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SPACE–TIME : ARENA OR ILLUSION ?*

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ABSTRACT

This paper develops a framework which allows us to treat the topology and dimension of the space-time continuum as dynamically generated. We present examples of quantum systems which are defined without a notion of space, but which nevertheless undergo a transition to a space-time phase. The dimension of the space is an integer valued order parameter which characterizes distinct phases of a single system. We also show the interactions between the low energy particles of the system are gauge-like. Finally, we discuss the computability of Newton's constant in this class of theories.

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1. INTRODUCTION

Self-consistency problems in many areas of physics have forced theorists to try to achieve a better understanding of phenomena at, or near, the Planck scale. Such considerations, whether they have to do with the structure of the early universe and the *flatness problem*,[1] or with applications of supergravity to solving the *hierarchy problem*,[2] necessitate that we learn to incorporate Einstein gravity into the structure of quantum field theory. Unfortunately, to date, no satisfactory way of doing this has been proposed.

Many talented physicists have failed to solve the problem of renormalizing a conventional field theory of gravity plus matter, and the concommitant problem of "fine tuning" the cosmological constant. For this reason we feel the time is ripe for a reappraisal of basic tenets with an eye towards eliminating unnecessary assumptions. This paper presents the result of one such reappraisal. We find there is a way of formulating quantum field theory which violates our most cherished beliefs and yet appears capable of describing physics as we know it. The notion abandoned as superfluous to a quantum field theory is that of the four dimensional space-time continuum. There seem to be quantum systems which start out with a well-defined notion of time but no notion of space, and dynamically undergo a transition to a space-time phase — a phase in which the physics of the low energy degrees of freedom of the system are best described by an effective Lagrangian written in terms of conventional relativistic fields. In this sense, the notion of the four dimensional space-time continuum as the arena within which the game of field theory is to be played is replaced by the notion of the space-time continuum as an illusion of low-energy dynamics.

On the face of it the idea that one can abandon relativity and the space-

time continuum as superfluous baggage seems ridiculous. For this reason we begin with a discussion of some of the ideas which led us to this approach. We go on to describe the class of Hamiltonians suggested by these considerations, and present the methods we will use to analyze such systems. In particular, we anticipate results of the sections to follow and discuss the notions of *dimension* and *scattering* for a theory formulated without reference to an underlying space-time. The next section introduces simple examples and argues that for these examples low energy physics is correctly described by effective 1+1–dimensional relativistic field theories. This is followed by a section devoted to the problem of obtaining higher dimensional space-times. We present toy Hamiltonians which lead to effective $2+1$ and $3+1$ relativistic space-times, thus demonstrating that this problem is solvable. We also discuss the relationship between the space-time dmensions, internal "flavor" symmetries, and of possible phases of a given Hamiltonian. The last section summarizes results and presents a list of caveats and speculations about those aspects of the problem not yet studied in detail.

2. FUNDAMENTALS

2.1 Mach's Principle and the Cosmological Constant.

A suggestive way of looking at the problem of "fine tuning the cosmological constant" is that it might be related to the fact that the Einstein theory does not successfully embody Mach's principle,[3] which asserts that the local inertial properties of matter should be determined by the other (in particular distant) matter in the universe. This assertion implies that in a really empty universe a "test particle" should not know how to move; as a corollary, an empty space

should have *no geometry at all.* On the other hand, the Einstein equations allow for a *flat space geometry*

$$g_{\mu\nu} = \eta_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

in a universe devoid of matter and radiation.

This failure of Einstein's theory to incorporate Mach's principle is likely to be related to the problem of "fine tuning" the cosmological constant, which arises when one attempts to couple quantum matter fields to general relativity. Quantum fields generally have a non-zero vacuum energy, whose contribution to the energy-momentum tensor is equivalent to an enormous cosmological constant. Unless this contribution is precisely cancelled by an equally enormous bare cosmological constant, *i.e.* "fine tuned away", Einstein's space-time curls up into a tightly curved manifold. In contrast to this situation, a theory which obeys Mach's principle and yields no geometry at all for a truly empty space, *i.e.* devoid of even quantum fluctuations, *might* yield a flat geometry[♯1] when quantum effects are taken into account. In this case the flatness problem would have a natural solution. However, now we have to solve another problem: *How to arrive at such a desirable state of affairs?*

As a starting point we focused upon the fact that our ideas of space-time are derived by carrying out measurements with clocks and rulers, the latter being replaced by radar sets in more modern expositions of the subject.[4] In other words, our notions of space and time are the result of carrying out a large class

♯1 We mean, a flat geometry for the background space-time metric of the low-energy physics.

of scattering experiments. The necessary input to this constructive approach is the existence of some objects that can be scattered and a Hamiltonian which describes the scattering process. Space-time, if it is an appropriate description of low energy physics, then emerges as a construct. This sort of argument naturally leads us to inquire as to what role the coordinates x_μ play in conventional field theories.

2.2 WHAT ARE THE x_μ'S?

Space-time is usually treated as an arena for dynamical theories, even when one takes into account the Einstein approach to gravity as geometry. To better appreciate this fact, let us reexamine the usual formulation of quantum field theory. The Hamiltonian for a self-interacting scalar field in m-dimensions is written as

$$H = \int d^m x \left[\Pi^2(x) + (\nabla\Phi)^2(x) + V(\Phi(x)) \right] \quad . \tag{2.1}$$

What is the significance of the variables 'x_μ' which appear in this formula? At the level of the canonical commutation relations the x_μ's are nothing but labels for independent operators $\Phi(x)$ and $\Pi(x)$ which define the theory. *A priori* there is no reason for x_μ's to have anything to do with space and time. However, the way the gradient terms appear in (2.1) ensures that space-time defined by this Hamiltonian leads us back to the same x_μ's (up to a Lorentz transformation). This rather non-trivial result implies that *a preconceived notion of the topological and differentiable structure is associated with the underlying variables* and that certain aspects of x_μ's must be taken into account before we define the quantum problem. In fact, great care must be taken in renormalizing the theory so as not to do great violence to this relationship.

At first glance Einstein's theory takes care of these troubles. In this case we consider an action of the form

$$L = \int d^m x \sqrt{-g} \left[R(x) + \text{matter terms} \right] , \qquad (2.2)$$

which is invariant with respect to arbitrary coordinate transformations of the variables x_μ. This invariance seems to reduce the significance of the x_μ's to their role as labels for independent quantum degrees of freedom. From a heuristic point of view this approach has another beautiful property to recommend it: if the background metric of space-time arises as a dynamical effect, then there is a natural reason why the space-time reconstructed from scattering experiments must come out intimately related to the variables x_μ. The equations of general relativity, and therefore the Heisenberg equations of the quantum theory (assuming that the theory can be quantized) form a set of hyperbolic partial differential equations. It is known[5] that for this set of differential equations the causal hypersurfaces of the combined system of fields are determined by the causal hypersurfaces of the universally coupled spin-two field, *i.e.* $g_{\mu\nu}$. (By the causal surfaces of the fields, we mean the submanifolds of the space-time across which the fields may be discontinuous, *i.e.* the light cones of the theory.) It follows that if $g_{\mu\nu}$ develops a ground state expectation value, $\tilde{g}_{\mu\nu}$, then solving the quantum problem for the matter fields will reconstruct a theory whose light cones are those specified by $\tilde{g}_{\mu\nu}$. If in particular $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$, the space-time constructed by scattering the matter fields will be the desired Minkowski space. This seems to be a highly attractive scenario, modulo difficulties in defining the quantum theory; unfortunately this heuristic picture isn't completely correct.

