SLAC-PUB-3212
September 1983
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STABILITY PROPERTIES OF AN ABELIANIZED CHROMOELECTRIC FLUX TUBE*

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ABSTRACT

The quantum fluctuations about a uniform chromoelectric field confined to a cylindrical region are computed. The effective Lagrangian determined from this procedure yields the following interesting results: (1) The chromoelectric flux contained in this region must be less than a calculable critical value to preclude the existence of unstable fluctuation modes which would render the calculation ill-defined. (2) At a flux below this critical value, the fluctuations tend to make it energetically favorable for the flux tube to collapse upon itself. (3) Making analogies between this chromoelectric flux tube model and physical hadronic strings, we estimate that the string of flux in a physical $Q\bar{Q}$ hadronic system is well within the region of stability. However, if the quark charges are increased by a factor of roughly three, the string would become unstable by the above mechanism. This is given as an argument against the existence of certain exotic quark states.

Submitted to Physical Review D

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* Work supported by the Department of Energy, contract DE-AC03-76SF00515.

1. Introduction

The string picture of hadrons within the framework of quantum chromodynamics (QCD) consists of quarks connected by tubes of chromoelectric flux.[1] The simplest hadron in this picture is a meson made up of a quark-antiquark pair connected by a single tube of conserved chromoelectric flux. The dynamics which constrain the fields to form tubes rather than spreading out in the usual electric dipole pattern of classical electrodynamics are supplied by the vacuum fields of QCD,[2] which exert an inward pressure on the flux tube. A stable radius for the tube is attained as a balance is reached between the outward pressure of the confined chromoelectric fields and the inward pressure of the vacum fields. To be more specific, consider a segment of flux tube as illustrated in Fig. 1. If a negative energy density, $-B$, is attributed to the vacuum as in standard bag models,[3] the classical radius of the tube is obtained by minimizing the energy ϵ:

$$\epsilon = \left(\frac{1}{2} | \underset{\sim}{E}_a |^2\right)\pi R^2 L + (B)\pi R^2 L \quad . \tag{1.1}$$

The first term is the classical energy of the flux fields (assumed uniform) and the second term is minus the energy of the displaced vacuum fields. The chromoelectric field strength, $| \underset{\sim}{E}_a |$, is determined from the magnitude of the flux generated by the quark-antiquark pair. For a fixed flux, ϕ_0, $| \underset{\sim}{E}_a | = \phi_0/\pi R^2$. Minimizing the energy of Eq. (1.1) gives a stable radius of

$$R = \left(\frac{\phi_0^2}{2\pi^2 B}\right)^{1/4} \quad . \tag{1.2}$$

Thus, a stable configuration can be found for arbitrary chromoelectric flux in this classical picture. In this work, it will be shown how these naive classical expectations are changed when the quantum fluctuations about the uniform chromoelectric fields contained in the flux tube are incorporated. The specific model investigated is in the configuration of Fig. 1 where the uniform flux field is in the abelianized form

$$A^a_\mu = -\frac{1}{2} F_{\mu\nu}\, x_\nu\, \delta^{a3} \qquad , \tag{1.3}$$

where $F_{\mu\nu}$ corresponds to a constant chromoelectric field along the tube axis. The effective Lagrangian (and thus the energy density) for the region inside the tube will be computed to one loop in the quantum fluctuations. This calculation is similar to previous computations of the fluctuations about uniform fields in QCD,[4] but the difference here is that the fluctuations must also satisfy the boundary condition that they vanish beyond the interior of the tube. The determination of this effective Lagrangian yields the following interesting results:

1. The chromoelectric flux must be less than a calculable critical value, ϕ_c, to preclude the existence of unstable fluctuation modes which would render the calculation meaningless.
2. At a flux below this critical value there is an interesting phenomenon. The quantum fluctuations tend to make it energetically favorable for the flux tube to collapse upon itself, eliminating the existence of a stable finite radius.
3. Using the linear potential inferred from heavy quarkonium spectroscopy,[5] estimates of the flux contained in a physical quark-antiquark string can be made. This flux strength is well within the stability region of this calculation, however if the quark charges were increased by a factor of ~ 3 our estimates would imply flux tube collapse. This may be an argument against the existence of some exotic multi-quark states.

In Section 2 of this paper, we compute the unrenormalized effective Lagrangian for the region inside the flux tube. The expression obtained is renormalized and simplified in Section 3, and the implications for tube stability are given in Section 4. In Section

5, some physical estimates are made. Finally, Section 6 contains a discussion of the results and approximations made, as well as concluding remarks.

2. Effective Lagrangian Inside the Flux Tube

To compute the effective Lagrangian for the region inside the flux tube, the quantum gluon fluctuations about the uniform chromoelectric field in the tube interior must be calculated. For simplicity we will restrict ourselves to the gauge theory of SU(2), and we perform the analysis of the fluctuations in Euclidean space using the fact that the Euclidean functional integral is a legitimate representation of physical amplitudes defined in Minkowski space.[6] The explicit connection between the effective Lagrangian, $\mathcal{L}_E^{eff}$, and the functional integral is

$$Z_E = N \int [DA]\, exp\Big(\int \; d^4x \; \mathcal{L}_E\Big) \equiv N'\, exp\Big(\int \; d^4x \; \mathcal{L}_E^{eff}\Big) \quad , \tag{2.1}$$

where N and N' are normalization constants. The Lagrangian for the pure SU(2) theory is given by

$$\mathcal{L}_E = -\frac{1}{4}\, F^a_{\mu\nu}\, F^a_{\mu\nu} \quad , \tag{2.2a}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc} A^b_\mu A^c_\nu \quad . \tag{2.2b}$$

The configuration we will study is an abelianized uniform chromoelectric field inside the tube of Fig. 1, with spatial direction parallel to the tube axis. The form of the field is explicitly given by

$$\bar{A}^a_\mu = -\frac{1}{2}\, \bar{F}_{\mu\nu}\; x_\nu\; \delta^{a3} \tag{2.3a}$$

for the region inside the tube. The constant matrix $\bar{F}_{\mu\nu}$ has components

$$\bar{F}_{0\|} = E \quad , \quad \text{all other } \bar{F}_{\mu\nu} = 0 \quad , \tag{2.3b}$$

where the index "$\|$" denotes the spatial direction parallel to the tube axis, and E is the magnitude of the chromoelectric field.

