SLAC-PUB-3203
September 1983
(T/E)

Covariant Polarization Bases for Spin = 1/2, 1, 3/2 Particles and Their Use*

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ABSTRACT

A simple method is presented for evaluating transition amplitudes between massive states with spin = 1/2, 1 and definite polarization. The spin 3/2 case is also briefly discussed. Applications to testing the decay and production of vector bosons are considered.

Submitted to Nuclear Physics B

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*Work supported by the Department of Energy, contract DE-AC03-76SF00515, and in part by the Instituto Nazionale di Fisica Nucleare (INFN).

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1. Introduction

Present day gauge theory represents a successful domain in the study of e.m., weak, strong and even gravitational interactions. With many fundamental problems solved or in the process of being solved we are very often faced with second generation problems, i.e., technical difficulties which prevent us from producing numbers out of the theory. To name only one of these problems we can say that the present generation of accelerators is very near to test weak effects in e^+e^- annihilation. Therefore radiative corrections must be available and among them hard bremsstrahlung corrections, which have been a difficult task for many years. In this respect a major achievement is represented by the techniques developed by the CALKUL collaboration.[1] In brief they have been able to produce an explicit representation for the photon wave function which allows to compute directly and with incredible simplicity the amplitude for processes like $e^+e^- \to X\gamma$. At the same time different efforts,[2] already present in the literature, for computing transition amplitudes between Dirac spinors have been itemized and developed in ref. 3.

All these methods use covariant polarization bases and we feel that these features should be extended to include massive particles with spin 1, 3/2 as well as the graviton.[4] Indeed a look at the relevant literature[5] shows that for explicit calculations involving vector bosons, the wave functions are still written component by component.*

The main result of our paper can be summarized as follows. A spin s massive particle is described by a totally symmetric $2s$ – spinor satisfying the correspondent Bargmann-Wigner equations. By a suitable generalization of the so called fusion method we can prove that a massive particle with spin $s = 1, 3/2$ and four momen-

* A notable exception is found in ref. 6. Reference 5 is by no means complete or exhaustive. Eventual omissions are purely casual.

tum p is equivalent to $2s$ Dirac particles, each in a state of four momentum p_i with $p_i^2 = -m_i^2$, $p_i = (m_i/m)p$ and $\Sigma m_i = m$. The total spin is the vector sum of the spins of the Dirac particles.

The Proca and the Rarita-Schwinger fields are then expressed in terms of the second and third rank spinors. As a result of the fusion method the corresponding wave functions can be entirely expressed in terms of Dirac spinors with arbitary polarization vectors.* Thus transition amplitudes between states with spin $\leq 3/2$ can be evaluated with the same methods of ref. 3 and in the final answer we have only external momenta, polarization vectors of the Dirac particles and longitudinal polarization vectors for the spin 1 particles. The feasibility of the method is illustrated in several examples of processes with external vector bosons. The outline of the paper is as follows. In sec. 2 we discuss the spinor equations for spin 1. The corresponding wave function is analyzed in detail in sec. 3. Section 4 contains explicit calculations for $Z^0 \to f^+ f^-$, $e^+e^- \to Z^0 H^0$, $e^+e^- \to Z^0\gamma$ and $e^+e^- \to W^+W^-$. Finally, in the appendix technical details are examined.

2. Spinor equations for a massive spin 1 particle

The spinor equations equivalent to the Proca equation for a massive vector particle were found a long time ago.[7)] Originally these equations and their solution when no e.m. field is present were obtained in the two component spinor formalism. A simple interpretation follows. A free particle of mass m and spin 1 is equivalent to a pair of Dirac particles of masses m_1 and m_2. If p, p_1 and p_2 are the four momenta describing the states of these particles then

$$p_i = \frac{m_i}{m} p \quad , \quad m_1 + m_2 = m$$

*The usual helicity amplitudes are obtained when we choose the polarization of the merging fermions to be longitudinal.

and the spin of the vector boson is the sum of the spins of the two Dirac particles.

The same result can be expressed in the four component formalism where a massive spin 1 particle is described by a symmetric second rank spinor $\Psi_{\alpha\beta}$. This can easily be derived by using the Bargmann-Wigner[8] approach as described by Luriè.[9]

The symmetric spinor $\Psi_{\alpha\beta}$ satisfies the equations

$$\begin{aligned} \not{\partial}_{\alpha\lambda}\Psi_{\lambda\beta} + m\Psi_{\alpha\beta} &= 0 \\ \not{\partial}_{\beta\lambda}\Psi_{\alpha\lambda} + m\Psi_{\alpha\beta} &= 0 \end{aligned} \tag{2.1}$$

A symmetric 4×4 matrix can always be decomposed as

$$\Psi = \frac{-1}{4} m A^{\mu}\gamma_{\mu} c + \frac{i}{2} F^{\mu\nu}\sigma_{\mu\nu} c \tag{2.2}$$

where $c = -c^{\intercal} = -c^{-1}$ is the charge conjugation matrix. The vector particle wave function is

$$A^{\mu} = \frac{1}{m} trc\gamma^{\mu}\Psi \tag{2.3}$$

Moreover any symmetric second rank spinor can be written as

$$\Psi_{\alpha\beta} = \frac{1}{2}(\psi_{1\alpha}\psi_{2\beta} + \psi_{2\alpha}\psi_{1\beta})$$

Equations (2.1) written in momentum space become*

$$(i\not{p} + m)\Psi(p) = \Psi(p)(i\not{p}^{\intercal} + m) = 0 \tag{2.4}$$

and they are satisfied by the ansatz

$$\begin{aligned} \Psi(p) &= 1/2[\psi(p_1)\psi(p_2) + \psi(p_2)\psi(p_1)] \\ i\not{p}_j\psi(p_j) &= -m_j\psi(p_j) \end{aligned} \tag{2.5}$$

if and only if $p_i = (m_i/m)p$ and $m_1 + m_2 = m$.

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* Our convention is $g_{\mu\nu} = g^{\mu\nu} = \delta_{\mu}^{\nu} = (+, +, +, +)$ (the Pauli metric).

In order to deal with polarized particles we consider the spinors $u(p,\ n,\ \lambda)$, $\lambda = \pm 1$. They are defined[3)] as the eigenstates of the operator $P_+(p,\ n,\ \lambda)$ corresponding to eigenvalue 1.

$$
\begin{aligned}
P_+(p,n,\lambda) &= \Lambda_+(p)\frac{1}{2}(1 + i\lambda\gamma^5 \not{n})\Lambda_+(p) \\
\Lambda_+(p) &= \frac{1}{2p_0}(-i \not{p} + m)\gamma^4
\end{aligned}
\tag{2.6}
$$

n is the polarization vector and $p \cdot n = 0$, $n^2 = 1$.

The wave function for an incoming spin 1 particle of four momentum p is thus reconstructed by the fusion method from eq. (2.3) using eq. (2.5) and the properties of the spinor $u(p,\ n,\ \lambda)$.

3. Polarization Bases for Vector Particles

Having discussed the preliminaries we proceed in the construction of a polarization basis for a massive vector boson. We first introduce

$$
\chi^\mu(p,\lambda) = \frac{1}{4m} trc\gamma^\mu \Sigma u(p_i,n,\lambda_l)u(p_j,n,\lambda_k)
\tag{3.1}
$$

where λ $=$l/2$(\lambda_1 + \lambda_2)$, $p_i = (m_i/m)p$ and the sum denotes symmetrization with respect to $p_1 \leftrightarrow p_2$ and $\lambda_1 \leftrightarrow \lambda_2$.

Taking into account the properties of c we find

$$
\begin{aligned}
\chi^\mu(p,\lambda) &= \frac{1}{4m}\Sigma tr u^{\mathsf{T}}(p_j,n,\lambda_k)c\gamma^\mu u(p_i,n,\lambda_l) \\
&= \frac{1}{4m}\Sigma\, \bar{v}\,(p_j,n,\lambda_k)\gamma^\mu u(p_i,n,\lambda_l)
\end{aligned}
\tag{3.2}
$$

$v(p,\ n,\ \lambda)$ can be introduced as follows.[3)]Let $u(p,\ n,\ \lambda)$ be an eigenstate of $P_-(p,n,\lambda)$ satisfying the Dirac equation for $p_0 = -(p^2 + m^2)^{1/2}$. P_- is given by an expression similar to the one in eq. (2.6) with

$$
\Lambda_-(p) = -\frac{1}{2p_0}\gamma^4(+i \not{p} + m)
$$

Thus $v(p, n, \lambda) = cu(p^*, n^*, \lambda)$ where $p^* = (-\vec{p}, p_0)$, $n^* = (\vec{n}, -n_0)$. χ^μ, as given in eq. (3.2), describes after a suitable normalization the vector boson wave function.