Although going over to the Einstein theory of gravity removes the geometrical significance of the coordinates x_μ, the fact that they specify the differential topology of the system remains. In other words, although the notion of distance has become a dynamical concept, we still need to introduce the notions of *'neighborhood'* and *'derivative'* as primitive, *i.e.* undefined, terms. When we assume the x_μ's to be coordinate patches on a certain manifold, we establish a certain topological and differentiable structure for the space-time. Since the primitive notion of *differentiability* must be defined before we specify an action or a Hamiltonian, the assumption that we are dealing with a theory defined on a differentiable manifold carries non-trivial content.

2.3 BASICS.

Motivated by these considerations, we explore the question of whether formulating a theory so that the analogs of the x_μ's are reduced to labels for quantum degrees of freedom[6] will allow us to progress towards the goal of incorporating Mach's principle into field theory. We begin by introducing an indexing set

$$J \;=\; \{\, l : l = 1, \ldots, N \,\} \quad .$$

We then introduce a set of quantum "harmonic oscillator", or "bose field", variables $\Pi(l)$ and $\Phi(l)$, satisfying canonical commutation relations

$$\Big[\, \Pi(l)\,,\, \Phi(m) \Big] \;=\; -\,i\,\delta_{lm} \quad ,$$

and multicomponent "Fermion field" variables $\Psi_\alpha(l)$ and $\Psi^\dagger_\beta(l)$, satisfying canon-

ical anti-commutation relations

$$\left\{ \Psi_\alpha^\dagger(l)\,,\, \Psi_\beta(m) \right\} \;=\; \delta_{\alpha\beta}\,\delta_{lm} \quad .$$

Intuitively, these quantum degrees of freedom are to be thought of as playing the role of *site variables* in some sort of lattice field theory.

In addition to these *site fields* we also introduce *link fields*, which for the sake of definiteness we choose to be harmonic oscillator variables $P_{\langle lm\rangle}^{\alpha\beta}$ and $X_{\langle lm\rangle}^{\alpha\beta}$. The notation '$\langle lm\rangle$' is introduced to stand for an arbitrary pair of integers $l, m \in J$, ordered such that $l < m$. These *link fields* are assumed to satisfy the canonical commutation relations

$$\left[P_{\langle lm\rangle}^{\alpha\beta}\,,\, X_{\langle l'm'\rangle}^{\alpha'\beta'} \right] \;=\; -\,i\,\delta_{ll'}\,\delta_{mm'}\,\delta_{\alpha\alpha'}\,\delta_{\beta\beta'} \quad .$$

The role of the *link fields* is to provide dynamical variables which allow us to write the analogue of a *kinetic term* for a Hamiltonian without requiring that the notion of differentiation or even neighborhood to be defined.

Starting with these objects we consider a Hamiltonian of the form

$$\begin{aligned} H \;=\; \sum_{\substack{l\in J \\ \langle lm\rangle}} \Big[& \tfrac{1}{2}\,\Pi^2(l) \;+\; X_{lm}^2\,\Phi(l)\,\Phi(m) \;+\; V(\Phi(l)) \\ & -\,i\,X_{lm}\cdot\Big\{\,\Psi^\dagger(l)\,\Psi(m) \;-\; \Psi^\dagger(m)\,\Psi(l)\,\Big\} \\ & +\,\tfrac{1}{2}\,P_{\langle lm\rangle}^2 \;+\; V\left(\cdots X_{\langle lm\rangle}\cdots\right)\Big] \quad . \end{aligned} \tag{2.3}$$

Note, to simplify the discussion we will temporarily treat the fermion fields as one component objects and drop the Dirac indices α and β. We also introduce a

matrix notation for the variables $X_{(lm)}$:

$$X_{lm} \;=\; -\,X_{ml} \;=\; \begin{cases} X_{(lm)} & \text{for} \quad l < m \quad , \\ X_{(ml)} & \text{for} \quad m < l \quad ; \end{cases} \tag{2.4}$$

and define X^2_{lm} to be

$$X^2_{lm} \;=\; \sum_{k \in J} X_{lk}\, X_{km} \quad . \tag{2.5}$$

As it stands, (2.3) is the Hamiltonian of a complicated quantum mechanical system all of whose couplings, for want of a better choice, are assumed to be of order unity. The idea we wish to pursue is that, for some dynamical reason, a subset of the variables $X_{(lm)}$ acquire non-vanishing ground state (*i.e. vacuum*) expectation values $\overline{X}_{(lm)}$. When this occurs, the terms quadratic in the fields $\Phi(l)$ and $\Psi(l)$ yield a solvable zeroth order Hamiltonian for the "matter fields" in the system and, to the extent that fluctuations in the fields $X_{(lm)}$ can be ignored, map the matter problem into some sort of lattice theory. The terms involving variables $X_{(lm)}$ which have non-vanishing expectation values are identifiable as the derivative term in this effective lattice Hamiltonian. Two main questions are: *"What drives the dynamical generation of expectation values of the fields $X_{(lm)}$?"*, and *"What sort of lattice forms?"*. While the meaning of the first question is self-evident, the second merits a brief discussion.

If the ground-state of a Hamiltonian of the form defined in (2.3) corresponds to the vanishing vacuum expectation value of the fields $\overline{X}_{(lm)}$, no zeroth order geometry (or effective lattice theory) — reasonable or unreasonable — forms. In other words there is no sense in which the matter fields scatter in a background determined by the vacuum expectation value of the link fields; everybody interacts on an equal footing with everyone else. On the other hand, if a pattern

of vacuum expectation values does develop, then there is a zeroth order matter field theory whose arena of definition is specified by those link variables with non-vanishing expectation values. Our problem is to determine if there is any reasonably simple way of choosing terms for (2.3) so that the resulting theory becomes equivalent to a Hamiltonian lattice theory with dynamics closely approximating that of a relativistic continuum theory. The aim of this paper is to show that for a large class of Hamiltonians of the type specified in (2.3), the second possibility is realized. In addition, we will show that simply by increasing the number of degrees of freedom the resulting lattice theory can be made to approximate a relativistic continuum theory arbitrarily well.

2.4 Bosons Versus Fermions as Fundamental Building Blocks

Before discussing specific calculations, let us see if we can intuitively understand those aspects of the theory which control the dynamical generation of ground-state expectation values for link-variables. Obviously the potential terms, $V(\cdots X_{(lm)} \cdots)$, can determine whether the theory develops non-zero expectation values of the link variables; however the vacuum energy of the matter fields can play a similar role. The possibility that matter fields cause spontaneous generation of non-vanishing link-field expectation values by themselves, *i.e.* irrespective of the details of the link Hamiltonian, is very appealing. In a moment we will argue this possibility is realized if the dominant matter fields in the system are fermionic. Unfortunately we will find that fermionic fields alone cannot stabilize the resulting space-time structure against quantum fluctuations of the link variables. This will lead us to consider *generalized plaquette terms* in the link-field potential (we will define them later), which can produce stable space-

times. These generalized plaquette terms will play a crucial role when we address the question of obtaining 3+1–dimensional space-times, and when we discuss the way in which these theories generate effective low energy gauge theories.