The functional integral will be analyzed in the region of the field configuration of Eq. (2.3), $\bar{A}^a_\mu$. The fields will be parameterized as

$$A^a_\mu(x) = \bar{A}^a_\mu(x) + b^a_\mu(x) \quad , \tag{2.4}$$

and the Lagrangian can be expanded in powers of the fluctuations b^a_μ. With this parameterization for the fields, and introducing a background gauge fixing term[7] with the associated Fadeev-Popov determinant Δ_{FP}, the Euclidean functional integral becomes

$$\begin{aligned} Z_E =& N \int [Db]\, \Delta_{FP}\, exp\Big\{\int d^4x\Big[-\frac{1}{4}\bar{F}^a_{\mu\nu}\bar{F}^a_{\mu\nu} + \frac{1}{2} b^a_\mu\Big(\delta_{\mu\nu}(\bar{D}_\sigma \bar{D}_\sigma)_{ac} \\ & -(\bar{D}_\mu \bar{D}_\nu)_{ac} - 2g\epsilon^{adc}\bar{F}^d_{\mu\nu}\Big) b^c_\nu + \frac{\alpha}{2} b^a_\mu (\bar{D}_\mu \bar{D}_\nu)^{ac} b^c_\nu + O(b^3)\Big]\Big\} \ , \end{aligned} \tag{2.5}$$

where "barred" terms depend only upon the "background" chromoelectric field, and $\bar{D}^{ac}_\mu = \delta^{ac}\partial_\mu - g\epsilon^{acb}\bar{A}^b_\mu$. Choosing the gauge parameter to be $\alpha = 1$, and rewriting the appropriate Fadeev-Popov term yields

$$Z_E = N \int [Db]\, exp\Big\{\int d^4x\Big[-\frac{1}{4}\bar{F}^a_{\mu\nu}\bar{F}^a_{\mu\nu} + \frac{1}{2} b^a_\mu\, \bar{\Theta}^{ac}_{\mu\nu}\, b^c_\nu + \ell n\, Det(-\bar{D}_\sigma \bar{D}_\sigma) + O(b^3)\Big]\Big\} \tag{2.6a}$$

with

$$\bar{\Theta}^{ac}_{\mu\nu} = \delta_{\mu\nu}(\bar{D}_\sigma \bar{D}_\sigma)^{ac} - 2g\epsilon^{adc}\bar{F}^d_{\mu\nu} \quad . \tag{2.6b}$$

The one loop approximation will be used in computing the effective Lagrangian from Eq. (2.6). This corresponds to retaining only the terms quadratic in b_μ in the exponent. Note that in order for the one loop computation to make any sense

$$\int d^4x \; b^a_\mu(x) \; \bar{\Theta}^{ac}_{\mu\nu} \; b^c_\nu(x) \; < \; 0 \quad , \tag{2.7}$$

where the integral extends over the cylinder of Fig. 1.

Formally, the Gaussian integration over the b^a_μ fields of Eq. (2.6) can be done, yielding for the effective Lagrangian defined in Eq. (2.1):

$$\mathcal{L}^{eff}_E = -\frac{1}{4}\,\bar{F}^a_{\mu\nu}\,\bar{F}^a_{\mu\nu} - \frac{1}{2}\,Tr\;\ell n(-\bar{\Theta}^{ac}_{\mu\nu}) + Tr\;\ell n(-\bar{D}_\sigma\,\bar{D}_\sigma)^{ac} \quad . \tag{2.8}$$

The traces can be most easily evaluated by determining the eigenvalues of the operators $-\bar{\Theta}^{ac}_{\mu\nu}$ and $-\bar{D}_\sigma\,\bar{D}_\sigma$ subject to the cylindrical boundary conditions, and summing. The eigenvalue equation to solve is

$$\bar{\Theta}^{ac}_{\mu\nu}\; b^c_\nu = \lambda\, b^a_\mu \tag{2.9}$$

with the constraint that $b^a_\mu(x) \to 0$ at the surface of the cylinder. From the explicit form of $\bar{A}^a_\mu$ from Eq. (2.3), it follows that the eigenvalue equation for b^3_μ does not depend upon the chromoelectric field and decouples from the problem after renormalization (see Section 3). The eigenvalue equations for the eigenmodes in the color directions orthogonal to the 3-direction are

$$\left\{\delta_{\mu\nu}\left(\nabla^2 - \frac{g^2E^2(x_0^2+x_{\parallel}^2)}{4} \mp igx_\alpha\,\bar{F}_{\alpha\sigma}\,\partial_\sigma\right) \mp 2ig\bar{F}_{\mu\nu}\right\} b^{\pm}_\nu = \lambda b^{\pm}_\mu \tag{2.10}$$

where $b^{\pm}_\mu = b^1_\mu \pm i b^2_\mu$. The equation is further diagonalized by considering the following linear combinations of Lorentz indices: $b^-_{0\pm i\parallel} = b^-_0 \pm i b^-_{\parallel}$, and $\underset{\sim}{b}^-_{\perp}$ where "$\perp$" signifies the two spatial directions orthogonal to the tube axis. This yields

$$\left(\nabla^2 - \frac{g^2E^2(x_0^2+x_{\parallel}^2)}{4} + igx_\alpha\,\bar{F}_{\alpha\sigma}\,\partial_\sigma\right)\underset{\sim}{b}^-_{\perp} = \lambda\,\underset{\sim}{b}^-_{\perp} \tag{2.11a}$$

$$\left(\nabla^2 - \frac{g^2E^2(x_0^2+x_{\parallel}^2)}{4} + igx_\alpha\,\bar{F}_{\alpha\sigma}\,\partial_\sigma \mp 2gE\right) b^-_{0\pm i\parallel} = \lambda b^-_{0\pm i\parallel} \quad . \tag{2.11b}$$