All the relevant formulas are explicitly proven in the appendix. Under the conditions required by the fusion method we get

$$\bar{u}\,(p_i, n, \lambda)(1; \not{n}; \not{n}')v(p_j, n, \lambda') \simeq (0; \frac{m}{E} P_-; \frac{m}{E} P_+) \tag{3.3}$$

with $E = p_0$ and

$$p_i \cdot n' = n \cdot n' = 0 \quad n' \cdot n' = 1$$

$$P_\pm = \frac{1}{2}(1 \pm \lambda\lambda')$$

The symbol $\simeq$ in eq. (3.3) indicates that an arbitrary phase has been neglected.[1)]

Introducing

$$\Gamma_{ij}(\lambda, \lambda') = u(p_i, n, \lambda)\,\bar{v}\,(p_j, n, \lambda') \tag{3.4}$$

we can multiply and divide by the appropriate term in eq. (3.3) to have

$$\begin{aligned} \Gamma_{ij}(\lambda, \lambda) &\simeq \frac{E}{m} U_i(\lambda)\, \not{n}'\, V_j(\lambda) \\ \Gamma_{ij}(\lambda, -\lambda) &\simeq \frac{E}{m} U_i(\lambda)\, \not{n}\, V_j(-\lambda) \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} U_i(\lambda) &= u(p_i, n, \lambda)\,\bar{u}\,(p_i, n, \lambda) = \frac{1}{4E}(-i\,\not{p} + m)(1 + i\lambda\gamma^5\,\not{n}) \\ V_i(\lambda) &= v(p_i, n, \lambda)\,\bar{v}\,(p_i, n, \lambda) = \frac{1}{4E}(-i\,\not{p} - m)(1 + i\lambda\gamma^5\,\not{n}) \end{aligned} \tag{3.6}$$

From these results we compute χ^μ.

$$\chi^\mu(p, \lambda) = \frac{1}{4m} tr\gamma^\mu[\Gamma_{12}(\lambda_1, \lambda_2) + \Gamma_{12}(\lambda_2, \lambda_1) + \Gamma_{21}(\lambda_1, \lambda_2) + \Gamma_{21}(\lambda_2, \lambda_1)] \tag{3.7}$$

Dropping an overall phase we get

$$\chi^\mu(p,0) = -\frac{1}{E}n^\mu$$
$$\chi^\mu(p,\pm 1) = -\frac{1}{E}(n'^\mu \mp N^\mu) \quad , \quad N^\mu = \frac{1}{m}\epsilon^{\mu\nu\alpha\beta}n'_\nu n_\alpha p_\beta \tag{3.8}$$

We easily derive

$$\chi^+_\mu(p,0) = \chi_\mu \qquad \chi^+_\mu(p,\pm 1) = \chi_\mu(p,\mp 1)$$

$\epsilon_\mu(p,\ n,\ \lambda)$, the spin 1 incoming wave function which satisfies

$$p\cdot\epsilon(p,n,\lambda) = 0 \qquad \epsilon^+(p,n,\lambda)\epsilon(p,n,\lambda') = \delta_{\lambda\lambda'}$$

is given by *

$$\epsilon_\mu(p,n,0) = n_\mu \qquad \epsilon_\mu(p,n,\pm 1) = \frac{1}{\sqrt{2}}(n'_\mu \mp N_\mu) \tag{3.9}$$

or

$$\epsilon_\mu(p,n,\lambda) = -\frac{1}{2}\frac{E}{m}c_\lambda[\bar{v}(p,n,\lambda_1)\gamma_\mu u(p,n,\lambda_2) + \bar{v}(p,n,\lambda_2)\gamma_\mu u(p,n,\lambda_1)] \tag{3.10}$$

$\lambda = \frac{1}{2}(\lambda_1+\lambda_2)$ and $c_0 = 1$, $c_{\pm 1} = \frac{1}{\sqrt{2}}$. An object of interest is the tensor $\epsilon^+_\mu(\lambda)\epsilon_\nu(\lambda)$. From the properties of the Levi-Civita tensor we get

$$N_\mu N_\nu = -\frac{1}{m^2}p_\mu p_\nu - \delta_{\mu\nu} + n_\mu n_\nu + n'_\mu n'_\nu$$

thus

$$\epsilon^+_\mu(0)\epsilon_\nu(0) = n_\mu n_\nu$$
$$\begin{aligned}\epsilon^+_\mu(\lambda)\epsilon_\nu(\lambda) &= \frac{1}{2}[\delta_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu - n_\mu n_\nu + \lambda(N_\mu n'_\nu - N_\nu n'_\mu)] \quad (\lambda = \pm 1)\\ &= \frac{1}{2}\Big[\delta_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu - n_\mu n_\nu + i\lambda(S_{\alpha\beta})_{\mu\nu}N^\alpha n'^\beta\Big]\\ &= \frac{1}{2}\Big[\delta_{\mu\nu} + \frac{1}{m^2}p_\mu p_\nu - n_\mu n_\mu + 2\frac{\lambda}{m}\epsilon_{\alpha\beta\mu\nu}n^\alpha p^\beta\Big]\end{aligned}$$

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* Compare with the vector boson polarizations in $e^+e^- \to W^+W^-$ of ref. 10.

where S is the spin operator, $(S_{\alpha\beta})_{\mu\nu} = -i(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\nu}\delta_{\beta\mu}) = -i\epsilon_{\rho\tau\alpha\beta}\epsilon^{\rho\tau}_{\mu\nu}$. For a longitudinal n the last term becomes $-2\lambda(\vec{S}\cdot\vec{p}\,/|\,\vec{p}\,|)_{ij}$ and

$$\Sigma_\lambda \epsilon^+_\mu \epsilon_\nu = \delta_{\mu\nu} + \frac{1}{m^2} p_\mu p_\nu$$

from the relation

$$\gamma^\mu \epsilon^{\mu\nu\alpha\beta} = +\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^5 + \delta^{\nu\alpha}\gamma^5\gamma^\beta - \delta^{\nu\beta}\gamma^5\gamma^\alpha + \delta^{\alpha\beta}\gamma^5\gamma^\nu$$

it follows

$$\not{\epsilon}(p, n, \pm) = \frac{1}{\sqrt{2}} \not{n}' \left(1 \mp \frac{1}{m}\gamma^5 \not{n} \not{p}\right)$$

ϵ_μ is given in terms of the particle four momentum and of a four vector n which has a clear physical meaning. It is the polarization vector relative to the pair of Dirac particles which merge into the vector boson.

The vector n' describes the degree of arbitrariness of the solution. Indeed we may go to the p rest frame and select n to be along the third direction with n' in the 1-2 plane. Thus

$$\epsilon^0_\mu = (0,0,1,0) \qquad \epsilon^\pm_\mu = \frac{1}{\sqrt{2}} e^{\pm i\phi}(1, \pm i, 0, 0)$$

which for $\phi = 0$ is the conventional result for spin up, down and zero along the third axis.

It is also seen that for a longitudinal $n(p)$ the vectors $\vec{\epsilon}\,(p, n, \lambda)$ are eigenstates of $\vec{s}\cdot\vec{p}\,/|\,\vec{p}\,|$.* For instance, if $\vec{p}$ is along the third direction and $\vec{n}\,//\,\vec{p}$ we may choose n' such that

$$\epsilon^0_\mu = \frac{E}{m}(0,0,1,\beta) \quad \epsilon^\pm_\mu = \frac{1}{\sqrt{2}}(1, \pm i, 0, 0,) \quad (|\,\vec{p}\,| = \beta E)$$

* Provided $\vec{n}'$, $\vec{N}$ and $\vec{n}$ are chosen to form a right-handed basis.

any other choice for n' consistent with $n' \cdot n' = 1$, $n' \cdot p = n' \cdot n = 0$ gives the same result up to phases.

