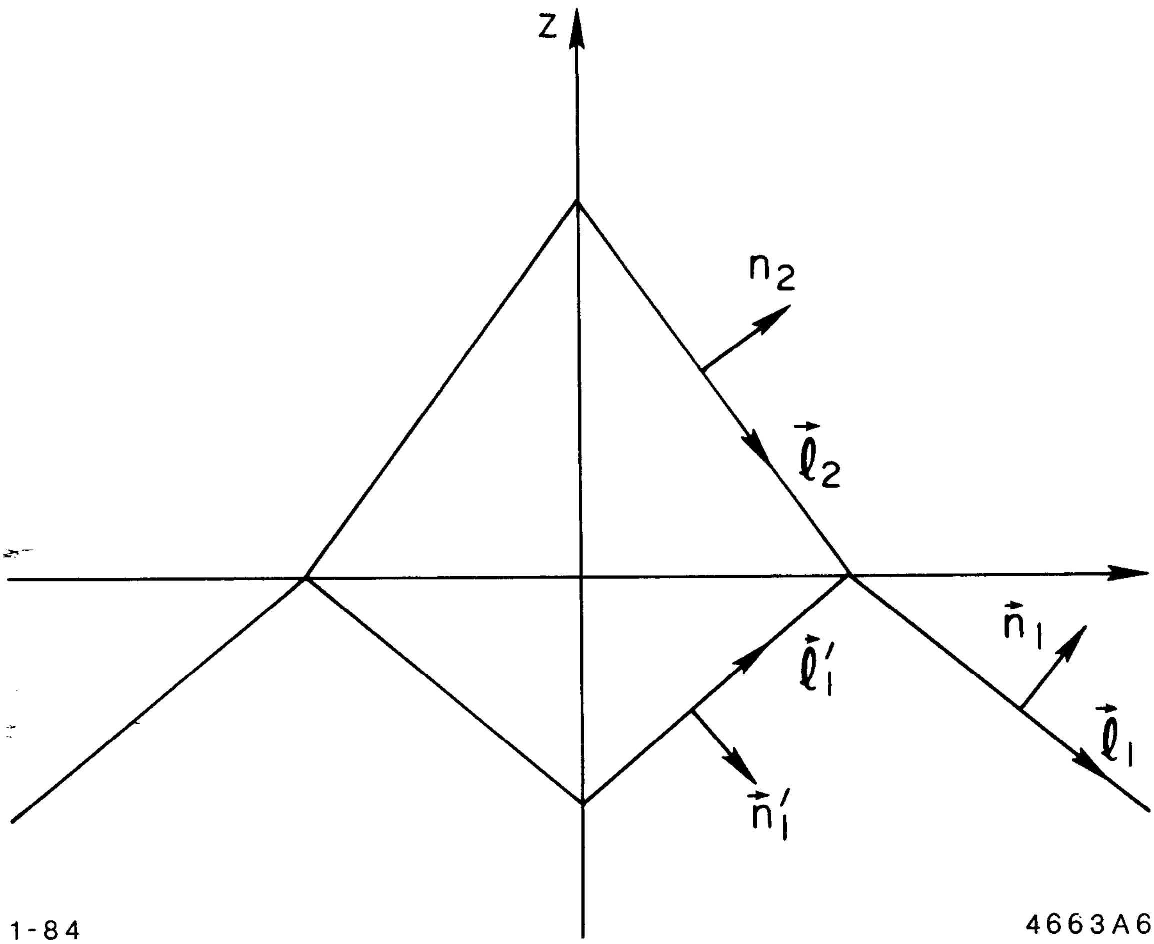
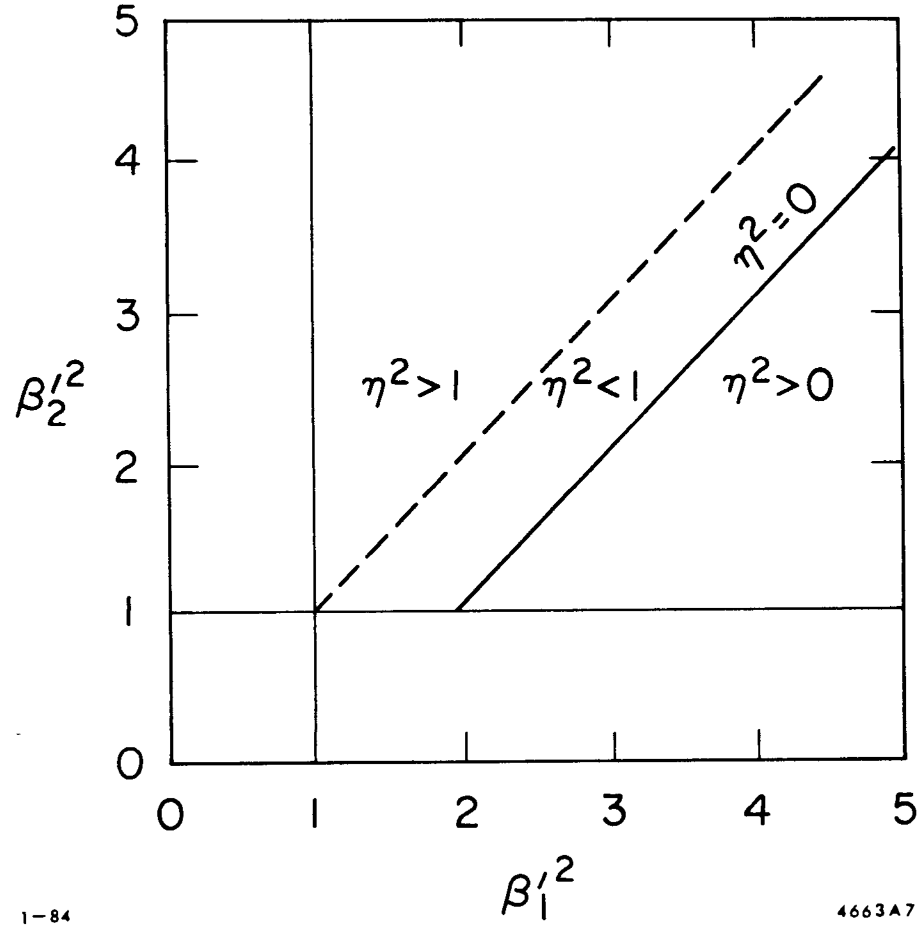