Complex conjugate equations exist for b_μ^+. These equations can be easily solved by the following procedure. Define the operators

$$a_\mu \equiv \partial_\mu + \frac{g}{2} E x_\mu \quad , \quad a_\mu^+ \equiv -\partial_\mu + \frac{g}{2} E x_\mu \tag{2.12a}$$

and form the linear combinations

$$C^+ \equiv a_0^+ + i a_{||}^+ \quad , \quad C \equiv a_0 - i a_{||} \tag{2.12b}$$

which satisfy the commutation relation

$$[C, C^+] = 2gE \quad . \tag{2.12c}$$

The eigenvalue equations can be rewritten as

$$\left(\underset{\sim}{\nabla}_\perp^2 - C^+ C - gE \right) \underset{\sim}{b}_\perp^- = \lambda \, \underset{\sim}{b}_\perp^- \tag{2.13a}$$

$$\left(\underset{\sim}{\nabla}_\perp^2 - C^+ C - gE \mp 2gE \right) b_{0 \pm i ||}^- = \lambda b_{0 \pm i ||}^- \quad . \tag{2.13b}$$

The commutation relation of Eq. (2.12c) quickly determines the eigenvalue spectrum of the $c^+ c$ term to be

$$\lambda_{||} = 2gE\ell \quad , \quad \ell = 0, 1, 2, \ldots \quad . \tag{2.14}$$

What remains to be computed is the contribution from the term $\underset{\sim}{\nabla}_\perp^2$. Writing this two-dimensional D'Alembertian in cylindrical coordinates, and recalling the boundary conditions on the quantum fluctuations implied by the flux tube boundaries, yields

$$\nabla_{2-d}^2 \psi(\rho, \phi) = -\lambda_\perp^2 \, \psi(\rho, \phi) \quad , \quad \psi(\rho = R, \phi) = 0 \quad . \tag{2.15}$$

The eigenfunctions for these transverse dimensions are clearly proportional to Bessel functions of integer order, $J_m(\rho\lambda_\perp)$, with the eigenvalue $\lambda_\perp$ fixed by the Bessel function zeros, $J_m(R\lambda_\perp) = 0$.[8] Thus, for the transverse dimensions, the eigenvalue spectrum is

$$\lambda_\perp \equiv \lambda_{mn} = \frac{c_{mn}}{R} \qquad , \qquad \begin{matrix} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \end{matrix} \tag{2.16}$$

where c_{mn} is the value of the argument of the nth zero of $J_m(x)$.[8] The full eigenvalue spectrum is now straightforward:

$$\begin{aligned} b^\pm_\perp : \quad \lambda &= -\left(\frac{c_{mn}}{R}\right)^2 - 2gE\ell - gE \\ b^\pm_{0\pm i\|} : \quad \lambda &= -\left(\frac{c_{mn}}{R}\right)^2 - 2gE\ell + gE \\ b^\pm_{0\mp i\|} : \quad \lambda &= -\left(\frac{c_{mn}}{R}\right)^2 - 2gE\ell - 3gE \qquad . \end{aligned} \tag{2.17}$$

Identical analysis goes through for the operator $\bar{D}_\sigma \bar{D}_\sigma$ with the eigenvalue equation

$$(\bar{D}_\sigma \bar{D}_\sigma)^{ac} \psi^c = \lambda \psi^a \qquad , \tag{2.18a}$$

yielding the spectrum

$$\psi^\pm \, : \lambda = -\left(\frac{c_{mn}}{R}\right)^2 - 2gE\ell - gE \qquad . \tag{2.18b}$$

Now, knowledge of the normal mode spectrum allows the evaluation of $\mathcal{L}_E^{eff}$ from Eq. (2.8) using the identity

$$\ell n\, a = -\int_0^\infty \frac{ds}{s}\, e^{-as} \qquad . \tag{2.19}$$

Ignoring constant terms that do not depend upon E,

$$\begin{aligned} \mathcal{L}_E^{eff} = -\frac{1}{4}\bar{F}^a_{\mu\nu}\bar{F}^a_{\mu\nu} + c \sum_{m,n,\ell} \int_0^\infty \frac{ds}{s} \Big\{ exp\Big[-s\Big(2gE\ell - gE + \frac{c^2_{mn}}{R^2}\Big)\Big] \\ + exp\Big[-s\Big(2gE\ell + 3gE + \frac{c^2_{mn}}{R^2}\Big)\Big]\Big\} \end{aligned} \tag{2.20}$$

where c is determined from the eigenmode normalization when taking the trace, and is easily shown to be $c = (gE/2\pi^2 R^2)$. Using the simple identity

$$\sum_{\ell=0}^{\infty} (x)^{\ell} = \frac{1}{(1-x)} \quad , \qquad \text{for} \quad x < 1 \tag{2.21}$$

$\mathcal{L}_E^{eff}$ becomes

$$\mathcal{L}_E^{eff} = -\frac{E^2}{2} + \frac{gE}{2\pi^2 R^2} \sum_{m,n} \int_0^{\infty} \frac{ds}{s} \, \frac{cosh(2sgE)}{sinh(sgE)} \, exp\left(-\frac{sc_{mn}^2}{R^2}\right) \quad . \tag{2.22}$$

This expression appears divergent for $s \to 0$, but this is the normal ultraviolet singularity removed by standard renormalization as will be shown in the next section. However, there potentially exists a serious infra-red divergence at $s \to \infty$ in Eq. (2.22) for certain values of E and R. More specifically, for large s, the integrand of Eq. (2.22) becomes

$$\text{(integrand)} \to exp\left[-s\left(\frac{c_{mn}^2}{R^2} - gE\right)\right] \quad . \tag{2.23a}$$

This means the integral is divergent [corresponding to unstable modes with eigenvalues which violate Eq. (2.7)] unless the chromoelectric flux satisfies the condition

$$\frac{\phi}{\pi} < c_{mn}^2 \tag{2.23b}$$

where $\phi \equiv g(\pi R^2)E$. Condition (2.23b) is just the point made in the introduction that the flux must be less than a critical value for the fluctuation calculation to be well defined. Here we see that the critical flux is given by

$$\phi < \phi_c = \pi c_{01}^2 \quad , \tag{2.23c}$$

and ϕ must always satisfy this condition.