Despite the fact that the matter fields cannot, by themselves, play the decisive role in determining the structure of space-time, it is pedagogically advantageous for us to begin our discussion of the general theory with an analysis of the role they play. This will allow us to introduce, in the simplest context, most of the important but unfamiliar concepts we will need in the discussions to follow. Let us begin by focusing on the effective quadratic problem defined for the fields $\Phi(l)$ in (2.3). Since we are dealing with Bose fields, their contribution to the vacuum energy is

$$\mathcal{E}_{boson} \;=\; \tfrac{1}{2} \sum \sqrt{\epsilon_\lambda^2} \quad , \tag{2.6}$$

where by ϵ_λ we mean the eigenvalues of the matrix $\overline{X}_{lm}^2$. (This is just the familiar sum over the zero-point energies of a set of independent harmonic oscillators.) The contribution coming from (2.6) is a positive function of the vacuum expectation values $\overline{X}_{(lm)}$, and is minimized when all ϵ_λ's are equal to zero. This occurs if and only if the ground-state expectation values of the link-variables vanish. Hence, the Bose site fields play no interesting role in our problem and we will ignore them in all discussions to follow.

The situation is quite different for fermions because their contribution to the vacuum energy is negative. Computing the fermionic contribution to the vacuum energy reduces to diagonalizing the quadratic form

$$-\,i\,\overline{X}_{lm} \cdot \Big[\, \Psi^\dagger(l)\,\Psi(m) \;-\; \Psi^\dagger(m)\,\Psi(l) \Big] \quad . \tag{2.7}$$

By definition, the matrix of expectation values $\overline{X}_{lm}$ is antisymmetric; hence, it

has an equal number of positive and negative eigenvalues (it may have some zero eigenvalues as well). Since the ground state of the theory is obtained by filling the negative energy sea, it follows that the fermionic contribution to the vacuum energy is the sum over the negative eigenvalues of $\overline{X}_{lm}$, or

$$\mathcal{E}_{\text{fermion}} = -\tfrac{1}{2} \sum \sqrt{\epsilon_\lambda^2} \quad . \tag{2.8}$$

From (2.8) we see that even if the potential terms in (2.3) do not lead to dynamical generation of non-vanishing vacuum values for link-variables, the fermions perforce generate such an effect. What we have to study is whether the fermionic contribution alone is sufficient to stabilize the effect against quantum fluctuations of the link variables.

2.5 Link Mean Field Theory.

Assuming that our generic Hamiltonian only includes link and fermion fields, we turn our attention to the tools we will use to analyze the theory. Determining the properties of the ground-state of a Hamiltonian like (2.3), even without the Bose-fields, is an extraordinarily difficult non-perturbative problem. Our approach is to adopt a variational technique which has proven to be quite effective when applied to lattice systems, namely, Hamiltonian mean-field theory.[7] While this method, a variant of the familiar Rayleigh-Ritz variational technique, is not particularly successful when applied to the calculation of critical exponents or the order of a phase transition, it is usually quite successful in determining what phases exist. Since this is the only way in which we will apply this method, we feel comfortable using it in its simplest incarnation.

The approach, as we will implement it, is to choose as a trial wave-function for the ground state of (2.3) (minus the Bose fields) a product state of the form

$$|\Psi_{\rm var}\rangle \;=\; \prod_{\langle lm\rangle} \exp\Big(-\tfrac{\gamma}{2}\,\big(X_{\langle lm\rangle} - C_{\langle lm\rangle}\big)^2\Big)\cdot|\Psi_{\rm fermion}\rangle \quad . \tag{2.9}$$

The variational parameters appearing in the wave-function are: the variables $C_{\langle lm\rangle}$, which determine the expectation values of the operators $X_{\langle lm\rangle}$; the variable γ, which determines the width of the gaussians, and therefore the expectation value of operators such as $P^2_{\langle lm\rangle}$ and $(X^2_{\langle lm\rangle} - C^2_{\langle lm\rangle})$; and the unspecified state $|\Psi_{\rm fermion}\rangle$. To determine whether or not the link variables have non-vanishing expectation values in the ground-state of the system we simply compute the ratio

$$\mathcal{E}_{\rm effective}\left(C_{\langle lm\rangle}\,,\,\gamma\,,\,|\Psi_{\rm fermion}\rangle\right) \;=\; \frac{\langle\Psi_{\rm var}|\;H\;|\Psi_{\rm var}\rangle}{\langle\Psi_{\rm var}|\Psi_{\rm var}\rangle} \quad , \tag{2.10}$$

and minimize it as a function of the parameters $C_{\langle lm\rangle}$, γ and $|\Psi_{\rm fermion}\rangle$.

2.6 Scattering Problem.

What do we mean by discovering a scattering problem hidden within the theory specified by Hamiltonians of the form (2.3)? Our approach is quite straightforward and not necessarily the most general one; nevertheless, it will suffice for our purposes. We will show that for the cases of interest there exists a function $F(l,\vec{x})$ of the Euclidian coordinates $\vec{x}$ and points $l \in J$, which allows us to define the effective fields $\Psi(\vec{x})$ as

$$\Psi(\vec{x}) \;=\; \sum_{l\in J} F(\,\vec{x},l\,)\,\Psi(l) \quad , \tag{2.11}$$

such that they have canonical equal time anticommutation relations in the limit of infinite number of quantum degrees of freedom. Furthermore, we will show

that the quadratic part of the effective fermionic Hamiltonian can be written in terms of $\Psi(\vec{x})$ as

$$H_{\text{fermion}} \;=\; \int dx\; \Psi^{\dagger}(\vec{x})\; \vec{\alpha}\cdot\vec{\partial}_{x}\,\Psi\,(\vec{x}) \quad . \tag{2.12}$$

Given these "continuum" fermionic fields, we define the time dependent operators

$$\Psi(\,\vec{x},t\,) \;=\; e^{-itH_{\text{fermion}}}\;\Psi(\vec{x})\;e^{itH_{\text{fermion}}} \tag{2.13}$$

and use these operators to define wave packets and scattering states, *etc.* The fact that the Heisenberg equations of motion for these fields comes out to be the usual relativistic equations guaranties that the space-time reconstructed using them will be of the desired type and that its dimension will be $(d+1)$, where d is the dimension of the variables $\vec{x}$, provided the residual interactions among the effective low energy degrees of freedom look approximately local in the same x-coordinates. It remains to be shown that this is the case.