Fig. 4



5
4
3
2
1
0
0
1
2
3
4
5
$\beta_2'^2$
$\beta_1'^2$
$\eta^2>1$
$\eta^2<1$
$\eta^2>0$
$\eta^2=0$
1-84
4663A7

Fig. 5

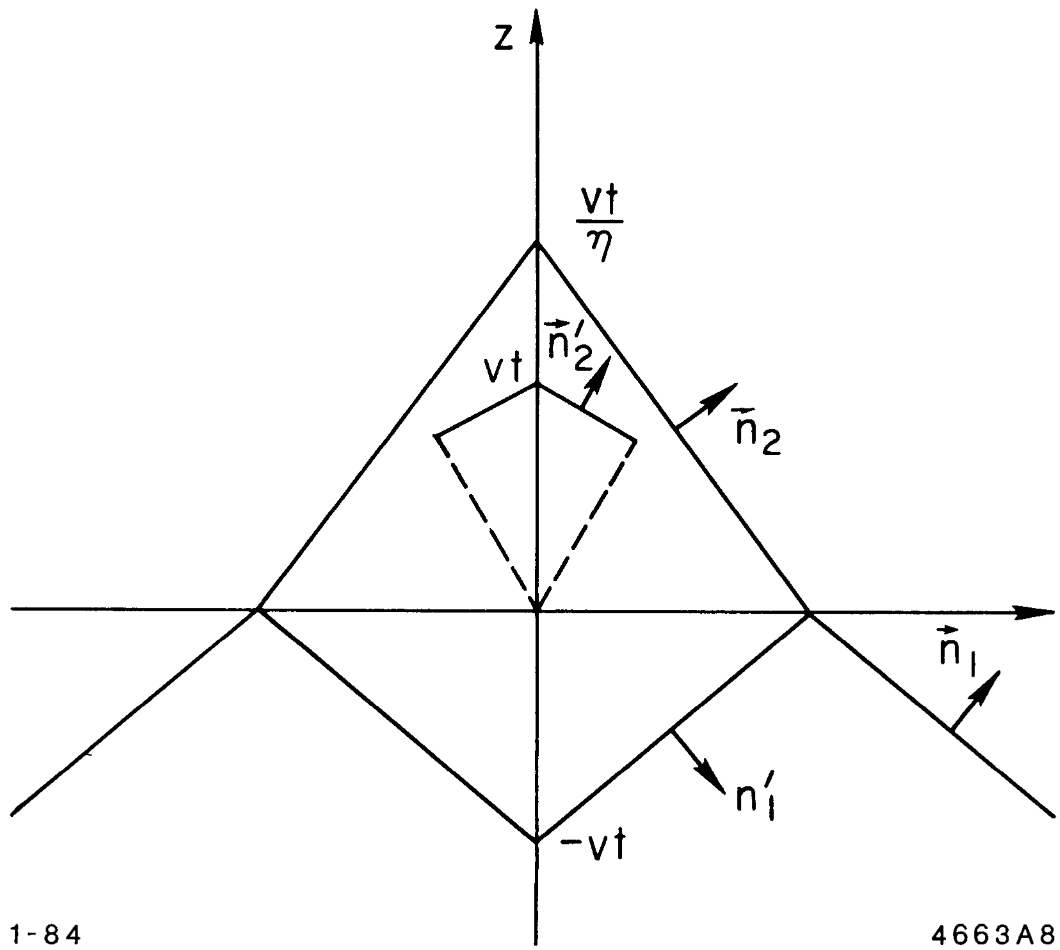


Fig. 6