3. Renormalization and Simplification

The expression for $\mathcal{L}_E^{eff}$, Eq. (2.22) contains the normal ultraviolet ($s \to 0$) singularities encountered in one loop computations. It can be renormalized in the usual way, and we choose the following physical renormalization conditions for the quantum corrections, $\mathcal{L}^{(1)}$,

$$\mathcal{L}^{(1)}\Big|_{E=0} = 0 \tag{3.1a}$$

$$\mathcal{L}^{(1)}\Big|_{R=0} = 0 \quad . \tag{3.1b}$$

The condition (3.1a) merely corresponds to demanding that the energy density in the absence of the chromoelectric field is zero. Condition (3.1b) states that as the volume of the flux tube vanishes, so does the contribution to the energy density from the quantum fluctuations. In Appendix A we show that these renormalization conditions are exactly equivalent to the usual Coleman-Weinberg conditions,[9] and thus have a field theoretic basis.

Imposing the renormalization conditions on the effective Lagrangian of Eq. (2.22) yields the finite expression

$$\begin{aligned}\mathcal{L} = &-\frac{E^2}{2} + \frac{gE}{2\pi^2 R^2}\sum_{m,n}\int_0^\infty \frac{ds}{s}\left(2sinh(sgE) + \frac{1}{sinh(sgE)} - \frac{1}{sgE}\right)exp\left(-\frac{sc_{mn}^2}{R^2}\right) \\ &-\frac{gE}{2\pi^2 \bar{R}^2}\sum_{m,n}\int_0^\infty \frac{ds}{s}\left(2sinh(sgE) + \frac{1}{sinh(sgE)} - \frac{1}{sgE}\right)exp\left(-\frac{sc_{mn}^2}{\bar{R}^2}\right)\end{aligned} \tag{3.2}$$

with the implicit limit $\bar{R} \to 0$. The integrals over s can be done quite easily. First note that although the entire integrand is finite as $s \to 0$, each of the individual terms is not. This can be remedied by multiplying the integrands by s^ϵ, integrating each term individually, then taking the finite ϵ-independent result from the limit $\epsilon \to 0$.

The result for $\mathcal{L}$ after the $\epsilon \to 0$ and $\bar{R} \to 0$ limits is[10]

$$\mathcal{L} = -\frac{E^2}{2} + \frac{gE}{2\pi^2 R^2} \sum_{m,n} \left\{ \ell n\left(\frac{\lambda_{mn}+1}{\lambda_{mn}-1}\right) + \lambda_{mn}\, \ell n \,\frac{2}{\lambda_{mn}} \right.$$
$$\left. + 2\ell n\,\Gamma\left(\frac{1}{2}(1+\lambda_{mn})\right) + \lambda_{mn} - \ell n(2\pi) \right\} - \frac{11gE}{12\pi^2 R^2} \sum_{m,n} \frac{1}{\lambda_{mn}} \tag{3.3}$$

where $\lambda_{mn} \equiv c_{mn}^2/gER^2$. Noting that the entire calculation is restricted to $gER^2 < c_{01}^2$ (or $\lambda_{mn} > 1$) from Eq. (2.23c), we can simplify our expression by using a convenient asymptotic expansion for $\Gamma(z)$:[10]

$$\Gamma(z) = z^{z-\frac{1}{2}} e^{-z} \sqrt{2\pi} \left\{ 1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \cdots \right\} \quad , \quad z \geq 1 \ . \tag{3.4}$$

The expression for $\mathcal{L}$ can then be rewritten as

$$\mathcal{L} = -\frac{E^2}{2} + \frac{gE}{2\pi^2 R^2} \sum_{m,n} \left\{ \ell n\left(\frac{\lambda_{mn}+1}{\lambda_{mn}-1}\right) + \lambda_{mn}\, \ell n\left(1 + \frac{1}{\lambda_{mn}}\right) - 1 - \frac{11}{6\lambda_{mn}} \right.$$
$$\left. + 2\ell n\left(1 + \frac{1}{6(1+\lambda_{mn})} + \frac{1}{72(1+\lambda_{mn})^2} - \cdots\right) \right\} \ . \tag{3.5}$$

In this form, it is easily seen that the ultraviolet singularities have been removed by the renormalization conditions of Eq. (3.1). It is explicitly shown in Appendix B that the sum over m and n has been rendered finite.

Equation (3.5) is the exact expression for the one loop effective Lagrangian of a cylinder of radius R containing a uniform chromoelectric field of strength E. In Section 5, this expression will be used to investigate the dynamical properties of a flux tube. First it is necessary to change to more physical variables. The effective Lagrangian above is written as a function of E and R. However, the relevant physical variables are the (conserved) flux generated by the (conserved) sources, $\phi = gE(\pi R^2)$, and the tube radius, R. Written in terms of these variables, Eq. (3.5) becomes

$$\mathcal{L} = -\frac{\phi^2}{2\pi^2 R^4 g^2} \left\{ 1 - \frac{g^2}{\pi\phi} \sum_{m,n} \left[\ell n\left(\frac{\lambda_{mn}+1}{\lambda_{mn}-1}\right) + \lambda_{mn}\, \ell n\left(1 + \frac{1}{\lambda_{mn}}\right) - 1 - \frac{11}{6\lambda_{mn}} \right.\right.$$
$$\left.\left. + 2\ell n\left(1 + \frac{1}{6(1+\lambda_{mn})} + \frac{1}{72(1+\lambda_{mn})^2} - \cdots\right) \right] \right\} \tag{3.6}$$

where $\lambda_{mn} = \pi c_{mn}^2/\phi$. In this form, the dynamical implications are more easily accessible.

4. Flux Tube Stability

The expression for the one loop effective Lagrangian gives the energy density of the flux tube configuration via the simple Euclidean space relation, $\epsilon = -\mathcal{L}$. If in addition a negative energy density, $-B$, is attributed to the vacuum as in the introduction, the total energy per unit length of the flux tube configuration takes the form

$$\epsilon = \frac{\phi^2}{2\pi g^2 R^2}\left\{1 - \alpha_s \sum_{m,n} f_{mn}(\phi)\right\} + B\pi R^2 \tag{4.1a}$$

with

$$\begin{aligned} f_{mn}(\phi) = \frac{4}{\phi}\Bigg\{ & \ell n\left(\frac{\lambda_{mn}+1}{\lambda_{mn}-1}\right) + \lambda_{mn}\,\ell n\left(1+\frac{1}{\lambda_{mn}}\right) - 1 - \frac{11}{6\lambda_{mn}} \\ & + 2\ell n\left(1 + \frac{1}{6(1+\lambda_{mn})} + \frac{1}{72(1+\lambda_{mn})^2} - \cdots\right)\Bigg\} \quad . \end{aligned} \tag{4.1b}$$

The first term in Eq. (4.1a) corresponds to the self-energy of the uniform chromoelectric field. It contains the classical term plus the contribution proportional to α_s due to the quantum fluctuations. The last term in Eq. (4.1a) corresponds to minus the energy of the displaced vacuum fields. If the quantum fluctuations are ignored, for a given flux we have a simple energy minimization problem where the outward pressure of the classical chromoelectric fields is balanced by the inward pressure of the vacuum fields, giving a finite stable radius. However, we will now investigate how this simple picture changes when the quantum fluctuations are included.