In many calculations we need $\not{\epsilon}$ and it turns out that a particularly elegant expression can be derived within the present formalism. Consider $\gamma \cdot \chi$ with χ defined in eq. (3.2).

$$\gamma \cdot \chi(p, \lambda) = \frac{1}{4m} \Sigma \gamma^{\mu} tr \gamma^{\mu} u(p_i, n, \sigma_l) \bar{v}(p_j, n, \sigma_k) \tag{3.11}$$

Inside the trace only the part of $u\,\bar{v}$ which contains an odd number of γ-matrices gives a non zero contribution. From eqs. (3.5)(3.6) we derive

$$[\Gamma_{ij}(\lambda, \lambda')]_{\substack{odd \\ even}} \simeq \mp [v(p_i, n, -\lambda)\, \bar{u}(p_j, n, -\lambda')]_{\substack{odd \\ even}} \tag{3.12}$$

Also reversing the order of the γ-matrices in the string $(u\,\bar{v})_{odd}$ leads to

$$[\Gamma_{ij}(\lambda, \lambda')]^{R}_{odd} \simeq [\Gamma_{ij}(\lambda', \lambda)]_{odd} \tag{3.13}$$

Where again the symbol $\simeq$ gives equality up to a phase. We may now use the identity[11)]

$$\gamma^{\mu} tr \gamma^{\mu} s = 2(s + s^{R})$$

and get

$$\gamma \cdot \chi(p, \lambda) = \frac{1}{4m} \Sigma[(u\,\bar{v})_{odd} + (u\,\bar{v})^{R}_{odd}]$$

In fixing the normalization of $\epsilon_{\mu}(p, n, 0)$ we have made the choice that $\bar{u}(p_i, n, +1)$ $v(p_j, n, -1)$ and $\bar{u}(p_i, n, -1)$ $v(p_j, n, +1)$ have the same phase. Therefore $\gamma \cdot \chi$ can be cast in the form

$$\begin{aligned} \gamma \cdot \chi(p, 1/2(\lambda \pm \lambda)) = \frac{1}{m} \{ & u(p, n, \lambda)\, \bar{v}(p, n, \pm\lambda) + u(p, n, \pm\lambda)\, \bar{v}(p, n, \lambda) \\ & - e^{i\psi_{\pm}(\lambda)} [v(p, n, -\lambda)\, \bar{u}(p, n, \mp\lambda) + v(p, n, \mp\lambda)\, \bar{u}(p, n, -\lambda)] \} \end{aligned} \tag{3.14}$$

and

$$\not{\epsilon}(p,n,\lambda) = -EC_\lambda \gamma \cdot \chi(p,\lambda).$$

Next we discuss the application of these results to the calculation of processes involving external vector bosons. The usual helicity basis for vector particles corresponds to longitudinal polarization for the merging Dirac particles or

$$n_\mu = \frac{1}{\beta m}(\vec{p}\,, \beta^2 E) \quad 1-\beta^2 = \frac{m^2}{E^2}$$

Helicity amplitudes can now be evaluated following two alternative paths. Whenever a vector boson is emitted by a fermion line is more convenient to start from eq. (3.10) and therefore to transform the amplitude into a trace of γ-matrices. Indeed in this case the $A\,\bar{\psi}\,\psi$ vertex is transformed into a four fermion vertex allowing us to use computational techniques already developed for spinor amplitudes.[3)] For longitudinal polarization ($\lambda = 0$) however a technical problem arises, due to the presence of undetermined phases. To overcome this point we always use $\epsilon^0_\mu = n_\mu$ as prescribed by eq. (3.9), while for $\epsilon^\pm_\mu$ we prefer the above mentioned procedure which avoid the introduction of an extra parameter through the vector n'_μ.

In this way any helicity amplitude is converted into an expression which only contains momenta, the polarization vector n_μ and polarization vectors for the fermions. When two or more vector bosons are present we find it more convenient to express n and n' in terms of the external momenta which specify the process. This approach is somehow similar in spirit to the one of refs. 1(and 6). In the next section we illustrate the two procedure by means of several examples.

4. Decay and Production of Vector Bosons

4.1 $Z^0 \to f^+ f^-$

First we analyze the Z^0 decay into a pair of massless fermions, $Z^0(q) \to f^+(k_1) + f^-(k_2)$. The amplitude for the process reads

$$A(\sigma, 1/2\lambda_1, 1/2\lambda_2) = \bar{u}\,(k_2, \lambda_2)\, \not{\epsilon}(q, n, \sigma)(v_0 + a_0\gamma^5)v(k_1, \lambda_1) \tag{4.1}$$

Introducing

$$p_i = \frac{m_i}{M_0} q \quad (m_1 + m_2 = M_0)$$

$$2\sigma = \sigma_1 + \sigma_2$$

we find

$$\begin{aligned} A(\sigma, 1/2\lambda_1, 1/2\lambda_2) = &\frac{1}{4}\frac{E}{M_0} C_\sigma \Sigma tr\gamma^\mu(v_0 + a_0\gamma^5)v(k_1, \lambda_1)\,\bar{u}\,(k_2, \lambda_2) \\ &\times\, tr\gamma^\mu u(p_i, n, \sigma_l)\,\bar{v}\,(p_j, n, \sigma_k) \end{aligned} \tag{4.2}$$

The matters of this example being simple, we discuss it in some details. A first procedure consists in starting from eq. (4.2) and in eliminating the repeated γ-matrices with a technique developed in ref. 3. Let Γ be defined as

$$\Gamma = v(k_1, \lambda_1)\,\bar{u}\,(k_2, \lambda_2)$$

It can be written[3)] as a product of two (three) γ-matrices for $\lambda_1 = \lambda_2 (\lambda_1 = -\lambda_2)$, but only the odd part will survive inside the trace in eq. (4.2). Hence $\lambda_1 = -\lambda_2 = \lambda$ as especied for massless fermions.

Reversing the order of the γ-matrices we get

$$\Gamma^R \simeq -\gamma^5 v(k_2, -\lambda)\,\bar{u}\,(k_1, \lambda)\gamma^5$$

thus

$$A(\sigma,1/2\lambda,-1/2\lambda)=\frac{1}{2}\frac{E}{M_0}C_\sigma\Sigma\Big\{tr(v_0+a_0\gamma^5)V(k_1,\lambda;p_j,n,\sigma_k)trU(p_i,n,\sigma_l;k_2,-\lambda)$$
$$-e^{i\psi}tr\gamma^5V(k_2,-\lambda;p_j,n,\sigma_k)tr(a_0+v_0\gamma^5)U(p_i,n,\sigma_l;k_1,\lambda)\Big\} \tag{4.3}$$

where we have denoted by ψ the overall undetermined phase and U,V are explicity computed in the appendix.

$$U(p_i,n,\sigma_j;k_l,\lambda)=u(p_i,n,\sigma_j)\,\bar{u}\,(k_l,\lambda)$$
$$V(k_l,\lambda;p_i,n,\sigma_j)=v(k_l,\lambda)\,\bar{v}\,(p_i,n,\sigma_j)$$

Moreover U and V can only be determined up to a another phase and the two terms in eq. (4.3) interfere with an unknown coefficient. However for this particular example we proceed assuming that all the phases can be fixed to 1. Thus

$$A(\sigma,1/2\lambda,-1/2\lambda)=-\frac{1}{8}(\frac{E}{M_0E_1E_2})^{1/2}C_\sigma\Sigma\Sigma_{a,b=\pm}(v_0+\lambda a_0)n_{ab}\mathcal{P}^a(\lambda,\sigma_1)\mathcal{P}^b(\lambda,\sigma_2) \tag{4.4}$$

$$\begin{aligned}M_0n_{ab}&=(aM_0k_1\cdot n-k_1\cdot q)^{1/2}(-bM_0k_2\cdot n-k_2\cdot q)^{1/2}\\&+(bM_0k_1\cdot n-k_1\cdot q)^{1/2}(-aM_0k_2\cdot n-k_2\cdot q)^{1/2}\end{aligned} \tag{4.5}$$

In the Z^0 rest frame n_{ab} reduces to

$$n_{ab}=M_0(1+acos\theta)^{1/2}(1+bcos\theta)^{1/2}$$

with f^+ produced along the positive third axis and θ being the polar angle of the spin direction. The projection operators $\mathcal{P}$ select the desired amplitude, and the result is

$$\begin{aligned}A(\pm1,\pm1/2\,,\mp1/2)&=-\frac{1}{\sqrt{2}}(v_0\pm a_0)(1+cos\theta)\\A(\pm1,\mp1/2\,,\pm1/2)&=-\frac{1}{\sqrt{2}}(v_0\mp a_0)(1-cos\theta)\\A(0,\pm1/2\,,\mp1/2)&=-(v_0\pm a_0)sin\theta\end{aligned} \tag{4.6}$$

Another way of computing the amplitude is the following

$$A(\sigma, 1/2\lambda_1, 1/2\lambda_2) = \frac{1}{4}\frac{E}{M_0} c_\sigma \Sigma tr \gamma^\mu (v_0 + a_0\gamma^5) V(k_1, \lambda_1; p_i, n, \sigma_l)\gamma^\mu U(p_j, n, \sigma_k; k_2, \lambda_2) \tag{4.7}$$

At this level there is no ambiguity due to arbitrary phases, and clearly this remains true for $\sigma = \pm 1$. For $\sigma = 0$ however the symmetrization in σ_1, σ_2 introduces such an ambiguity, as it will be clear from the following example.