Despite the fact that we have no definitive answers to the question of the residual interactions, it is worth noting that there are in principle two ways in which they can be approximately local. In any lattice approximation to a continuum theory there appear non-local, non-linear interactions of the form

$$\kappa\cdot\int\prod_{i}dx_{i}\;\Psi^{\dagger}(x_{1})\,\Psi(x_{2})\;\ldots\;\Psi(x_{n})\cdot\Delta(x_{1},x_{2},\ldots,x_{n}) \quad , \tag{2.14}$$

where the support of the function $\Delta(x_1,x_2,\ldots,x_n)$ extends over a set

$$S_{\Delta} \;=\; \{\,(x_{1},x_{2},\ldots,x_{n})\colon\; |x_{i}-x_{j}| \le \delta \;\text{ for }\; i,j=1,\ldots,n\,\}$$

One possibility for interactions (2.14) to look approximately local is to include only terms with small values of δ (as measured in the physical units), so that

all violations of locality occur at separations too small to have been resolved to date. The second possibility allows for *large* values of δ, but only for terms which have very small magnitude κ. These terms are potentially more dangerous, since if the support of $\Delta(x_1, \ldots, x_n)$ extends over the whole set, then the magnitude of κ must be *very small.*

3. TWO-DIMENSIONAL SPACE-TIMES

3.1 THE SIMPLEST SPACE.

To get a feeling for the variational procedure and the way in which we will discuss the results of such a calculation, let us begin with the simple Hamiltonian.

$$H = \sum_{\langle lm \rangle} \left[\tfrac{1}{2} P^2_{\langle lm \rangle} + \tfrac{\mu^2}{2} X^2_{\langle lm \rangle} - i X_{\langle lm \rangle} \cdot \left\{ \Psi^\dagger(l) \Psi(m) - \Psi^\dagger(m) \Psi(l) \right\} \right] . \tag{3.1}$$

The virtue of this Hamiltonian is that we can completely analyze our problem using straightforward analytic techniques; the drawback is that the results are uninteresting. Using the general trial wavefunction specified in (2.9), we obtain

$$\langle \Psi_{var} | H | \Psi_{var} \rangle = \sum_{\langle lm \rangle} \left[\frac{\gamma}{4} + \frac{\mu^2}{2} \cdot \left(C^2_{\langle lm \rangle} + \frac{1}{2\gamma} \right) - i C_{\langle lm \rangle} \cdot \langle \Psi_f | \Psi^\dagger(l) \Psi(m) - \Psi^\dagger(m) \Psi(l) | \Psi_f \rangle \right] . \tag{3.2}$$

The variation over γ is easily carried out, yielding $\gamma_{min} = \mu$, which leaves us with the problem of minimizing over the class of all purely fermionic functions, $|\Psi_f\rangle$. Since the fermionic term in (3.2) is purely quadratic in the fields, the best $|\Psi_f\rangle$ is obtained in the usual way, namely by diagonalizing the matrix $C_{\langle lm \rangle}$,

expanding the fields $\Psi(l)$ in terms of the eigenfunctions $u_\lambda(l)$ and filling the negative energy sea. To be specific, if the normalized functions $u_\lambda(l)$ satisfy

$$\sum_{m\in J} C_{lm}\, u_\lambda(m) \;=\; i\,\epsilon_\lambda\, u_\lambda(l) \quad ,$$

where $i\epsilon_\lambda$ are the eigenvalues of the matrix C_{lm}, we introduce orthogonal linear combinations of the operators $\Psi(l)$

$$\Psi(\lambda) \;=\; \sum_{l\in J} u^*_\lambda(l)\,\Psi(l) \quad .$$

In terms of these operators the fermionic Hamiltonian can be written as

$$H \;=\; \sum_\lambda \epsilon_\lambda\, \Psi^\dagger(\lambda)\,\Psi(\lambda) \quad ,$$

where the variables $\Psi^\dagger(\lambda)$ and $\Psi(\lambda')$ satisfy the same anti-commutation relations as the original fields. If we define $|0\rangle$ to be the state annihilated by all of the operators $\Psi(\lambda)$, then the ground state of the Hamiltonian is obtained from $|0\rangle$ by applying to it all of the operators $\Psi^\dagger(\lambda)$ for which the eigenvalue ϵ_λ is negative. The ground state energy is

$$\mathcal{E}_{vacuum} \;=\; -\tfrac{1}{2}\textstyle\sum_\lambda \sqrt{\epsilon^2_\lambda} \quad . \tag{3.3}$$

Returning to the discussion of (3.2), we observe that the matrix C_{lm} is antisymmetric and can, by means of a real similarity transformation, be brought

into the form

$$\begin{pmatrix} 0 & \epsilon_1 & 0 & 0 & \cdots \\ -\epsilon_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \epsilon_2 & \cdots \\ 0 & 0 & -\epsilon_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad . \tag{3.4}$$

It then follows that the variational energy can be written as

$$\langle \Psi_{\rm var}|\ H\ |\Psi_{\rm var}\rangle \ = \ \frac{N(N-1)}{8}\cdot\left(\gamma + \frac{\mu^2}{\gamma}\right) + \frac{\mu^2}{2}\sum_\lambda \epsilon_\lambda^2 - \frac{1}{2}\sum_\lambda \sqrt{\epsilon_\lambda^2} \quad . \tag{3.5}$$

From (3.5) it is clear that not only does the variable γ come out independent of N, but that the minimum is achieved when all of the eigenvalues have the common N-independent absolute value $|\epsilon|^{-1} = \sqrt{2}\mu^2$. This analysis shows that, as advertised, the fermionic contribution to the ground state energy forces at least some link variables to develop non-zero vacuum expectation values. The question *"to what space-time does the resulting theory correspond?"* can be partially answered by saying that to the degree this theory has a space-time, its dimension is $1 + \infty$. Let us explain this rather cryptic remark.

Following the discussion in the preceding section, we extract the space-time dimension from knowledge of the energy spectrum by inverting the usual Fourier transform procedure. This works because it is in the momentum basis that the fermionic Hamiltonian (2.12) is diagonal. For a massless theory $\epsilon(k) = \sqrt{k^2}$, so a d-dimensional Hamiltonian has a density of eigenstates which behaves like $\rho(\epsilon) \propto \epsilon^{d-1}$. Hence, if we order the eigenvalues of the fermionic Hamiltonian C_{lm} in ascending value and plot them versus the integers $j = -n, \ldots, n$,[♯2] we can read off the dimension from the detailed shape of the plot: in a d-dimensional case

♯2 We have chosen an odd $N = 2n + 1$.

$\epsilon(j)$ is proportional to $\sqrt[d]{j}$. It follows that as $d \to \infty$ the energy $\epsilon(j)$ tends to the same value for all positive j; hence this simple system is infinite-dimensional. There are two major lessons to be extracted from this exercise: first, *fermions behave as expected and cause the vacuum expectation value of the link variables to become non-zero*; and second, *fermions by themselves tend to push the system to high dimensions*. The real problem is to understand how to get a reasonable system of low space-time dimension.

3.2 Making a One-Dimensional Space.

The results to be presented in the rest of this paper were obtained by means of a mixture of numerical and analytic techniques. Although the results of analytic methods alone would suffice at this stage of the discussion, we feel that comparing them with the results of computer calculations may lead to some useful insights. Moreover, some of the speculations which are presented in the last section of this paper are based solely on the computer results obtained for small N. For this reason we have chosen to present the material to follow in semi-historical fashion.

Inspection of the simple Hamiltonian (3.1) reveals why the eigenvalues tend to come out equal in absolute magnitude: the vacuum energy is a sum of invariants of the matrix C_{lm}. The term multiplying the variable μ^2 is the trace of the matrix C_{lm}^2, and the fermionic energy is proportional to the logarithm of the determinant of C_{lm}^2. This leads to the conjecture that the situation will change if one destroys this property of the Hamiltonian, and a more connected pattern of vacuum expectation values will develop. In order to test this hypothesis we

study a slightly more complicated Hamiltonian, namely

$$H = \sum_{\langle lm \rangle} \left[\tfrac{1}{2} P^2_{\langle lm \rangle} + \eta X^{2r}_{\langle lm \rangle} - i X_{\langle lm \rangle} \cdot \left\{ \Psi^\dagger(l) \Psi(m) - \Psi^\dagger(m) \Psi(l) \right\} \right] \quad . \tag{3.6}$$

We are interested in this problem for $r > 1$, since the case $r = 1$ reduces to (3.6). Our early attempts to understand this problem began by assuming that the parameter γ in our trial wavefunction was large and constant. We then studied the vacuum energy of the system as as function of the parameters $C_{\langle lm \rangle}$ alone. In this case finding the minimum of the function

$$\eta C^{2r}_{\langle lm \rangle} - \tfrac{1}{2} \textstyle\sum_\lambda \sqrt{\epsilon^2_\lambda} \tag{3.7}$$

is no longer a procedure which can be carried out analytically: unlike the preceding case, the term

$$\eta \sum_{\langle lm \rangle} C^{2r}_{\langle lm \rangle}$$

cannot be expressed as a function of the eigenvalues alone. For this reason we turned to a computer analysis of the problem for values of $11 \leq N \leq 121$.