The contribution from the quantum fluctuations is proportional to the sum over m and n of $f_{mn}(\phi)$. We recall from Appendix B that the series is rapidly convergent, and thus can perhaps be approximated by the first one or two terms. This is verified in

Fig. 2 where f_{01}, f_{11}, f_{21} are plotted over the allowable (calculable) range of ϕ [recall Eq. (2.23c)]. The full sum is well approximated by the first term, f_{01}, to within 10% over the entire range of ϕ, and even better for the interesting region of $\phi \to \pi c_{01}^2$. The important characteristics of $f_{01}(\phi)$ to note are the following: (i) $f_{01}(\phi)$ is a positive quantity over the entire range of ϕ, (ii) $f_{01}(\phi)$ monotonically increases with ϕ, (iii) $f_{01}(\phi)$ increases like a logarithmic singularity as ϕ approaches the critical value, $\phi = \pi c_{01}^2$.

The physical implications of the quantum fluctuations as embodied in Eq. (4.1a) can now be understood. Characteristic (i) points to the fact that the quantum fluctuations tend to decrease the coefficient of the classical self-energy term. Physically, this corresponds to a decrease of the classical outward pressure of the uniform chromoelectric field. Characteristic (ii) implies that this decrease in the outward pressure becomes more pronounced for stronger chromoelectric flux fields. Finally, characteristic (iii) states that for a chromoelectric flux with magnitude sufficiently close to the critical value of πc_{01}^2, the coefficient of the flux self-energy term flips sign. The naive interpretation of this phenomenon is that the outward pressure which was necessary to sustain a finite radius, becomes an inward pressure forcing the flux tube to collapse upon itself for sufficiently strong chromoelectric flux fields. This interpretation will be discussed more in Section 6.

5. Physical Estimates

In the last section it was shown that if the magnitude of the chromoelectric flux contained in a cylindrical region comes sufficiently close to a critical value, the flux tube will collapse due to the effects of its quantum fluctuations. In this section, it will be assumed that ordinary heavy quark-antiquark ($Q\bar{Q}$) states can be described as flux tubes (see Fig. 3), and the tube parameters estimated from relevant data. These tube parameters will then be employed in the formalism of the previous sections, and interpreted where possible.

The linear potential determined from heavy quarkonium spectroscopy has the form[5]

$$V(r) = \kappa r \quad , \tag{5.1}$$

where κ is determined to be roughly .15 GeV^2. If it is assumed that the linear potential is generated by a tube of conserved chromoelectric flux between the $Q\bar{Q}$ pair, the potential would have the form

$$V(r) = \left(\frac{E^2}{2}\right)(\pi a^2) r \quad , \tag{5.2}$$

where $(E^2/2)$ is the energy density of the physical chromoelectric flux field, (πa^2) is the tube cross-sectional area, and r is the tube length. Comparison of Eq. (5.1) and Eq. (5.2) yields

$$E = \left(\frac{2\kappa}{\pi a^2}\right)^{1/2} \tag{5.3}$$

for the chromoelectric field strength in a physical flux tube (meson). With this relation, to completely specify all the parameters in a meson flux tube, the tube radius must be estimated. For this we use the typical hadronic scale of .5 fermi. This yields the parameters for a physical meson flux tube

$$E = .12\,\mathrm{GeV}^2 \quad , \qquad \phi = 6.46 \quad . \tag{5.4}$$

The magnitude of f_{01} can be determined from Fig. 2 for this value of ϕ ($\phi = .36\pi c_{01}^2$). It is interesting to note that f_{01} at this value is $\sim .02$, which implies that the one loop quantum fluctuations change the outward pressure of this physical flux tube by a term $\sim .02\alpha_s$ compared to the leading order of one. It appears in this crude model of meson flux tubes that the quantum fluctuations are very small for the physically implied parameters.

Another interesting feature of this analysis is that although the quantum corrections to the physical $Q\bar{Q}$ flux tube are extremely small, if the flux (quark charge) were increased by a factor of less than 3, the critical flux strength would be approached, giving extremely large destabilizing quantum corrections. This is the origin of the argument that perhaps some exotic quark states could be ruled out in this picture. More specifically, if instead of a $Q\bar{Q}$ state, the analysis was performed for a $(3Q)(3\bar{Q})$ state where the charges are additive, the conserved flux would be increased toward the $\phi \sim 1$ (πc_{01}^2) region. In this situation, the quantum fluctuations as given by f_{01} become large, and the flux tube would collapse as described in the last section.

It is important to note that these physical estimates were made for the gauge group SU(2) rather than the physical group SU(3). However, this has no effect on the qualitative conclusions, and very little effect on the numerical results. This can be easily seen by expanding the degrees of freedom in Eq. (2.10) to those of SU(3). This yields additional terms from fluctuations outside the SU(2) subgroup which couple more weakly to the chromoelectric field due to the magnitudes of the SU(3) structure constants. In Fig. 4 we plot the magnitude of these corrections to $f_{01}(\phi)$ that come from expanding the gauge group to SU(3). The corrections are small over the entire allowable region of ϕ.

6. Summary and Conclusions

In this work, the one loop effective Lagrangian for a tube of uniform chromoelectric flux has been analyzed. It was first noted that the calculation could only be performed for a flux magnitude, ϕ, less than a critical value due to the existence of ill-defined unstable modes. However, within this well-behaved region there exist some interesting implications. We found that for small ϕ the quantum fluctuation corrections are small, but increase rapidly and become very large as the critical flux magnitude is approached. The physical manifestation of these large fluctuations is that the classical outward pressure of a confined flux of chromoelectric field strength changes sign for sufficiently large flux field strength, causing the tube to tend to collapse on itself. It is then shown that when this model is applied to physical $Q\bar{Q}$ meson states, the meson states are well within the region where the quantum fluctuations are small. However, if the physical meson states had their color charge increased by a factor of ~ 3, the flux tube states would tend to collapse on themselves via the above mechanism. This is used as an argument against the existence of certain exotic multiquark states.