4.2 $e^+e^- \to Z^0H^0$

As a first generalization of the previous example we consider $e^+(k_1) + e^-(k_2) \to Z^0(q) + H^0(Q)$ where H^0 is the neutral Higgs particle of the standard model.

The amplitude is

$$A(\lambda_1, \lambda_2, \sigma) = \frac{i}{4} g^2 \frac{M_0}{cos^3\theta_W} \frac{1}{s - M_0^2} \bar{A}(\lambda_1, \lambda_2, \sigma) \tag{4.8}$$

with $s = -(k_1 + k_2)^2$ and θ_W is the weak mixing angle. Also

$$\bar{A}(\lambda_1, \lambda_2, \sigma) = \bar{v}(k_1, \lambda_1)\gamma^\mu (v_0 + a_0\gamma^5) u(k_2, \lambda_2)\epsilon_\mu^+(q, n, \sigma) \tag{4.9}$$

Using $\epsilon^+(\sigma) = \epsilon(-\sigma)$ and following the path of the least number of undetermined phases we have

$$\bar{A}(\lambda_1, \lambda_2, -\sigma) = \frac{1}{4}\frac{E}{M_0} C_{+\sigma} \Sigma tr(v_0 - a_0\gamma^5)\gamma^\mu W(k_2, \lambda_2; p_i, n, \sigma_l)\gamma^\mu W(p_j, n, \sigma_k; k_1, \lambda_1) \tag{4.10}$$

W is defined and computed in the appendix.

From the discussion of sec. 4.1 we know that W can be evaluated only up to a phase. For $\sigma = \pm 1 (\sigma_1 = \sigma_2 = \pm 1)$ there is actually only an overall phase in $\bar{A}$ which we may neglect. For $\sigma = 0$ we have to symmetrize, i.e. to sum the results with $\sigma_1 = +1$, $\sigma_2 = -1$ and $\sigma_1 = -1$, $\sigma_2 = +1$, which produces an intrinsically undetermined interference. For this reason we choose to compute

$$\bar{A}(\lambda_1, \lambda_2, 0) = tr \not{A} (v_0 + a_0\gamma^5) u(k_2, \lambda_2)\bar{v}(k_1, \lambda_1) \tag{4.11}$$

and

$$\bar{A}(\lambda_1,\lambda_2,\pm 1)=\frac{1}{2\sqrt{2}}\frac{E}{M_0}tr(v_0-a_0\gamma^5)\gamma^\mu W(k_2,\lambda_2;p_i,n,\mp 1)\gamma^\mu W(p_j,n,\mp 1;k_1,\lambda_1) \tag{4.12}$$

The advantage of a formulation based on eq. (4.12) and not on an expression similar to the one in eq. (4.11), with $\not{p}$ replaced by $\not{p}' \pm \not{N}$, is the possibility of expressing the result in terms of k_1, k_2, q and n directly.

After some algebra we find that only $\bar{A}(\lambda,-\lambda,\pm 1)$ survives

$$\bar{A}(\lambda,-\lambda,\pm 1)=-\frac{1}{4\sqrt{2}}\frac{1}{M_0^2(E_1E_2)^{1/2}}(v_0+\lambda a_0)\Sigma_{a,b=\pm}n_{2a}n_{1b}f_{ab}^{\lambda}p^a(\lambda,\mp 1)p^b(\lambda,\mp 1) \tag{4.13}$$

$$n_{1a}=M_0^{1/2}(-aM_0k_1\cdot n-k_1\cdot q)^{-1/2} \quad n_{2a}=M_0^{1/2}(aM_0k_2\cdot n-k_2\cdot q)^{-1/2} \tag{4.14}$$

$$f_{ab}^{\lambda}=(1-ab)M_0^2k_1\cdot k_2-2(1+ab)k_1\cdot qk_2\cdot q+4abM_0^2k_1\cdot nk_2\cdot n$$
$$+(3a+b)M_0k_1\cdot qk_2\cdot n-(a+3b)M_0k_1\cdot nk_2\cdot q+\lambda(a-b)M_0\epsilon_{\mu\nu\alpha\beta}q^\mu n^\nu k_1^\alpha k_2^\beta \tag{4.15}$$

From eq. (4.13) follows that $a=b$ is selected in the sum. Also

$$f_{++}^{\lambda}=-4M_0^2(n_{1+}n_{2+})^{-2} \quad f_{--}^{\lambda}=-4M_0^2(n_{1-}n_{2-})^{-2} \tag{4.16}$$

the result becomes particularly simple

$$\bar{A}(\lambda,-\lambda,\pm 1)=\frac{1}{\sqrt{2}}\frac{1}{M_0(E_1E_2)^{1/2}}(v_0+\lambda a_0)$$
$$(k_1\cdot q\mp\lambda M_0k_1\cdot n)^{1/2}(k_2\cdot q\pm\lambda M_0k_2\cdot n)^{1/2} \tag{4.17}$$

to compute $\bar{A}(\lambda,-\lambda,0)$ we use[3)]

$$u(k_2,-\lambda)\,\bar{v}(k_1,\lambda)=\frac{1}{u(k_2-\lambda)\not{n}v(k_1,\lambda)}u(k_2,-\lambda)$$
$$\bar{u}(k_2,-\lambda)\not{n}v(k_1,\lambda)\,\bar{v}(k_1,\lambda)$$
$$\simeq\frac{1}{4\sqrt{2}}(E_1E_2)^{-1/2}(2k_1\cdot nk_2\cdot n-k_1\cdot k_2)^{-1/2}$$
$$\not{k}_2\not{n}\not{k}_1(1-\lambda\gamma^5) \tag{4.18}$$

It follows

$$\bar{A}(\lambda,-\lambda,0)=\frac{1}{\sqrt{2}}(E_1E_2)^{-1/2}(v_0+\lambda a_0)(2k_1\cdot nk_2\cdot n-k_1\cdot k_2)^{1/2} \tag{4.19}$$

To check the correctness of the result we evaluate

$$\Sigma_\sigma|\bar{A}(\lambda,-\lambda,\sigma)|^2=$$

$$\frac{1}{2}\frac{1}{E_1E_2}|v_0+\lambda a_0|^2(\frac{2}{M_0^2}k_1\cdot qk_2\cdot q-k_1\cdot k_2)$$

which is independent from n, as it should be.

For helicity amplitudes we use

$$n=\frac{1}{\beta M_0}(\vec{q},\beta^2E) \quad \beta^2=1-\left(\frac{M_0}{E}\right)^2$$

In terms of invariants

$$s=-(k_1+k_2)^2, t=-(k_1-q)^2, u=-(k_2-q)^2 \tag{4.20}$$

we obtain (m_H = Higgs mass)

$$\begin{aligned}
&2E\sqrt{s}=s+M_0^2-m_H^2\\
&k_1\cdot n=\frac{1}{2\beta M_0}\left(t-M_0^2+2\frac{M_0^2s}{s+M_0^2-m_H^2}\right)\\
&k_2\cdot n=\frac{1}{2\beta M_0}\left(u-M_0^2+2\frac{M_0^2s}{s+M_0^2-m_H^2}\right)
\end{aligned} \tag{4.21}$$

Consider now an arbitrary diagram where a vector boson is attached to a fermion line. The corresponding amplitude will be for instance

$$A(\lambda_1,\lambda_2,\sigma,...)=\frac{1}{2k_1\cdot q+m^2}\bar{v}(k_1,\lambda_1)\not{\epsilon}(q,n,\sigma)(v+a\gamma^5)(\not{k}_1-\not{q})Su(k_2,\lambda_2) \tag{4.22}$$

where S is the remainder of the amplitude. Helicity amplitudes are extracted by means of the following two equations

$$
\begin{aligned}
A(\lambda_1,\lambda_2,0,\ldots) &= \frac{1}{2k_1\cdot q+m^2} tr(v-a\gamma^5)\,\not{n}(\not{k}_1-\not{q})Su(k_2,\lambda_2)\,\bar{v}(k_1,\lambda_1)\\
A(\lambda_1,\lambda_2,\sigma,\ldots) &= \frac{1}{\sqrt{2}}\frac{E}{m}\frac{1}{2k_1\cdot q+m^2} tr(v-a\gamma^5)\gamma^\mu(\not{k}_1-\not{q})\\
&\quad\times SW(k_2,\lambda_2;q,n,\sigma)\gamma^\mu W(q,n,\sigma;k_1,\lambda_1)(\sigma=\pm 1)
\end{aligned}
\tag{4.23}
$$

The W are defined and evaluated in the appendix.