This analysis led to the surprising result that the configurations which minimized (3.7) have non-vanishing vacuum expectation values *of all link variables* $X_{\langle lm \rangle}$; moreover, absolute magnitudes of the expectation values tend to be equal: $\langle vac | \, X_{\langle lm \rangle} \, | vac \rangle = \pm \overline{X}$. On the one hand, this means that our original conjecture was correct and breaking the invariance of the Hamiltonian forced the formation of more interesting highly connected configurations; on the other hand, these configurations seem to have nothing in common with any easily recognizable lattice theory. On the contrary, these systems appear to describe a set

of fermions living upon what a mathematician would call an *"N–simplex" i.e.* an object for which every point l, is a nearest neighbor of every other point l'. Naively an object of this sort appears to have dimension $(N-1)$.

At this point, it is important to recall that the dimension of the corresponding space-time picture can be obtained only from an investigation of the spectrum of eigenvalues of the matrix $\langle \text{vac}|\ X_{(lm)}\ |\text{vac}\rangle$. In order to better understand what is happening we must study the spectrum of eigenvalues for fermionic Hamiltonians of the form

$$H_{simplex} \;=\; i \sum_{\langle lm \rangle} C_{\langle lm \rangle} \cdot \Big\{ \Psi^{\dagger}(l)\, \Psi(m) \;-\; \Psi^{\dagger}(m)\, \Psi(l) \Big\} \quad , \qquad (3.8)$$

where the variables $C_{\langle lm \rangle}$ are randomly assigned the values ± 1. The result of carrying out this eigenvalue analysis on a large sample of matrices is the remarkable fact that over most of its range, the spectrum of eigenvalues is linear when plotted versus the integers $i = -n, \ldots, n$. This means that, contrary to our intuition, theories defined by Hamiltonians of the type specified in (3.8) are equivalent to 1+1–dimensional field theories.

Understanding the Pattern of Eigenvalues. It is obviously impossible to diagonalize all antisymmetric matrices of type (3.8) for arbitrary large N, so we cannot be absolutely sure that we are dealing with 1+1–dimensional object solely on the basis of computer calculations for $N \leq 121$. However, Dyson's results[8] concerning the behavior of eigenvalues of random matrices enables us to understand our computer results and to explore the *pattern* of eigenvalues for almost all matrices $C_{\langle lm \rangle}$.

Our problem is to find the distribution of eigenvalues for a real antisymmetric matrix, whose [independent] matrix elements are essentially random. Let us

follow F. Dyson in our definitions: a random antisymmetric matrix is a random element of an ensemble A_G of all real antisymmetric $(2n+1)\times(2n+1)$ matrices with a gaussian probability distribution

$$P(A_{(1,2)}\cdots A_{(2n,2n+1)}) = C\exp\Big[-\sum_{\langle l,m\rangle}\Big(A^2_{\langle l,m\rangle}/2a^2\Big)\Big]\,dA_{(1,2)}\cdots dA_{(2n,2n+1)} \quad ,$$

where a and C are constants. The major result which we will need for our discussion is the following

Theorem I: *A random element of A_G has eigenvalues $0, \pm iE_1, \ldots, \pm iE_n$,*[‡3] *whose probability distribution is given by*

$$P(E_1,\ldots,E_n) = C\left[\prod_{i=1}^{n}E_i^2\right]\cdot\left[\prod_{i<j}\left(E_i^2-E_j^2\right)^2\right] \times\exp\left[-\sum_{i=1}^{n}\left(E_i^2/4a^2\right)\right]\cdot dE_1\cdots dE_n \quad .$$

Given the probability distribution P, we can compute the function of interest, namely the *eigenvalue density* $\rho(E)$ which tells us the expected number of eigenvalues in the interval $[E, E+dE]$. This eigenvalue density (*i.e.* density of states) can be found by Dyson's method and in our case of an imaginary antisymmetric $(2n+1)\times(2n+1)$ matrix, all of whose entries are ± 1, it comes out to be

$$\rho(E) = \frac{2n+1}{4\pi\, n}\cdot\sqrt{8n-E^2} \quad \text{for} \quad -\sqrt{8n}\le E\le\sqrt{8n} \quad ,$$

and vanishes for $|E|\ge\sqrt{8n}$. (See the appendix for a simple derivation of $\rho(E)$ due to Brezin, Itzykson and Zuber.) The spacing between eigenvalues is given

‡3 *This is the case for any real antisymmetric $(2n+1)\times(2n+1)$ matrix.*

by

$$\Delta(E) \equiv \frac{1}{\rho(E)} = \frac{4\pi n}{2n+1} \cdot \frac{1}{\sqrt{8n - E^2}} \quad ,$$

which in the large n limit becomes

$$\Delta(E) = \frac{\pi}{\sqrt{2n}} \cdot \left(1 - \frac{E^2}{8n}\right)^{-1/2} \quad .$$

We see that near $E = 0$ eigenvalues are spaced fairly uniformly, but near the endpoints of the spectrum the spacing becomes thinner. These results are in complete agreement with the numerical results.

Why Is This a One-Dimensional Space ? The Dyson calculation tells us that our computer result is not an artifact of small n, but a behavior one expects to hold for $n \to \infty$. For energies small on the scale of $\sqrt{n}$ eigenvalues are uniformly spaced; therefore the index $j = -n, \ldots, n$, which labels them, can be interpreted as a momentum variable for an effective one dimensional theory. In order to make this interpretation more physical, let us multiply the Hamiltonian (3.8) with a scale factor $\Lambda = a_0^{-1}$ having the dimension of a mass. Then we can write the spectrum of eigenvalues which are relatively close to zero as

$$E(k) = k + O\left(\frac{|k|^3}{n\Lambda^2}\right) \quad , \tag{3.9}$$

where

$$k = \frac{2\pi}{L} \cdot j \quad , \qquad j = -n, \ldots, n \quad \text{and} \quad L = \sqrt{8n}\, a_0 \quad . \tag{3.10}$$

Momenta defined by (3.10) are characteristic for 1+1–dimensional lattice theories defined in a spatial volume L. Equation (3.10) allows for maximal momentum

$k_{max} = \frac{2\pi n}{L} = \frac{\pi}{2}\sqrt{2n}\cdot\Lambda$, which corresponds to the lattice spacing $a = \pi/k_{max} = a_0\sqrt{2/n}$. As we take the limit $n \to \infty$, the ratio of L to a grows like n, and the equivalent lattice theory passes over to a continuum theory defined in infinite volume. This limit is only unusual in that the lattice theory simultaneously gets both finer and larger, as measured in units of the scale parameter Λ.