It must be remembered that there are two major caveats in the above picture: (1) The restriction to one loop in the quantum fluctuations is not justified in the region $\phi \rightarrow \phi_c$ where the fluctuations are becoming extremely large. The only rigorous statement that can be made is that as the tube flux becomes larger, the outward pressure is very rapidly decreasing (going against one's classical intuition). However, the one loop approximation is perhaps not adequate to prove that the sign of the pressure changes. (2) The computation is precise for a confined tube of uniform abelianized chromoelectric flux, but this is at best a crude model of physical mesons. The boundary conditions imposed are similar to those of a rigid bag model, and no attempt is made to incorporate boundary fluctuations.

Even with these reservations, the qualitative behavior of this idealized cylinder of chromoelectric flux may lend some insight into the physics of hadronic strings.

ACKNOWLEDGEMENTS

I would like to thank R. Blankenbecler, W. Dittrich and S. Drell for useful discussions.

Appendix A

Here it will be shown that the renormalization conditions used in Section 3, Eq. (3.1), are equivalent to the usual Coleman-Weinberg renormalization conditions,[9] and thus have a field theoretic interpretation. As a first step, we recall that renormalization is relatively simple in background gauge.[7] In these gauges the gluon wave function and vertex function renormalizations are equal and cancelling, leading to a simple over-all renormalization of the action. The counterterm has the universal form of $ZS^{YM}_{classical}$, with Z being independent of the choice of gauge function. As a result, the renormalization conditions can be expressed by means of the function $\mathcal{L}$ only. Indeed, the renormalization conditions of Coleman and Weinberg are just

$$\mathcal{L}\,|_{\mathcal{F}=0} = 0 \tag{A.1a}$$

$$\left.\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right|_{\mathcal{F}=\mu^2/2} = -1 \tag{A.1b}$$

where $\mathcal{F} \equiv \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}$, and $\mathcal{F} = E^2/2$ for this calculation. Condition (A.1a) merely specifies the zero of the energy density, and condition (A.1b) is the usual coupling constant renormalization.

The conditions of Eq. (A.1) when applied to the unrenormalized Lagrangian L, yield the renormalized expression $\mathcal{L}$:

$$\mathcal{L} = L - L\,|_{E=0} - E^2 \left[\frac{\partial L}{\partial E^2}\right]_{E^2=\mu^2} - \frac{E^2}{2} \quad . \tag{A.2a}$$

Application of our renormalization conditions, Eq. (3.1), yields for the renormalized Lagrangian

$$\mathcal{L} = L - L\,|_{E=0} - L\,|_{R=0} - \frac{E^2}{2} \quad . \tag{A.2b}$$

Thus, the problem of showing the equivalence of the two sets of renormalization conditions has been reduced to showing the equivalence of the counterterms $E^2[\partial L/\partial E^2]_{E^2=\mu^2}$ and $[L]_{R=0}$.

From the unrenormalized Lagrangian of Eq. (2.22) we find

$$[\mathcal{L}]_{R=0} = \lim_{R\to 0} \frac{gE}{2\pi^2 R^2} \sum_{m,n} \int_0^\infty \frac{ds}{s} \left(2sinh(sgE) + \frac{1}{sinh(sgE)} - \frac{1}{sgE} \right) \times exp\left(-\frac{sc_{mn}^2}{R^2} \right) . \tag{A.3}$$

In the limit $R \to 0$, the exponential factor strongly damps the integrand for finite s. Thus, expanding the rest of the integrand about $s = 0$, integrating, and taking the $R \to 0$ limit yields

$$[\mathcal{L}]_{R=0} = \frac{11g^2E^2}{12\pi^2} \sum_{m,n} \frac{1}{c_{mn}^2} . \tag{A.4}$$

From the unrenormalized Lagrangian of Eq. (2.22) we also find

$$E^2 \left[\frac{\partial \mathcal{L}}{\partial E^2} \right]_{E^2=\mu^2} = \frac{gE^2}{4\pi^2 R^2 \mu} \sum_{m,n} \int_0^\infty \frac{ds}{s} \left(1 + \frac{sc_{mn}^2}{R^2} \right) \left(2sinh(sg\mu) + \frac{1}{sinh(sg\mu)} - \frac{1}{sg\mu} \right) \times exp\left(-\frac{sc_{mn}^2}{R^2} \right) . \tag{A.5}$$

In the limit $\mu \to 0$, Eq. (A.5) reduces to

$$E^2 \left[\frac{\partial \mathcal{L}}{\partial E^2} \right]_{E^2=\mu^2=0} = \frac{11g^2E^2}{12\pi^2} \sum_{m,n} \frac{1}{c_{mn}^2} . \tag{A.6}$$

A comparison of Eq. (A.4) and Eq. (A.6) shows that our physical renormalization conditions, Eq. (3.1), are equivalent to the Coleman-Weinberg renormalization conditions in the limit of the arbitrary renormalization scale going to zero.

