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A QUANTUM EFFECT IN GENERAL RELATIVITY*

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ABSTRACT

Using the general covariant Dirac equation, the gravitational correction to the anomalous magnetic moment of electron is calculated. It is shown that this quantum effect in general relativity, due to the earth gravity, could be measured by means of modern experiment devices.

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I.

At the present time, most of the effects useful for experimental test of general relativity only involve two constants, namely, the gravitational constant G and the velocity of light c. They are called the classical effects in general relativity. The gravitational deflection of light, the perihelion motion of a planet, the time-delay of a radar signal and the gravitational damping of the binary pulsar PSR 1913 + 16 are well known examples of this kind of effects.

In this paper, we shall investigate the quantum effects in general relativity. They involve not only the constants G and c, but also the planck's constant $\hbar$. An important problem relating to this kind of effects is to verify the general covariant Dirac equation, which is considered as an important foundation of the unified field theory involving the gravitational field.

Until now, the study of the quantum effects in general relativity only situated in the early stages. Using the neutron interferometer, the quantum mechanical phase-shift of neutron wave function induced by the gravitational field on the earth and by the earth's rotation[1] have been observed recently.

The purpose of this paper is to study the behavior of the general covariant equation of charged fermion with one-half spin when the gravitation field and the magnetic field exist simultaneously, especially in the non-relativistic case. We find that the gravitational field of the earth induces a correction to the anomalous magnetic moment of electron. The calculation shows that this correction has the same order of magnitude as the photon-photon scattering correction of the anomalous magnetic moment of electron due to the sixth order contribution from QED. We expect this new quantum effect in general relativity to be verifiable experimentally.

II.

We start by considering the general covariant Dirac equation in the case of existing electro-magnetic field

$$\Gamma^{\mu}\left(D_{\mu} - \frac{ie}{c\hbar} A_{\mu}\right)\psi + \frac{mc}{\hbar}\psi = 0 \tag{1}$$

Here A_{μ} is the electromagnetic potential, D_{μ} is the covariant derivative with respect to the local Lorentz group

$$D_{\mu} = \partial_{\mu} - \frac{1}{2}\omega_{\mu(ab)} I_{ab} \tag{2}$$

where $I_{ab} = \frac{1}{4}[\gamma_a \gamma_b - \gamma_b \gamma_a]$ and $\omega_{\mu(ab)}$ the gauge potential of the local Lorentz group. In terms of the vierbeins $\lambda^{\mu}_{(a)}$ and $\lambda_{\mu\,(a)}$

$$\omega_{\mu\,(ab)} = \lambda_{\mu\,(b)}\left(\lambda^{\nu}_{(a)}\right)_{;\mu}, \quad \omega_{\mu(ab)} = -\omega_{\mu\,(ba)} \tag{3}$$

and

$$\Gamma^{\mu} = \lambda^{\mu}_{(a)}\gamma_a \tag{4}$$

The $\lambda^{\mu}_{(a)}$ and $\lambda_{\mu(a)}$ are related to the Riemannian metric by

$$\lambda_{\mu(a)}\lambda_{\mu(a)} = g_{\mu\nu}, \quad \lambda^{\mu}_{(a)}\lambda^{\nu}_{(a)} = g^{\mu\nu}$$

and

$$\lambda_{\mu(a)}\lambda^{\mu}_{(b)} = \delta_{ab}, \quad \lambda_{\mu(a)}\lambda^{\nu}_{(a)} = \delta^{\nu}_{\mu} \tag{5}$$

Using (2), (3) and (4), the general covariant Dirac equation (1) can be rewritten in the following form

$$\Gamma^{\mu}\left(\delta_{\mu} - \frac{ie}{c\hbar} A_{\mu}\right)\psi + \frac{1}{4}\left(\gamma_{a}\,\omega_{(a)} + \gamma_{0}\gamma_{5}\,\tilde{\omega}_{(a)}\right)\psi + \frac{mc}{\hbar}\psi = 0 \qquad (6)$$

where

$$\omega_{(b)} = \omega_{(aba)},\ \tilde{\omega}_{(e)} = \varepsilon_{abce}\,\omega^{A}_{(abc)} \qquad (7)$$

$\omega^{A}_{(abc)}$ is anti-symmetric for all indexes a, b and c defined by

$$\omega^{A}_{(abc)}\ \frac{1}{6}\left[\omega_{(abc)} - \omega_{(acb)} + \omega_{(bca)} - \omega_{(bac)} + \omega_{(cab)} - \omega_{(cba)}\right]$$

and

$$\omega_{(abc)} = \lambda^{\mu}_{(a)}\,\omega_{\mu(bc)}\ ,\ \omega_{(abc)} = -\,\omega_{(acb)} \qquad (8)$$

In the case that the metric tenser is diagonal, it can be proved that $\tilde{\omega}_{(a)} = 0$, the general covariant Dirac equation (6) can be reduced to the form

$$\Gamma^{\mu}\left(\partial_{\mu} - \frac{ie}{c\hbar} A_{\mu}\right)\psi + \frac{1}{4}\gamma_{a}\,\omega_{(a)}\Psi + \frac{mc}{\hbar}\Psi = 0 \qquad (9)$$

$$\omega_{(a)} = \left(\lambda^{\mu}_{(a)}\right)_{;\mu} = \frac{1}{\sqrt{-g}}\,\partial_{\mu}\left(\sqrt{-g}\ \lambda^{\mu}_{(a)}\right)$$