The explicit unitary mapping of the N-simplex theory into an equivalent lattice theory whose kinetic term is given by a *generalized SLAC-derivative* can now be explicitly given: we define transformed fields $\Psi(x)$ as

$$\Psi(x) \;=\; \Lambda^{1/2} \sum_{\substack{l\in J \\ -n\le j\le n}} e^{izk(j)}\, u_j^*(l)\, \Psi(l) \quad , \tag{3.11}$$

where the function $u_j(l)$ is the eigenfunction of the matrix $C_{(lm)}$ having eigenvalue E_j. The variable x has the dimensions of length; for x taking discrete values $x_p = p\cdot a$, (3.11) is a unitary equivalence. Actually, it is possible to think of the variable x as a continuous variable, and think of the fields $\Psi(x)$ as an overcomplete set of fields satisfying anticommutation relations which are non-local over a distance scale on the order of $a \sim a_0/\sqrt{n}$.[9] This amounts to defining a set of fields which interpolate between the variables $\Psi(x_p)$ and which, in the sense of weak limits, pass over to the continuum fields in the limit $\sqrt{n} \to \infty$.

The momentum-space form of the fermionic Hamiltonian is given by

$$H \;=\; \sum_j k_j\, \Psi^\dagger(j)\, \Psi(j) \quad ,$$

which in x–space becomes

$$H \;=\; \int_{-L/2}^{L/2} dx\; \Psi^\dagger(x)\, i\partial_x\, \Psi(x) \quad . \tag{3.12}$$

Thus we have established that that the dynamical system (3.6) generates a theory whose low energy degrees of freedom organize themselves into an apparent 1+1–dimensional *relativistic* field theory. We see that this approximation becomes arbitrarily good as the number of degrees of freedom tends towards infinity.

Understanding The Minimum. Having obtained an understanding of fermions on an N-simplex, we are now in a position to understand why this highly connected pattern of expectation values is favored for the case $r > 1$. Let us compare values of the effective potential (3.7) obtained with different patterns of $C_{\langle lm \rangle}$. In the *"infinite-dimensional"* case of (3.4), (3.7) becomes

$$\mathcal{E}_\infty = \sum_{i=1}^{n} \left[\eta \mid \epsilon_i \mid^{2r} - \tfrac{1}{2} \mid \epsilon_i \mid \right] \quad . \tag{3.13}$$

Minimizing (3.13) with respect to variables ϵ_i, we find that all $|\epsilon_i|$ are equal and the vacuum energy $\mathcal{E}_\infty$ is negative and proportional to n. On the other hand, in the *"one-dimensional"* case of $C_{\langle lm \rangle} = \pm \overline{C}$, (3.7) reduces to

$$\begin{aligned} \mathcal{E}_1 &= n\,(2n+1)\;\eta \cdot \overline{C}^{2r} \;-\; \tfrac{4}{3\pi}\sqrt{2n}\;(2n+1) \cdot \overline{C} \\ &\approx 2n^2\,\eta \cdot \overline{C}^{2r} \;-\; \tfrac{8}{3\pi}\;n^{3/2} \cdot \overline{C} \quad . \end{aligned} \tag{3.14}$$

The factor of $n^{3/2}$ appearing in (3.14) comes from the fact that the density of states factor is proportional to $\sqrt{n}$ and that the fermionic ground-state energy is, in the limit of large n, well approximated by

$$\mathcal{E}_{\text{fermi}} \;\propto\; -\frac{\sqrt{2n}}{\pi} \int_0^{\sqrt{8n}} dk \cdot k \cdot \sqrt{1 - k^2/8n} \quad .$$

Minimizing $\mathcal{E}_1$ with respect to $\overline{C}$ yields

$$\overline{C}_{min} \propto n^{-1/(4r-2)} \quad \text{and} \quad \mathcal{E} \propto -n^{(3/2-1/(4r-2))} \quad . \tag{3.15}$$

From this we see that for $r > 1$ the energy of the highly connected configuration grows like n to a power between unity and 3/2; whereas for the simpler configuration it only behave like n. This explains why the numerical calculations select the highly connected configurations over the less connected ones. It is also not important that *all* $C_{\langle lm \rangle}$ do not vanish. Numerical computations show that if we allow $C_{\langle lm \rangle}$ to take the values $\pm\overline{C}$ and 0 with varying probabilities, the one-dimensional nature of the spectrum remains intact when as many as forty percent of the variables $C_{\langle lm \rangle}$ vanish.

What About Fluctuations in the Link Variables ? Up to this point we have been interested in establishing the relationship between numerical studies and the analysis based upon Dyson's theorems for random matrices. Having seen that these techniques reproduce all of the essential features of the computer results obtained for small values of n, we now return to the question of how quantum fluctuations in the link-field variables affect our conclusions. This means restoring all of the γ–dependent terms to our expression for the expectation value of the Hamiltonian. For the case $r = 2$, *i.e.* the case of a Hamiltonian having a quartic potential in the fields $X_{\langle lm \rangle}$, this expectation value has the form

$$\mathcal{E} = n(2n+1) \cdot \left[\frac{\gamma}{4} + \frac{6\eta}{4\gamma^2} + \frac{6\eta\,\overline{C}^2}{2\gamma} + \eta\,\overline{C}^4 \right] - \alpha\,\overline{C}\,n^{3/2} \quad , \tag{3.16}$$

where α is an undetermined constant of proportionality of the order of unity. Inspection of (3.16) reveals that in the limit of large n, the variable γ tends to

a constant value and $\overline{C}$ tends to zero. This means two things: first, that this case reduces to an effective "infinite dimensional" theory; and second, that the fluctuations in the field dominate the vacuum expectation values, since γ tends to an n-independent constant but $\overline{C}$ vanishes in the limit of large n. For this reason we will refer to this case as being *fluctuation dominated.*

We can rescue all of our previous results if we extend our basic Hamiltonian to include what we will call generalized *plaquette* terms, *i.e.* if we add to (3.6) a term of the form

$$V^2_{triangle} = - \sum_{k<l<m} X^2_{\langle kl \rangle} X^2_{\langle lm \rangle} X^2_{\langle km \rangle} \quad , \tag{3.17}$$

which is independent of the signs of the individual factors X_{lm}; or a term of the form

$$V^1_{triangle} = - \sum_{k<l<m} X_{kl} X_{lm} X_{mk} \quad , \tag{3.18}$$

which, as we will see in the next section, forces the signs (or phases) of the X_{lm}'s to satisfy exactly the constraints which are required for the effective theory to be equivalent to a lattice Z_2–gauge theory. Terms of the form (3.17) or (3.18) tend to favor the formation of *triangles* of non-vanishing expectation values. The conditions $i < j < k$ tell us that there are three different link-variables appearing in each such term. The important feature of this class of terms, which avoids the difficulty of the *fluctuation dominated case* is that there are $\frac{1}{6}n^3$ such terms. If, for the sake of argument we choose to add (3.18) to (3.6), then (3.16) becomes

$$\begin{aligned} \mathcal{E} = n(2n+1) \cdot \left[\frac{\gamma}{4} + \frac{6\eta}{4\gamma^2} + \frac{6\eta\,\overline{C}^2}{2\gamma} + \eta\,\overline{C}^4 \right] \\ - \frac{n(n-1)(n-2)}{6} \cdot \left[\overline{C}^3 + \frac{3\overline{C}}{2\gamma} \right] - \cdots \quad , \end{aligned} \tag{3.19}$$

where the dots stand for the fermion terms, and we have implicitly assumed that the signs of the variables X_{lm} have been chosen so all possible triangle terms contribute a negative term to the expectation value of V^1. It is easy to convince oneself that the minimum of (3.19) occurs when $\overline{C} \sim n$ and $\gamma \sim n$. Thus, the addition of a simple term to (3.6) of the form (3.17) or (3.18) both solves the problem of forcing the theory to develop a pattern of connected non-vanishing expectation values and controls the fluctuations of the link fields, so that the mean-field approximation is valid.