Appendix B

Here it will be verified that the sum over m and n for the effective Lagrangian of Eq. (3.5) has been made convergent by the renormalization prescription of Eq. (3.1). The expression for $\mathcal{L}$ is

$$\mathcal{L} = -\frac{E^2}{2} + \frac{gE}{2\pi^2 R^2} \sum_{m,n} \left\{ \ell n\left(\frac{\lambda_{mn}+1}{\lambda_{mn}-1}\right) + \lambda_{mn}\, \ell n\left(1+\frac{1}{\lambda_{mn}}\right) - 1 \right.$$
$$\left. + 2\ell n\left(1+\frac{1}{6(1+\lambda_{mn})}+\cdots\right)\right\} - \frac{11gE}{12\pi^2 R^2} \sum_{m,n} \frac{1}{\lambda_{mn}} \tag{B.1}$$

where $\lambda_{mn} = c_{mn}^2/gER^2$. We begin by investigating the asymptotic behavior of c_{mn}, the Bessel function zeros. The m*th* order Bessel function has the asymptotic behavior[8]

$$J_m(x) \underset{x\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \quad . \tag{B.2}$$

This means the large order zeros occur for

$$x \equiv c_{mn} = \frac{m\pi}{2} + n\pi + \frac{3\pi}{4} \quad . \tag{B.3}$$

For large m and n, c_{mn} and thus λ_{mn} in Eq. (B.1) become large. Expanding Eq. (B.1) asymptotically in $(1/\lambda_{mn})$ yields

$$\mathcal{L} = -\frac{E^2}{2} + \frac{127gE}{360\pi^2 R^2} \sum_{m,n} \left[\frac{1}{\lambda_{mn}^3} + O\left(\frac{1}{\lambda_{mn}^5}\right)\right] \quad . \tag{B.4}$$

The leading order term in the sum has the form

$$\begin{pmatrix}\text{leading}\\ \text{order}\end{pmatrix} \sim \sum_{m,n} \left(\frac{1}{c_{mn}^2}\right)^3 \quad . \tag{B.5}$$

For m and n large, we can use Eq. (B.3) to rewrite the sum as

$$\begin{pmatrix}\text{leading}\\ \text{order}\end{pmatrix} \sim \sum_{m,n} \frac{1}{\left(\frac{m\pi}{2} + n\pi + \frac{3\pi}{4}\right)^6} \quad , \tag{B.6}$$

which is a well behaved, rapidly convergent sum.

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Figure Captions

1. Segment of chromoelectric flux tube of length L and radius R. The negative energy density of the vacuum is denoted by −B.
2. The terms $f_{mn}(\phi)$, where ϕ is in units of πc_{01}^2.
3. Chromoelectric flux tube generated by a quark-antiquark pair.
4. The upper curve is f_{01} computed for SU(2) as before. The lower curve is the contribution to f_{01} from the additional degrees of freedom when the gauge group is expanded to SU(3).

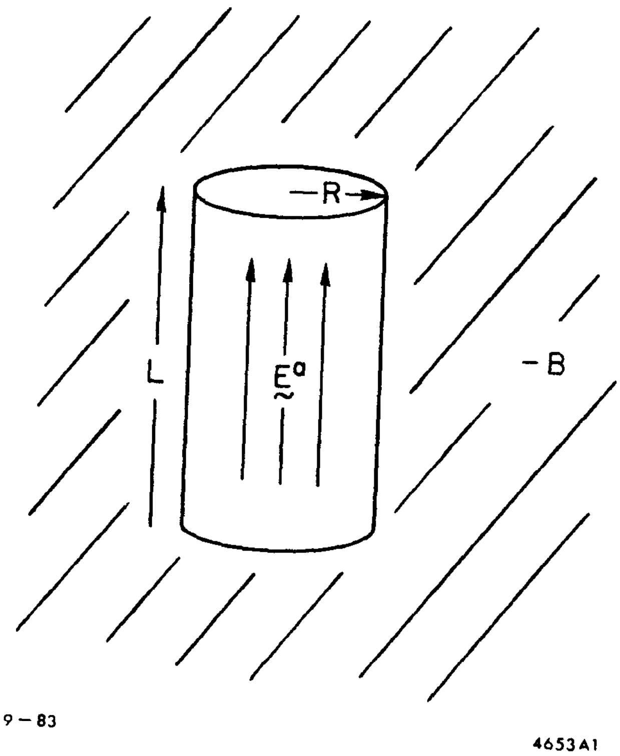
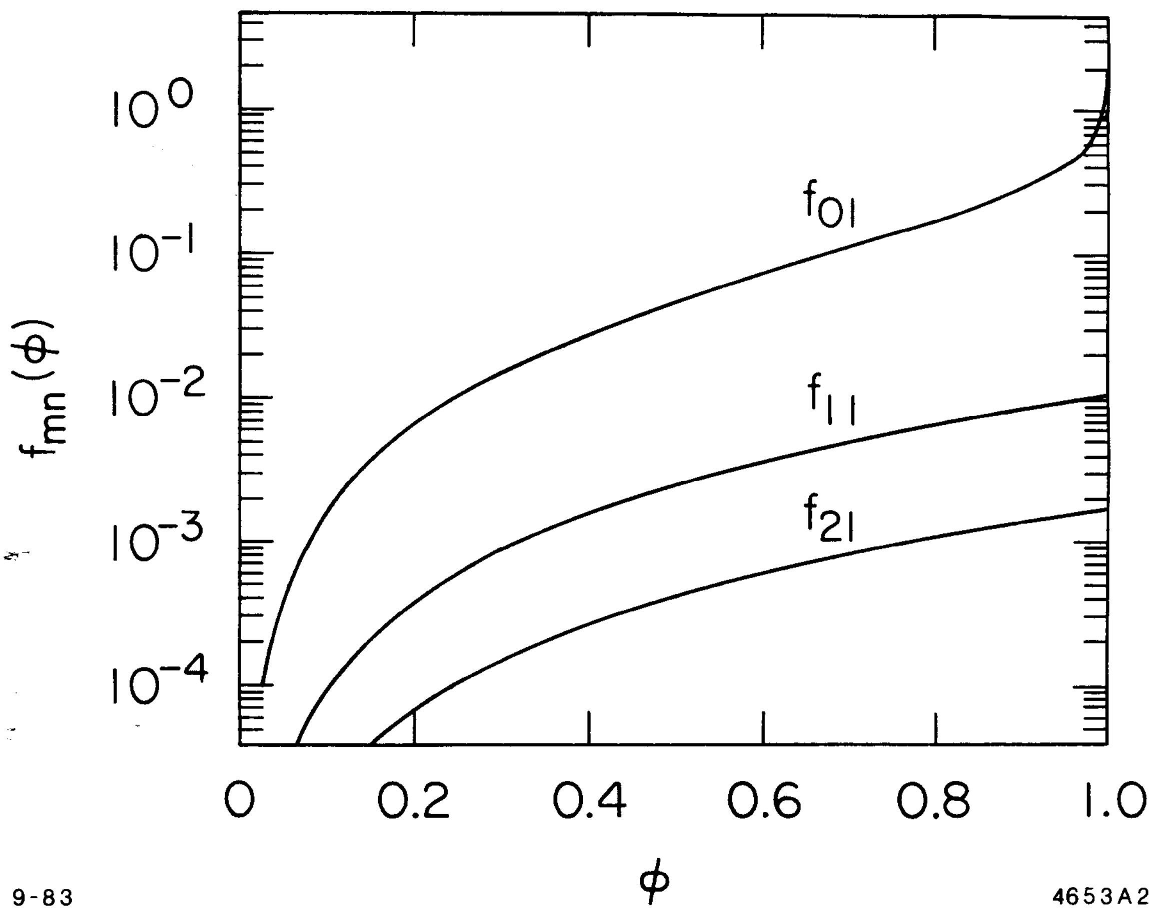