In the weak-field approximation, making use of the Post-Newton metric given by

$$-\ ds^{2} = \left(1 + \frac{2r_{0}}{r}\right)\left(dx_{1}^{2} + dx_{2}^{2} + dx_{3}^{2}\right) - \left(1 - \frac{2r_{0}}{r}\right)c^{2}dt^{2}$$

where

$$r^{2} = x_{1}^{2} + x_{2}^{2} + x_{3}^{2}$$

$$r_{0} = \frac{GM}{c^{2}}$$

M is the mass of the spherical source. One can show that, when $r_0/r \ll 1$, the Dirac equation (9) has the form

$$i\hbar \frac{\partial\psi}{\partial t} = c\vec{\alpha} \cdot \left[\left(1 - \frac{2r_0}{r}\right)\left(\vec{p} - \frac{e}{e}\vec{A}\right)\right] + \\ + \beta\left[mc^2 + v(r)\right] + e\phi \Big\}\psi = 0, \tag{10}$$

omitting the terms higher than the first order in r_0/r . Here ϕ denotes the electric potential and $v(r)$ the gravitational potential

$$v(r) = -\frac{GmM}{r} \tag{11}$$

$$\vec{a} = \frac{1}{4}\frac{\hbar}{i}\nabla\left(\frac{r_0}{r}\right) \tag{12}$$

and

$$\beta = \gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\alpha} = i\beta\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

In the non-relativistic approximation, the Pauli equation corresponding to (10) can be expressed as

$$i\hbar \frac{\partial\Psi}{\partial t} = \left\{\frac{1}{2m}\left[\sigma \cdot \left(1 - \frac{r_0}{r}\right)\left(\vec{p} - \frac{e}{c}\vec{A}\right) + \vec{a}\right]^2 + v(r) + e\phi\right\}\Psi = 0 \tag{13}$$

where Ψ is the Pauli's wave function.

From Eq. (13) we see that when $r_0/r \to 0$, this equation is just the usual Pauli equation with gravitational potential $v(r) = -\frac{GmM}{r}$. Moreover, we conclude that the inertial mass m of the electron in Dirac's equation (1) is exactly equal to the gravitational mass of the electron given by (11), i.e., the equivalence principle holds for Dirac particle. The factor $\left(1 - \frac{2r_0}{r}\right)$ and the vector $\vec{a}$, with $\nabla \times \vec{a} = 0$, in Eq. (13) are due to the curved space-time.

If we choose the Coulomb gauge $\nabla \cdot \vec{A} = 0$, in the weak-field and the non-relativistic approximation, omitting the term higher than the first order in r_0/r and the term

$$\frac{r_0}{r}\frac{\vec{p}^2}{m} \simeq v(r)\left(\frac{v^2}{c^2}\right)$$

the Pauli equation (13) can be further expressed as

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} = &\left[\frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{H} + e\phi\right]\Psi + \\ &+ v(r)\Psi + \frac{4r_0}{r}\frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{H}\,\Psi + \frac{4r_0}{r}\frac{e}{mc}\vec{A}\cdot\vec{p}\,\Psi + \\ &+ \left[-\frac{2r_0}{r}\frac{e^2}{mc^2}\vec{A}^2 - \frac{1}{i}\frac{e\hbar}{2mc}\frac{r_0}{r^3}(\vec{r}\cdot\vec{A}) - \frac{e}{mc}\frac{r_0}{r^3}\vec{\sigma}\cdot\vec{r}\times\vec{A}\right. \\ &\left. + \frac{\hbar}{m}\frac{r_0}{r^3}\vec{\sigma}\cdot\vec{r}\times\vec{p}\right]\Psi \end{aligned} \tag{14}$$

In the case of homogeneous magnetic field, $\vec{A}$ is determined by

$$\vec{A} = \frac{1}{2}\vec{H}\times\vec{r}$$

and from (14) we have

$$\begin{aligned} i\hbar\frac{\partial\Psi}{\partial t} = &\left[\frac{1}{2m}(\vec{P} - \frac{e}{c}\vec{A})^2 - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{H} + e\,\phi\right]\Psi \\ &+ \left[v(r) + \frac{2r_0}{mr^3}\frac{\hbar}{2}\vec{\sigma}\cdot\vec{L}\right]\Psi \\ &+ \left[\frac{3r_0}{r}\frac{e}{2mc}(\vec{\sigma}\cdot\vec{H}) + \frac{r_0}{r}\frac{e}{2mc}(\vec{\sigma}\cdot\vec{n})(\vec{n}\cdot\vec{H})\right]\Psi \\ &+ \frac{2r_0}{r}\frac{e}{mc}\vec{L}\cdot\vec{H}\,\Psi \\ &+ rr_0\frac{e^2}{2mc^2}\left[\vec{H}^2 - (\vec{H}\cdot\vec{n})^2\right]\Psi \end{aligned} \tag{15}$$

where $L = \vec{r} \times \vec{p}$ and $\vec{n} = \vec{r}/r$.