4.3 $e^+e^- \rightarrow Z^0\gamma$

In this example we work out the general formulas given in the last subsection. The process is $e^+(k_1)+e^-(k_2)\rightarrow Z^0(q)+\gamma(Q)$. Two diagrams contribute to the amplitude

$$
\begin{aligned}
A(\lambda_1,\lambda_2,\sigma,\rho) = \bar{v}(k_1,\lambda_1)\Big[&\frac{1}{2k_2\cdot Q}\,\not{\epsilon}^+(q,n,\sigma)(v_0+a_0\gamma^5)(\not{k}_2-\not{Q})\,\not{\eta}(Q,\rho)\\
&-\frac{1}{2k_1\cdot Q}\,\not{\eta}(Q,\rho)(\not{k}_1-\not{Q})\,\not{\epsilon}^+(q,n,\sigma)(v_0+a_0\gamma^5)\Big]u(k_2,\lambda_2)
\end{aligned}
\tag{4.24}
$$

where η is the photon polarization and overall factors have been neglected. For $\not{\eta}$ we use the expression of ref. 1

$$
\not{\eta}(Q,\rho) = \frac{1}{2\sqrt{2}N}[\not{k}_1\,\not{k}_2\,\not{Q}(1-\rho\gamma^5) - \not{Q}\,\not{k}_1\,\not{k}_2(1+\rho\gamma^5)] \tag{4.25}
$$

Where N is a computable normalization factor. It follows

$$
\begin{aligned}
A(\lambda,-\lambda,\rho,\sigma) =& \frac{1}{2\sqrt{2}N}[(v_0-\rho a_0)tr(1+\rho\gamma^5)\,\not{\epsilon}^+(\not{k}_2-\not{Q})\,\not{k}_1\,u(k_2,-\lambda)\,\bar{v}(k_1,\lambda)\\
&+(v_0+\rho a_0)tr(1-\rho\gamma^5)\,\not{k}_2(\not{k}_1-\not{Q})\,\not{\epsilon}^+\,u(k_2,-\lambda)\,\bar{v}(k_1,\lambda)]
\end{aligned}
\tag{4.26}
$$

where as expected $A(\lambda,\lambda,\rho,\sigma)=0$. Since the main purpose of these examples is to show the feasibility of the method, we concentrate on Z^0 longitudinally polarized.

Thus $\epsilon^0 = n$ and we evaluate $u_2 \bar{v}_1$ in the usual way, i.e. we multiply and divide by the bilinear form $\bar{u}_2 \not{n} v_1$.

It follows

$$A(\lambda, -\lambda, \rho, 0) = \frac{1}{16N}(E_1E_2)^{-1/2}(2k_1 \cdot nk_2 \cdot n - k_1 \cdot k_2)^{-1/2}$$
$$\times \Big\{(1-\lambda\rho)(v_0 - \rho a_0)tr(1-\lambda\gamma^5) \not{n} (\not{k}_2 - \not{Q}) \not{k}_1 \not{k}_2 \not{n} \not{k}_1 \tag{4.27}$$
$$+ (1+\lambda\rho)(v_0 + \rho a_0)tr(1-\lambda\gamma^5) \not{k}_2 (\not{k}_1 - \not{Q}) \not{n} \not{k}_2 \not{n} \not{k}_1\Big\}$$

Notice that the two terms in the previous expression never interfere. The traces are easily computed and give

$$\begin{aligned} A(\lambda, -\lambda, \rho, 0) = {} & \frac{1}{2N}(E_1E_2)^{-1/2}(2k_1 \cdot nk_2 \cdot n - k_1 \cdot k_2)^{-1/2} \\ & \times \Big\{(1-\lambda\rho)(v_0 - \rho a_0)[(k_1 \cdot k_2 - k_1 \cdot Q)(k_1 \cdot nk_2 \cdot n - k_1 \cdot k_2) \\ & + (k_1 \cdot n)^2(k_2 \cdot Q - k_1 \cdot k_2) - \lambda k_1 \cdot n\epsilon(Q_1, k_1, k_2, n)] \\ & + (1+\lambda\rho)(v_0 + \rho a_0)[(k_1 \cdot k_2 - k_2 \cdot Q)(k_1 \cdot nk_2 \cdot n - k_1 \cdot k_2) \\ & + (k_2 \cdot n)^2(k_1 \cdot Q - k_1 \cdot k_2) - \lambda k_2 \cdot n\epsilon(Q, k_1, k_2, n)]\Big\} \end{aligned} \tag{4.28}$$

where $\epsilon(Q, k_1, k_2, n) = \epsilon_{\mu\nu\alpha\beta}Q^\mu k_1^\nu k_2^\alpha n^\beta$.

Things further simplify where we allow for arbitrary phases. Any expression $a + b\epsilon$ where ϵ is a saturated Levi-Civita symbol is equal, modulo a phase, to $(a^2 - b^2\epsilon^2)^{1/2}$. After some algebraic manipulations over a product of two ϵ tensors we find

$$\begin{aligned} A(\lambda, -\lambda, \rho, 0) = {} & \frac{1}{2N}(E_1E_2)^{-1/2}(k_1 \cdot k_2)^{1/2} \\ & \times \Big\{(1-\lambda\rho)(v_0 - \rho a_0)[2(k_1 \cdot n)^2(k_1 \cdot Q + k_2 \cdot Q) \\ & + k_1 \cdot Q(2k_1 \cdot k_2 - k_1 \cdot Q)]^{1/2} \\ & + (1+\lambda\rho)(v_0 + \rho a_0)[2(k_2 \cdot n)^2(k_1 \cdot Q + k_2 \cdot Q) \\ & + k_2 \cdot Q(2k_1 \cdot k_2 - k_2 \cdot Q)]^{1/2}\Big\} \end{aligned} \tag{4.29}$$

This cancellation does not come unexpected since the final result cannot depend on the procedure used to evaluate $u_2 \bar{v}_1$ and the square root in the denominator of eq. (4.28) is the remainder of this arbitrariness.

The extension to transverse polarization for the Z^0 is straightforward since everything is reduced to a trace of γ-matrices.

This process could also be analyzed with the polarization basis of ref. 6 which however apply only to a specific class of processes, while there is no restriction for our formulas.

4.4 $e^+e^- \to W^+W^-$

Whenever two or more vector bosons appear as external particles in a given process, the number of terms generated by the use of eq. (3.10) makes the procedure non-competitive.

Instead we may express n_μ and n'_μ in terms of the external momenta and use directly eq. (3.9).

We study the process $e^+(p_1) + e^-(p_2) \to W^+(q_1) + W^-(q_2)$. The corresponding amplitude is given by

$$\begin{aligned} A = & \bar{v}\,\gamma^\mu(A_\gamma + A_z v_0 + A_z a_0 \gamma^5) \\ & \times u[2q_2 \cdot \epsilon_1 \epsilon_{2\mu} - 2q_1 \cdot \epsilon_2 \epsilon_{1\mu} + (q_1 - q_2)_\mu \epsilon_1 \cdot \epsilon_2] \\ & + A_\nu\, \bar{v}\,\gamma^\alpha (1+\gamma^5)(\not{p}_1 - \not{q}_1)\gamma^\beta u \epsilon_{1\alpha}\epsilon_{2\beta} \end{aligned} \tag{4.30}$$

where A_γ, A_z refer to a γ, Z^0 exchange in the s-channel and A_ν to the t-channel diagram with an internal neutrino line. In the standard model $v_0 = 4sin^2\theta_W - 1$ and $a_0 = -1$.

The polarization vectors are defined by

$$\epsilon^0_\mu(q_i) = n_\mu(q_i) \qquad \epsilon^\pm_\mu(q_i) = \frac{1}{\sqrt{2}}[n'_\mu(q_i) \mp N_\mu(q_i)] \tag{4.31}$$

with $N_\mu = 1/M\epsilon_{\mu\nu\alpha\beta}n'^\nu n^\alpha q_i^\beta$. A convenient choice for $n_\mu(q_i)$, which satisfies both $q_i \cdot n = 0$ and $n^2 = 1$, is*

$$
\begin{aligned}
n_\mu(q_1) &= -\frac{1}{Mn_0}(q_1 \cdot q_2 q_{1\mu} + M^2 q_{2\mu}) \\
n_\mu(q_2) &= -\frac{1}{Mn_0}(q_1 \cdot q_2 q_{2\mu} + M^2 q_{1\mu})
\end{aligned}
\tag{4.32}
$$

where the normalization gives $n_0^2 = (q_1 \cdot q_2)^2 - M^4$. In the e^+e^- c.m.s. $\vec{n}\,(q_i)$ is directed along $\vec{q}_i$, the direction of motion.