Fig. 1



10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$f_{mn}(\phi)$
f_{01}
f_{11}
f_{21}
0
0.2
0.4
0.6
0.8
1.0
ϕ
9-83
4653A2

Fig. 2

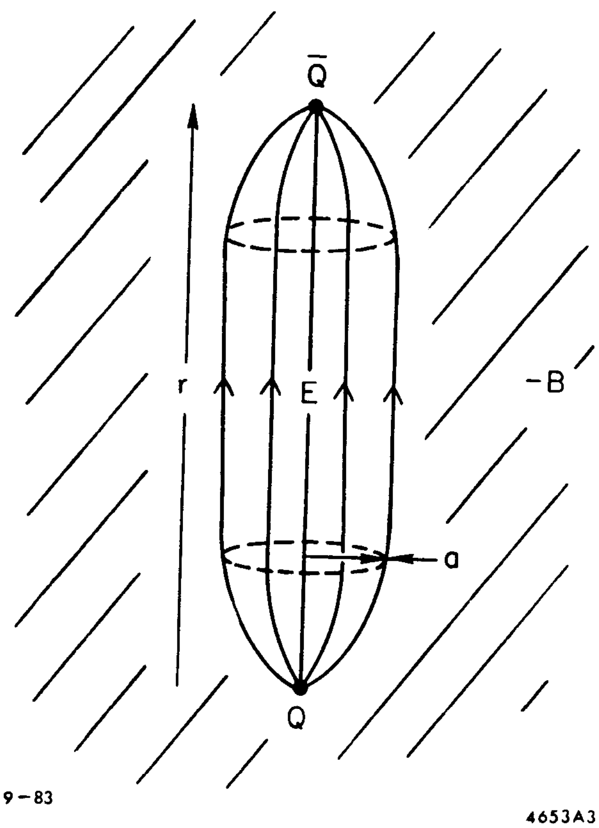
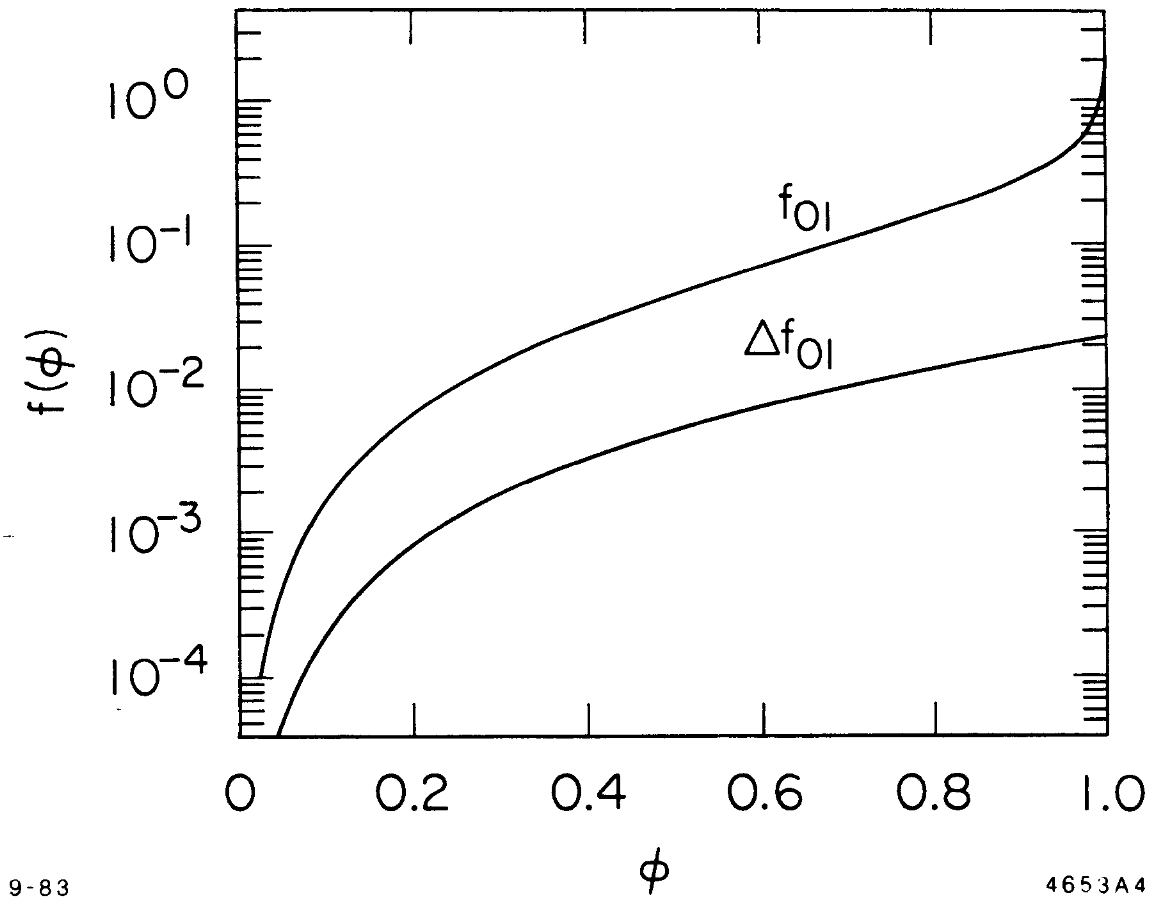


Fig. 3



10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$f(\phi)$
f_{01}
Δf_{01}
0
0.2
0.4
0.6
0.8
1.0
ϕ
9-83
4653A4

Fig. 4