Note that at this point the role of the fermions has changed. From the fluctuation analysis we see that the fermions, by themselves, are not capable of driving the theory to the desired form; nevertheless they will play an important role in determining the structure of the full theory. This is because there are many mean-link configurations which are degenerate at the level of the effective potentials which we have been considering. The fermion term in the energy provides a mechanism for lifting much of this degeneracy by effectively providing gradient terms which distinguish between the various cases. We will see an example of this sort of thing when we discuss the question of computing Newton's constant of gravitation in the higher dimensional theories. Although the role played by the fermions is different than we thought at first, they still play an important role in selecting among possible configurations and, of course as probes they define what we mean by our effective space-time picture. Despite the fact that at present we do not have analytic tools for finding the true minimum selected by the fermion terms, we wish to emphasize at this time that all of these quasi-degenerate configurations define 1+1–dimensional theories.

3.3 The Question of Locality.

The preceding discussion has established the following results: first, we have seen from the example of the theory of fermions defined on an N-simplex that it is easy for a system of this type to have a very different geometry than one intuitively expects; second, we have seen that a very general class of Hamiltonians can define a system which, at least at the level of our mean-link theory analysis, dynamically produces vacuum expectation values which lead to a 1+1–dimensional relativistic field theory. This interpretation is incomplete in that it uses only the analysis of the resulting free fermion theory; it remains to be shown that this is compatible with the scattering processes which occur due to quantum fluctuations of the link variables. The key question is whether these processes will look local (always up to some cutoff scale) with respect to the effective variables x. Unfortunately, we do not know the answer to this question for the general Hamiltonians we have been discussing; however, there are some things which we can say about this question.

For the case of the N-simplex theory it is relatively simple to invent *e.g.* four-fermion terms which will be guaranteed to look local in the variables x. One need only take these interaction terms to be of the form

$$\sum_{l_1,\ldots,l_4 \in J} F(l_1, \ldots, l_4) \cdot \Psi^\dagger(l_1)\, \Psi(l_2)\, \Psi^\dagger(l_3)\, \Psi(l_4) \quad ,$$

where the function $F(l_1, l_2, l_3, l_4)$ is defined to be

$$F(l_1, l_2, l_3, l_4) \;=\; \sum_{k_1,\ldots,k_4} \delta(k_1 + k_2 + k_3 + k_4) \cdot u_{k_1}(l_1)\, u^*_{k_2}(l_2)\, u_{k_3}(l_3)\, u^*_{k_4}(l_4) \quad .$$

It is easy to verify that changing over to the variables $\Psi(x)$ yields an interaction

of the form

$$\int dx \left(\Psi^{\dagger}(x)\, \Psi(x) \right)^2 \quad .$$

Numerical evaluation of the function $F(l_1, l_2, l_3, l_4)$ for small values of N shows that this function is as non-local on the simplex as the "derivative". While this is an interesting result and tells us that we expect interactions which look local in our effective space-time to look highly non-local in terms of our indexing set labels, it introduces highly artificial interaction terms which have nothing to do with our original Hamiltonian. It is by no means clear that our results remain stable if we enlarge our scheme to include four-point functions as dynamical variables (in analogy to the link variables). Nevertheless, this is an interesting question and merits further study.

A more interesting possibility is that by slightly complicating our Hamiltonian we can obtain a scheme which naturally evolves, in addition to its effective lattice, a leftover gauge theory which is responsible for all low energy scattering processes. The hope is that an effective lattice gauge interaction will tend to be better behaved with respect to the locality issue. This situation automatically occurs if we relax the condition that the fields $X_{\langle lm \rangle}$ are real. In this case we consider a Hamiltonian of the form

$$\begin{aligned} H \;=\; \sum_{\langle lm \rangle} \Big[& P^{\dagger}_{\langle lm \rangle}\, P_{\langle lm \rangle} \;+\; \eta\, V \left(X^{\dagger}_{\langle lm \rangle}\, X_{\langle lm \rangle} \right) \\ & - i \left\{ X_{\langle lm \rangle} \cdot \Psi^{\dagger}(l)\, \sigma_3\, \Psi(m) \;-\; X^{\dagger}_{\langle lm \rangle} \cdot \Psi^{\dagger}(m)\, \sigma_3\, \Psi(l) \right\} \Big] \quad ; \end{aligned} \tag{3.20}$$

for the obvious reasons, we have now made our fermionic fields two-component objects. In this case we see that a general mean-link wavefunction will yield an

expectation value for the variables $X_{\langle lm \rangle}$ which is a complex number, and which can be suggestively written as

$$\langle \text{vac} |\, X_{\langle lm \rangle} \,| \text{vac} \rangle \;=\; \left| X_{\langle lm \rangle} \right| \cdot e^{iA_{\langle lm \rangle}} \quad . \tag{3.21}$$

Not surprisingly, this pattern of expectation values also yields a relativistic 1+1–dimensional theory. It follows from another theorem by F. Dyson[8] that considers the probability distribution of eigenvalues of Hermitian matrices.

Theorem II: *Random elements of a gaussian ensemble of Hermitian $N \times N$ matrices have probability distribution of eigenvalues as given by*

$$\begin{aligned} \mathcal{P}(E_1, \ldots, E_N) \;=\; C & \left[\prod_{i<j} \left(E_i - E_j \right)^2 \right] \\ & \times \exp \left[- \sum_{i=1}^{N} \left(E_i^2 / 4a^2 \right) \right] \cdot dE_1 \cdots dE_N \quad . \end{aligned}$$

If this distribution is treated the same way we have treated the case of anti-symmetric matrices, then modulo unimportant details the conclusions are the same.

Equation (3.21) is suggestive in that the variables $A_{\langle lm \rangle}$ look suspiciously like gauge fields. This impression is strengthened by the fact that (3.20) is invariant with respect to a large class of transformations of the form

$$\begin{aligned} \Psi(l) \;&\rightarrow e^{i\alpha(l)}\, \Psi(l) \\ A_{\langle lm \rangle} \;&\rightarrow\; A_{\langle lm \rangle} \;+\; \alpha(l) \;-\; \alpha(m) \quad . \end{aligned} \tag{3.22}$$

Hence, the mean link minima for this theory are highly degenerate, and the true ground state must be a linear combination of these minima in order to

become the state of lowest energy. This in effect says that the ground state of the system is "gauge invariant" and that only "gauge invariant operators" can have vacuum expectation values. In other words, one has to restrict attention to gauge invariant Green's functions. Of course, at this stage of our discussion this "gauge-invariance" is in terms of the labels of the indexing set and there is no *a priori* reason why it must reflect itself as a local gauge invariance in terms of the effective coordinates x. This is a statement which requires a proof, and as we have already noted the question remains open. At this juncture the index-set gauge invariance is important in that it helps to guarantee that the residual low energy interactions of the theory involve only as many dynamical degrees of freedom as there would be in any lattice gauge theory.