Introducing the usual invariants

$$
s = -(q_1 + q_2)^2, t = -(p_1 - q_1)^2, u = -(p_2 - q_1)^2 \tag{4.33}
$$

we get $4n_0^2 = (s - 4M^2)s$. A solution for n'_μ which fulfill the prescribed requirements can also be found

$$
\begin{aligned}
n'_\mu(q_i) &= \frac{1}{n_1}\epsilon_{\mu\nu\alpha\beta}p_1^\nu p_2^\alpha q_i^\beta \\
|n_1|^2 &= -M^2(p_1 \cdot p_2)^2 - 2p_1 \cdot p_2 p_1 \cdot q_i p_2 \cdot q_i
\end{aligned}
\tag{4.34}
$$

or

$$
4|n_1|^2 = (t - M^2)(u - M^2)s - M^2 s^2
$$

From n_μ and n'_μ we construct N_μ

$$
N_\mu(q_i) = \frac{1}{4n_0 n_1}(N_{i1}p_{1\mu} + N_{i2}p_{2\mu} + N_i q_{i\mu})
$$

$$
\begin{aligned}
N_{22} = -N_{11} &= (2M^2 - s)(u - M^2) + 2M^2(t - M^2) \\
N_{12} = -N_{21} &= (2M^2 - s)(t - M^2) + 2M^2(u - M^2) \\
N_1 = -N_2 &= (t - M^2)^2 - (u - M^2)2
\end{aligned}
\tag{4.35}
$$

* Compare with eq. (6.10) of ref. 6.

In the e^+e^- c.m.s. $\vec{N}(q_i)$ lies in the scattering plane and $\vec{n}'(q_i)$ is orthogonal to it (with $N_0 = n'_0 = 0$). The polarization vectors $\epsilon_\mu^\lambda(q_i)$ are expressed in terms of p_1,p_2 and q_1,q_2. As a next step we eliminate Dirac spinors and γ-matrices from the amplitude. To keep things as general as possible the calculation will be carried on allowing for an arbitrary degree of transverse polarization in both the e^+ and e^- beams. As usual we derive[3)]

$$\begin{aligned} u_2\,\bar{v}_1 &= u(p_2,a_2,\lambda_2)\,\bar{v}(p_1,a_1,\lambda_1) \\ &= \frac{1}{8}(E_1E_2)^{-1/2}\Sigma_{a=\pm}K_a^{-1/2}S(\lambda_1,\lambda_2)P^a(\lambda_1,\lambda_2) \end{aligned} \tag{4.36}$$

where$E_i = p_{i0}$, $a_i^2 = 1$, $p_i \cdot a_i = 0$. Moveover

$$\begin{aligned} K_a &= -(p_1\cdot p_2 + m^2)(1 + a a_1\cdot a_2) + a p_1\cdot a_2 p_2\cdot a_1 \\ S(\lambda_1,\lambda_2) &= (-i\not{p}_2 + m)(1 + i\lambda_2\gamma^5\not{a}_2)(1 + i\lambda_1\gamma^5\not{a}_1)(-i\not{p}_1 - m) \end{aligned} \tag{4.37}$$

m being the electron mass. In general a_i will be the sum of two terms.

$$a_i = \frac{cos\psi_i}{\beta m}a_i^L + sin\psi_i a_i^T \tag{4.38}$$

However in this example we assume $\psi_1 = \psi_2$ and

$$a_i^L = (\vec{p}_i\,,\beta^2E) \qquad a_{1,2}^T = \pm(\vec{a}_T\,,0) \tag{4.39}$$

with $a_T^2 = 1$, $\vec{p}_i\cdot\vec{a}_T = 0$. Any generalization is straightforward. Neglecting the electron mass whenever possible we obtain

$$K_+ = -2cos^2\psi p_1\cdot_2 \qquad K_- = -2sin^2\psi p_1\cdot p_2 \tag{4.40}$$

and

$$\begin{aligned} S(\lambda,\lambda) &= -2cos\psi\,\not{p}_2\,(cos\psi - \lambda\gamma^5 + isin\psi\,\not{a}_T)\,\not{p}_1 \\ S(\lambda,-\lambda) &= -2sin\psi\,\not{p}_2\,[sin\psi - i(cos\psi - \lambda\gamma^5)\,\not{a}_T]\,\not{p}_1 \end{aligned} \tag{4.41}$$

Therefore collecting the various results we have

$$u_2\,\bar{v}_1 = -\frac{1}{4}(2E_1E_2p_1\cdot p_2)^{-1/2}\,\not{p}_2\,[\sin\psi - i(\cos\psi - \lambda\gamma^5)\,\not{s}_T]\,\not{p}_1 \qquad \text{for } \lambda_1 = -\lambda_2 = \lambda$$

$$= -\frac{1}{4}(2E_1E_2p_1\cdot p_2)^{-1/2}\,\not{p}_2\,[\cos\psi - \lambda\gamma^5 + i\sin\psi\,\not{s}_T] \qquad \text{for } \lambda_1 = \lambda_2 = \lambda \tag{4.42}$$

The amplitude A can now be given in terms of the quantities L_μ, L^5_μ, $L_{\mu\nu}$ and $L^5_{\mu\nu}$

$$\begin{aligned} L_\mu &= tr\gamma^\mu u_2\,\bar{v}_1 \qquad & L^5_\mu &= tr\gamma^\mu\gamma^5 u_2\,\bar{v}_1 \\ L_{\mu\nu} &= tr\gamma^\mu(\not{p}_1 - \not{q}_1)\gamma^\nu u_2\,\bar{v}_1 \qquad & L^5_{\mu\nu} &= tr\gamma^\mu\gamma^5(\not{p}_1 - \not{q}_1)\gamma^\nu u_2\,\bar{v}_1 \end{aligned} \tag{4.43}$$

as a result

$$\begin{aligned} A =& [(A_\gamma + v_0A_z)L^\mu + a_0A_zL^{5\mu}][2q_2\cdot\epsilon_1\epsilon_{2\mu} - 2q_1\cdot\epsilon_2\epsilon_{1\mu} + (q_1 - q_2)_\mu\epsilon_1\cdot\epsilon_2] \\ &+ A_\nu(L^{\mu\nu} + L^{5\mu\nu})\epsilon_{1\mu}\epsilon_{2\nu} \end{aligned} \tag{4.44}$$

Where the polarization vectors $\epsilon_i (i = 1,2)$ are known functions of the momenta and the leptonic tensors L are simply given in terms of traces of γ-matrices. If $\psi_1 \neq \psi_2$, as in any realistic case, eqs. (4.42) get modified but still remain simple in the limit of massless electrons. This example is only indicative of the procedure to be used for arbitrary e^+e^- polarizations and small mass approximation.

The final expression of the amplitude for $e^+e^- \to W^+W^-$ becomes rather complicated when we take into account an arbitrary transverse component of the beam polarization.* Therefore in the following we discuss effects for massless and longitudinal polarized e^+ and e^-. We start by rewriting the amplitude

$$\begin{aligned} A = tr[&(A_\gamma + A_zv_0 - A_za_0\gamma^5)\gamma^\mu W_\alpha \\ &+ A_\nu\gamma^\alpha(1+\gamma^5)(\not{k}_1 - \not{q}_1\,\gamma^\beta W_{\alpha\beta}]u(k_2,-\lambda)\,\bar{v}\,(k_1,\lambda \end{aligned} \tag{4.45}$$

---

* However we mention that this amplitude is in a form which can be easily handled by Schoonschip.

with

$$
\begin{aligned}
W_\alpha &= 2q_2 \cdot \epsilon_1 \epsilon_{2\alpha} - 2q_1 \cdot \epsilon_2 \epsilon_{1\alpha} + (q_1 - q_2)_\alpha \epsilon_1 \cdot \epsilon_2 \\
W_{\alpha\beta} &= \epsilon_{1\alpha} \epsilon_{2\beta}
\end{aligned}
\tag{4.46}
$$

We now multiply and divide $u\,\bar{v}$ by some bilinear invariant. Since the result is independent from the explicit form of this invariant a convenient choice will be the following. Let t be $t_\mu = (\vec{t}, 0)$ with $\vec{t}$ normal to the scattering plane. Thus

$$
u(k_2, -\lambda)\,\bar{v}(k_1, \lambda) \simeq \frac{1}{4\sqrt{2}} (E_1 E_2 k_1 \cdot k_2)^{-1/2} \not{k}_2 \not{t} \not{k}_1 (1 - \lambda\gamma^5) \tag{4.47}
$$

Thanks to the properties of the vector t_μ the leptonic parts of the amplitude can be expressed as a combination of terms where t appears only with a free index or saturated with an ϵ-symbol. These expressions are lengthy and insignificant at this level, and will not be presented here.