We notice that the first term at the right-hand side of Eq. (15) is just the hamiltonian of the usual Pauli equation. The second term involves the gravitational potential and the spin-orbital angular momentum coupling due to the curved space-time. The third term is the gravitational correction to the magnetic moment of electron. The fourth term is the interaction between $\vec{L}$ and $\vec{H}$, and the last term the non-linear effect of magnetic field as the space-time is curved.

III.

Henceforth we will concentrate on the third term at the right-hand side of the Pauli equation (15), i.e.,

$$\frac{r_0}{r} \cdot \frac{e\hbar}{2mc} \left[3 \vec{\sigma} \cdot \vec{H} + (\vec{\sigma} \cdot \vec{n}) (\vec{n} \cdot \vec{H})\right] \tag{16}$$

It gives a correction to the magnetic moment of electron:

$$\delta\mu = a_g \mu_0$$

with

$$a_g = \begin{cases} -\dfrac{4r_0}{r} & \text{when } \vec{H} \parallel \vec{n} \\ -\dfrac{3r_0}{r} & \text{when } \vec{H} \perp \vec{n} \\ -\dfrac{2r_0}{r} & \text{when } \vec{H} \parallel -\vec{n} \end{cases} \tag{17}$$

where $\mu_0 = \dfrac{e\hbar}{2mc}$ is the Bohr magnetron. This correction is a quantum effect in general relativity due to the curved space-time.

From (17) we find that when a magnetic field exists on the surface of the earth, the gravitational field of the earth should give a contribution to the magnetic moment with the numerical value

$$a_g = -\frac{4r_0}{R} = -2.78 \times 10^{-9} = -0.222 \left(\frac{\alpha}{\pi}\right)^3, \text{ when } \vec{H} \parallel \vec{n}$$

$$a_g = -\frac{3r_0}{R} = -2.09 \times 10^{-9} = -0.167 \left(\frac{\alpha}{\pi}\right)^3, \text{ when } \vec{H} \perp \vec{n}$$

$$a_g = -\frac{2r_0}{R} = -1.39 \times 10^{-9} = -0.111 \left(\frac{\alpha}{\pi}\right)^3, \text{ when } \vec{H} \parallel -\vec{n} \qquad (18)$$

where R is the radius of the earth. For the convenience of comparison with the result in QED, we have given the equivalent value of a_g in terms of the third power of the fine structure constant α.

It is well-known that the determination of the lepton anomalous magnetic moment is one of the most significant challenges to both theoretical and experimental physicists. The anomalous magnetic moment of electron has been calculated up to the sixth-order contribution in QED, the recent result is[2]

$$\mu = (1 + a_{QED})\,\mu_0$$

$$a_{QED} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) - 0.328479 \left(\frac{\alpha}{\pi}\right)^2 + (1.1765 \pm 0.0013)\left(\frac{\alpha}{\pi}\right)^3 \qquad (19)$$

in which the contribution of photon-photon scattering is given by[3]

$$a_{\gamma\gamma} = (0.400 \pm 0.006)\left(\frac{\alpha}{\pi}\right)^3 \qquad (20)$$

The contribution of weak- and strong-interaction effects, and the contribution of μ's and τ's,[2] amount to less than 0.005×10^{-9}.

Comparing (18) with (20), we find that the correction due to the gravitational field of the earth a_g is of the same order of magnitude as $a_{\gamma\gamma}$.

The best experimental value for the anomalous factor a is[4]

$$a_{exp} = \left[\frac{g-2}{2}\right]_{exp} = (1159652.200 \pm 40) \times 10^{-9} \qquad (21)$$

That is to say, by modern experimental devices the accuracy for measurement of the anomalous factor of magnetic moment of electron has reached the order of 1×10^{-9}, which is of the same order as a_g given by (18). Therefore, there is real possibility of testing this new quantum effect in general relativity experimentally.

We emphasize that the direct verification of the existence of the existence of the gravitational effect a_g would be most decisive by using the devices which are capable of changing the direction of the magnetic field $\vec{H}$, and measuring the corresponding variation of anomalous factor $a = \frac{g-2}{2}$. This method reduces the dependence on the accurate calculation of a_{QED} and the effects of other interactions, and the experimental value of the fine structure constant α. From (18) it can be shown that if we change the direction of magnetic field from $\vec{H} \parallel \vec{n}$ to $\vec{H} \parallel -\vec{n}$, there should exist a corresponding variation of anomalous factor

$$\delta a = -\frac{2r_0}{R} = -1.39 \times 10^{-9} \tag{22}$$

and from $\vec{H} \parallel \vec{n}$ to $\vec{H} \perp \vec{n}$, the corresponding variation should be

$$\delta a = -\frac{r_0}{R} = -0.70 \times 10^{-9} \tag{23}$$

We expect that this new quantum effect in general relativity will be verified experimentally in the near future.

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