Next we use polarization vectors ϵ as given in eqs. (4.32) (4.34) (4.35). When the leptonic parts of the amplitude are saturated with W_α and $W_{\alpha\beta}$ the following happens. Whenever there is an ϵ^0 only the terms with a Levi-Civita symbol survive since $t \cdot n(q_i) = 0$. On the other end for an $\epsilon^\pm$ there will be terms with an ϵ- symbol and an N[eq. (4.35)] or terms proportional to $t \cdot n'(q_i)$ which in turn contains again a Levi-Civita tensor.

To see in practice how this works we consider longitudinal polarized W's. In this case we have

$$
\begin{aligned}
A = \frac{1}{\sqrt{2}} (E_1 E_2 k_1 \cdot k_2)^{-1/2} \epsilon_{\mu\nu\alpha\beta} \Big\{ &-\lambda (A_\gamma + A_z v_0 + \lambda A_z a_0) W^\mu k_1^\nu k_2^\alpha \\
&+ (1+\lambda) A_\nu [k_1 \cdot k_2 n_1^\mu (k_1 - q_1)^\nu n_2^\alpha + n_1 \cdot n_2 k_2^\mu k_1^\nu q_1^\alpha \\
&+ k_1 \cdot n_1 k_2^\mu k_1^\nu n_2^\alpha - k_2 \cdot n_2 k_2^\mu k_1^\nu n_1^\alpha] \Big\} t^\beta
\end{aligned}
\tag{4.48}
$$

Using the polarization basis given in eq. (4.32) we get after some algebra

$$
A = \frac{1}{\sqrt{2}} (E_1 E_2 k_1 \cdot k_2)^{-1/2} \bar{A}\, \epsilon_{\mu\nu\alpha\beta} q_1^\mu k_1^\nu k_2^\alpha t^\beta \tag{4.49}
$$

$$\bar{A} = \frac{1}{M^2}\{-4\lambda(A_\gamma + A_z v_0 + \lambda A_z a_0)M^2 + (1+\lambda)A_\nu f(s,t)\} \tag{4.50}$$

Before giving f we notice one nice feature in the last equation: the sum of a large number of terms is now contained in a single saturated ϵ tensor. Moreover

$$\epsilon_{\mu\nu\alpha\beta} k_1^\mu k_1^\nu k_2^\alpha t^\beta \simeq (k_1 \cdot k_2)^{1/2}(2k_1 \cdot q_1 k_2 \cdot q_1 + M^2 k_1 \cdot k_2)^{1/2} \tag{4.51}$$

which gives

$$A = \frac{1}{2}(E_1 E_2)^{-1/2}\bar{A}[(t-M^2)(u-M^2) - M^2 s]^{1/2} \tag{4.52}$$

with

$$f(s,t) = \frac{ts + 4M^4}{s - 4M^2} \tag{4.53}$$

For transverse polarized W's the procedure works in the same way. The whole amplitude is proportional to a single ϵ-symbol which can be eliminated by allowing for an overall arbitrary phase. Notice also that the vector t drops from the final answer, as it should be since it has no physical meaning. However the choice we made for it at the beginning turns out to be very convenient in combining together the 3 diagrams for $e^+e^- \to W^+W^-$. Finally we mention that a covariant polarization basis for $W^+(p_1) + W^-(p_2) \to W^+(q_1) + W^-(q_2)$ can be derived along the same lines.

5. Wave Function for a Massive Spin 3/2 Particle

The formalism developed in the previous sections for vector particles can easily be extended to include the Rarita-Schwinger field. Following the notations of Luriè[9] we describe a spin 3/2 particle by means of a completely symmetric third rank spinor $\Psi_{\alpha\beta\lambda}$. The Bargmann-Wigner equations for spin 3/2 are

$$\begin{aligned} \not{\partial}_{\alpha\rho}\Psi_{\rho\beta\lambda} + m\Psi_{\alpha\beta\lambda} &= 0 \\ \not{\partial}_{\beta\rho}\Psi_{\alpha\rho\lambda} + m\Psi_{\alpha\beta\lambda} &= 0 \\ \not{\partial}_{\lambda\rho}\Psi_{\alpha\beta\rho} + m\Psi_{\alpha\beta\lambda} &= 0 \end{aligned} \tag{5.1}$$

We look for a solution of the form

$$\Psi_{\alpha\beta\lambda}(p) = \frac{1}{6}\Sigma_{perm}\psi_\alpha(p_i)\psi_\beta(p_j)\psi_\lambda(p_l) \tag{5.2}$$

where the sum is over the permutations of the indices ijl, and each ψ satisfies a Dirac equation

$$(i \not{p}_j + m_j)\psi(p_j) = 0 \qquad (\Sigma_i p_i = p)$$

This ansatz satisfies the Bargmann-Wigner equations (5.1) provided that

$$p_i = \frac{m_i}{m} p \qquad \Sigma_i m_i = m \tag{5.3}$$

Indeed with this choice $i \not{p}_l \psi(p_j) = -m_l \psi(p_j)$ and from the first equation in (5.1)

$$\begin{aligned}(i \not{p} + m)_{\alpha\rho}\Psi_{\rho\beta\lambda}(p) &= \frac{1}{6}\Sigma_{perm}\Big\{\Sigma_k[i \not{p}_k \psi(p_i)]_\alpha + m\psi_\alpha(p_i)\Big\}\psi_\beta(p_j)\psi_\lambda(p_l) \\ &= 0\end{aligned}$$

with identical results from the remaining two equations. The spin 3/2 wavefunction can be expressed by

$$\psi^\mu_\alpha = \gamma^\mu_{\rho\beta}\Psi_{\beta\lambda\alpha}C_{\lambda\rho} \tag{5.4}$$

with the constraint $\gamma_\mu\psi^\mu_\alpha = 0$. C is the charge conjugation matrix. Using eq. (5.2) inside eq. (5.4) and the properties of the spinor $u(p, n, \lambda)$ we arrive at the expression for the wavefunction of an incoming spin 3/2 particle of four momentum p. This synthetizes the fusion method for spin 3/2.

ψ^μ_α is directly related to

$$\chi^\mu_\alpha(p, \lambda_1, \lambda_2) = \frac{1}{3}C(\lambda_1, \lambda_2)\Sigma_{i=1,3}\chi^\mu(p - p_i, \lambda_1)u_\alpha(p_i, n, \lambda_2) \tag{5.5}$$

where χ^μ has been defined in sec. 2 and the coefficient C has to be fixed.

$$\chi^\mu[p, \frac{1}{2}(\lambda_1 + \lambda_2)] = \frac{1}{4}\Sigma\,\bar{v}\,(p_i, n, \lambda_l)\gamma^\mu u(p_j, n, \lambda_k)$$

$$p = p_1 + p_2 \qquad \lambda_i \epsilon(\lambda_1, \lambda_2)$$

Due to the constraints (5.3) we have

$$(p - p_i)^2 = -(m - m_i)^2 \qquad p_j = \frac{m_j}{m - m_i}(p - p_i)$$

Thus

$$\chi^\mu_\alpha(p, \lambda_1, \lambda_2) = -\frac{1}{3}\frac{m}{E}\frac{C(\lambda_1, \lambda_2)}{C(\lambda_1)}\sum_{i=1,3}\epsilon^\mu(p - p_i, n, \lambda_1)u_\alpha(p_i, n, \lambda_2) \tag{5.6}$$

where $C(0) = 1$, $C(\pm 1) = 1/\sqrt{2}$ and ϵ^μ denotes the wavefunction for a spin 1 particle of four momentum $p - p_i$.

ψ^μ_α will be the appropriate linear combination of the χ^μ_α and all we have to do is fixing the coefficients.

Again we have been able to reduce a polarization basis for $s > 1/2$ to ordinary Dirac spinors. *

Define

$$\psi^\mu_\alpha(\pm\frac{3}{2}) = \chi^\mu_\alpha(\pm 1, \pm 1) \quad \psi^\mu_\alpha(\pm\frac{1}{2}) = \chi^\mu_\alpha(0, \pm 1) + \chi^\mu_\alpha(\pm 1, \mp 1) \tag{5.7}$$

We require $\psi^+(i)\psi(j) = \delta_{ij}$. Using

$$u^+(p_i, n, \lambda)u(p_j, n, \lambda') = \delta_{\lambda\lambda'}$$

$$\epsilon^+(p - p_i, n, \lambda)\epsilon(p - p_j, n, \lambda') = \delta_{\lambda\lambda'}$$

* For a formal theory of arbitrary spin matrices see ref. 12.

which hold for $p_i = (m_i/m)p$, we obtain

$$|C(\pm 1, \pm 1)|^2 = \frac{1}{2}(\frac{E}{m})^2 \qquad |C(0, \pm 1)|^2 + 2|C(\pm 1, \mp 1)|^2 = (\frac{E}{m})^2 \tag{5.8}$$

The second set is fixed by the condition $\gamma \cdot \psi = 0$. It is not possible to solve for the constraint without an explicit reference to spinor components. The best we can do (with the use of bilinear forms) is to derive $C(i, j), i \neq j$ up to undertermined phases. The relevant formulas are

$$\not{\epsilon}(p - p_i, n, \sigma)u(p_i, n, \lambda) = \sqrt{2}\, e^{i\phi_\sigma} P^-(\lambda, \sigma)v(p - p_i, n, -\sigma) \quad (\sigma = \pm 1)$$

$$\not{\epsilon}(p - p_i, n, 0)u(p_i, n, \lambda) = e^{i\psi}\Sigma_{a=\pm} P^a(1, \lambda)v(p - p_i, n, -a)$$

Thus

$$\gamma \cdot \psi(\pm\frac{1}{2}) = -\frac{1}{3}\frac{m}{E}\Sigma_{i=1,3}[2e^{i\phi_\pm}C(\pm 1, \mp 1) + e^{i\psi}C(0, \pm 1)]v(p - p_i, n, \mp 1)$$

and $\gamma \cdot \psi(\pm 1/2) = 0$ together with eq. (5.8) gives

$$C(\pm 1, \mp 1) = \frac{1}{\sqrt{6}}\frac{E}{m} \qquad C(0, \pm 1) = \sqrt{\frac{2}{3}}\, e^{i\theta_\pm}\frac{E}{m}$$

In the p rest frame with n along the third axis and using hermitean γ-matrices we recover the usual result[9] $C(o, \pm 1) = \mp\sqrt{2/3}\,E/m$.

Acknowledgements

I wish to express my gratitude to Professor S. Drell for the hospitality at the Stanford Linear Accelerator Center.

APPENDIX A

In this appendix we explicity evaluate the traces which appear within our formalism.

Let U and V be defined by

$$U(p,n,\sigma;k,\lambda) = u(p,n,\sigma)\,\bar{u}\,(k,\lambda)$$

$$V(k,\lambda;p,n,\sigma) = v(k,\lambda)\,\bar{v}\,(p,n,\sigma) \qquad p^2 = -m^2, k^2 = 0$$

Thus

$$U(p,n,\sigma;k,\lambda) = [\bar{u}\,(p,n,\sigma)u(k,\lambda)]^{-1}U(p,n,\sigma)U(k,\lambda)$$

with

$$U(p,n,\sigma) = \frac{1}{4p_0}(-i\not{p} + m)(1 + i\sigma\gamma^5\not{n})$$

$$U(k,\lambda) = \frac{1}{4k_0}(-i\not{k})(1 + \lambda\gamma^5)$$

We drop arbitrary phases in front of terms with different projection operators.

$$U(p,n,\sigma;k,\lambda) = -\frac{1}{8}(mp_0k_o)^{-1/2}$$

$$\Sigma_{a=\pm}n_a(p,k)(-i\not{p} + m)(1 + ia\not{n})\not{k}(1 + \lambda\gamma^5)P^a(\lambda,\sigma)$$

$$mn_a^{-2} = amk\cdot n - p\cdot k \quad , \quad P^a(\lambda,\sigma) = \frac{1}{2}(1 + a\lambda\sigma)$$

A similar result holds for V

$$V(k,\lambda;p,n,\sigma) = \frac{i}{8}(mp_0k_0)^{-1/2}$$

$$\Sigma_{a=\pm}n_a(p,k)(1 + \lambda\gamma^5)\not{k}(1 - ia\not{n})(i\not{p} + m)P^a(\lambda,\sigma)$$

The traces are now easily computed

$$trU(p,n,\sigma;k,\lambda) = \frac{1}{2}m(mp_0k_0)^{-1/2}\Sigma_{a=\pm}n_a^{-1}P^a(\lambda,\sigma)$$

$$tr\gamma^5U(p,n,\sigma;k,\lambda) = \lambda trU$$

$$trV(k,\lambda;p,n,\sigma) = -\frac{1}{2}m(mp_0k_0)^{-1/2}\Sigma_{a=\pm}n_a^{-1}P^a(\lambda,\sigma)$$

$$tr\gamma^5V(k,\lambda;p,n,\sigma) = \lambda trV$$

Similarly we define

$$W(p,n,\sigma;k,\lambda) = u(p,n,\sigma)\,\bar{v}(k,\lambda)$$
$$W(k,\lambda;p,n,\sigma) = u(k,\lambda)\,\bar{v}(p,n,\sigma)$$

Thus

$$W(p,n,\sigma;k,\lambda) = [\bar{u}(p,n,\sigma)v(k,\lambda)]^{-1}U(p,n,\sigma)V(k,\lambda)$$
$$= -\frac{i}{8}(mp_0k_0)^{-1/2}$$
$$\Sigma_{a=\pm}n_a(-p,-k)(-i\not{p}+m)(1-ia\not{n})\not{k}(1-\lambda\gamma^5)P^a(\lambda,\sigma)$$

and

$$W(k,\lambda;p,n,\sigma) = \frac{i}{8}(mp_0k_0)^{-1/2}$$
$$\Sigma_{a=\pm}n_a(-p,-k)(1-\lambda\gamma^5)\not{k}(1+ia\not{n})(i\not{p}+m)P^a(\lambda,\sigma)$$

The traces are

$$trW(p,n,\sigma;k,\lambda) = -trW(k,\lambda;p,n,\sigma)$$
$$= \frac{1}{2}m(mp_0k_0)^{-1/2}\Sigma_{a\pm}n_a^{-1}(-p,-k)P^a(\lambda,\sigma)$$
$$tr\gamma^5W = -\lambda trW$$

Other useful formulas are

$$\gamma^\mu U(p,n,\sigma;k,\lambda)\gamma^\mu = \frac{1}{4}(mp_0k_0)^{-1/2}\Sigma_{a=\pm}[n_a(p,k)\not{k}(m+a\not{n}\not{p})$$

$$-2imn_a^{-1}(p,k)](1-\lambda\gamma^5)P^a(\lambda,\sigma)$$

$$\gamma^\mu V(k,\lambda;p,n,\sigma)\gamma^\mu = -\frac{i}{4}(mp_0k_0)^{-1/2}\Sigma_{a=\pm}(1-\lambda\gamma^5)[n_a(p,k)(m+a\not{p}\not{n})\not{k}$$

$$+2imn_a^{-1}(p,k)]P^a(\lambda,\sigma)$$

$$\gamma^\mu W(p,n,\sigma;k,\lambda)\gamma^\mu = +\frac{i}{4}(mp_0k_0)^{-1/2}\Sigma_{a=\pm}[n_a(p,k)\not{k}(m+a\not{n}\not{p})$$

$$-2imn_a^{-1}(p,k)](1+\lambda\gamma^5)P^{-a}(\lambda,\sigma)$$

$$\gamma^\mu W(k,\lambda;p,n,\sigma)\gamma^\mu = -\frac{i}{4}(mp_0k_0)^{-1/2}\Sigma_{a=\pm}(1+\lambda\gamma^5)[n_a(p,k)(m+a\not{p}\not{n})\not{k}$$

$$+2imn_a^{-1}(p,k)]P^{-a}(\lambda,\sigma)$$

Finally we notice that for $p_i = (m_i/m)p$

$$U(p_i,n,\lambda) = U(p,n,\lambda)\,\text{etc}\ldots$$

If n' satisfies $n'\cdot n' = 1$, $n'\cdot p = n'\cdot n = 0$ we have

$$tr\gamma^\mu U(p_i,n,\lambda)\not{n}'V(p_j,n,\lambda) = -\frac{1}{4}\frac{m}{E^2}tr\gamma^\mu(m-i\not{p})\not{n}'(1+i\lambda\gamma^5\not{n})$$

$$= -\frac{1}{4}(\frac{m}{E})^2\left(n'^\mu - \frac{\lambda}{m}\epsilon^{\mu\nu\alpha\beta}n'_\nu n_\alpha p_\beta\right)$$

$$tr\gamma^\mu U(p_i,n,\lambda)\not{n}V(p_j,n,-\lambda) = -\frac{1}{4}\frac{m}{E^2}tr\gamma^\mu(m-i\not{p})\not{n}(1-i\lambda\gamma^5)$$

$$= -\frac{1}{4}(\frac{m}{E})^2 n^\mu$$

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