Let us now turn to exactly this question, namely "*How many quantum degrees of freedom participate in low-energy interactions?*". We begin by noting that what keeps us from interpreting the fields $A_{\langle lm\rangle}$ as the sort of gauge degrees of freedom we are used to in ordinary lattice gauge theories is that there are too many of them. Recall that a lattice gauge theory defined on a $(2n+1)$–point lattice should have only $2n+1$ independent gauge degrees of freedom. While there are $n(2n+1)$ link terms which can be written down in such a theory, they satisfy constraints of the form

$$e^{iA_{\langle kl\rangle}} \cdot e^{iA_{\langle lm\rangle}} \cdot e^{-iA_{\langle lm\rangle}} = 1 \quad . \tag{3.23}$$

This is just the statement that for an abelian theory there are only as many independent link-variables as site variables, and that the remaining link-variables can be written as "path-ordered" products of the independent variables.

Let us now recall the *triangle* terms V^1 which we added to (3.6) in order to control the fluctuations of the link fields. It is easy to see that these terms are precisely what we need to enforce constraints (3.23) on the dynamical level. Consider terms of the form

$$- \sum_{k<l<m} \left[X_{\langle kl \rangle} X_{\langle lm \rangle} X^{\dagger}_{\langle km \rangle} \quad + \quad \text{H.c.} \right] \quad . \tag{3.24}$$

When rewritten in terms of the absolute values of the link variables $X_{\langle lm \rangle}$ and their phases $A_{\langle lm \rangle}$, (3.24) becomes

$$- \sum_{k<l<m} \left| X_{\langle kl \rangle} X_{\langle lm \rangle} X_{\langle km \rangle} \right| \cdot \cos \left(A_{\langle kl \rangle} + A_{\langle lm \rangle} - A_{\langle km \rangle} \right) \quad . \tag{3.25}$$

Since, as we discussed in the preceding section, the expectation values of the fields grow like n, we see that the argument of the cosine term wants to be zero. Hence, we see that those configurations which are not like *gauge-fields* pay a price on the order of n^3 in energy, whereas *gauge-like* excitations pay no such price. Combining this fact with the index set-gauge invariance, that guaranties that there are exactly the same number of *classical* variables which one would encounter in any formulation of a theory in $A_0 = 0$ gauge, it appears that the effective low-energy theory is a $1+1$ dimensional gauge theory. Since in the case under consideration the phases $A_{\langle lm \rangle}$ of the link variables are no longer random, the effective coordinates $\vec{x}$ are related to the lattice sites l in an approximately local way. Therefore, in the low energy/long wavelength limit our theory reduces to the well-known Schwinger model, which is indeed a local relativistic theory.

4. BEYOND ONE SPATIAL DIMENSION

In the preceding discussion of fermions on N-simplices we learned two important lessons: first, our naive notions of the dimension which should be assigned to a particular fermionic Hamiltonian tend to be incorrect; second, one component fermions tend to view almost any lattice as a relativistic 1+1–dimensional spacetime. Since, as we know, the real world does not quite fit into $1 + 1$ dimensions, producing a realistic theory becomes a challenging task.

Before proceeding to a detailed discussion of satisfactory theories with multi-component fermions let us try to get an intuitive feeling for why theories with one-component fermions do not work. Multi-component fermions are different in that they admit a set of d anticommuting Dirac matrices α, where d is the spatial dimension of a theory. In continuum theories Dirac matrices constitute velocity operators; in lattice theories α–matrices play the role of corner operators and tell the fermions when they are changing direction. This strongly suggests that in a theory without Dirac matrices fermions only know if they are moving forwards or backwards; as a result they only perceive one spatial dimension. The above reasoning may sound simplistic, but it is most probably true. At least no counterexample has been found to date.

We should be careful to point out that it is not absolutely necessary to have multi-component fermions in order to write lattice theories whose continuum limit is higher-dimensional. Exploiting the phenomenon of species doubling, Kogut and Susskind introduced fermions which had only one component per lattice site (modulo gauge degrees of freedom, *etc.*) in order to produce theories whose continuum limit was a 2+1 or 3+1–dimensional relativistic theory. Utilization of species doubling, however, requires careful control of the signs of the

lattice links. Even minor violations of the proper pattern of signs lead to major distortions in the fermion spectrum. Computer simulations show that randomly reversing the signs of 10% of the links of a Kogut-Susskind lattice changes the fermion spectrum from two-dimensional to one-dimensional. Since we have no way of analyzing dynamical generation of the link expectation values to the level of precision required by Kogut–Susskind fermions, we prefer to consider a less sensitive scheme. The scheme we present in this section exploits multi-component fermions that live on a *generalized SLAC lattice*.

4.1 Generalized SLAC Lattice: the Design.

To understand why we have to go to a generalized SLAC lattice, one must realize that multi-component fermions constitute a necessary, but insufficient condition for obtaining more than one space dimension. Basically the problem is that simply having multi-component fermions in a theory is more likely to lead to an internal symmetry than to a higher-dimensional space. In order to illustrate this point consider the *a priori* simplest object one can build with two-component fermions, namely an N-simplex with all three Pauli σ-matrices randomly assigned to its links. Our original conjecture was that the introduction of the σ-matrix algebra would by itself make the fermions see this object as three-dimensional (or at least two-dimensional, if chiral fermions do not work). The computer calculations surprised us: the spectrum of fermion energies came out linear and doubled — a clear indication of a one-dimensional theory with $SU(2)$ internal symmetry. It turns that such a behavior is an unavoidable consequence of yet another Dyson theorem concerning eigenvalue distribution random hermitian quaternionic matrices. This theorem applies to the case at hand because the Pauli

σ-matrices form a representation of the quaternion algebra.

Since a completely random pattern of link expectations values is inadequate for constructing a higher-dimensional space-time, we would like to follow a different line of approach. To begin, let us consider an example of a lattice which does describe a 2+1-dimensional space-time. Sites on this lattice are labelled by pairs of integers, p and q, so our indexing set is $J_2 = \{(p,q) : p,q = 1,\ldots,n\}$. Fermion fields are $\Psi(p,q)$ and the Hamiltonian has the form

$$H = \sum_{p,q} \left[-i X_{p'p} \Psi^\dagger(p',q)\, \sigma_x\, \Psi(p,q) - i Y_{q'q} \Psi^\dagger(p,q')\, \sigma_y\, \Psi(p,q) \right] \quad . \quad (4.1)$$

$X_{p'p}$ and $Y_{q'q}$ are random antisymmetric $n \times n$ matrices, which can be independently diagonalized. Following the arguments of the preceding section we rewrite H as

$$H = \sum_{k_x,k_y} \Psi^\dagger(k_x,k_y)\{k_x\sigma_x + k_y\sigma_y\}\Psi(k_x,k_y) \quad , \quad (4.2)$$

which looks like a momentum space form of the 2+1–dimensional Dirac Hamiltonian. In the usual way, H is rewritten in terms of creation and annihilation operators for positive and negative energy solutions of (4.2), and its ground state is obtained by filling the Dirac sea of negative energies.

The eigenvalues of (4.2) are

$$\mathcal{E}(k_x,k_y) = \pm\sqrt{k_x^2 + k_y^2} \quad . \quad (4.3)$$

Since each of the variables k_x and k_y is a one-dimensional momentum variable, we see from (4.3) that the energy spectrum of H has the desired 2+1–dimensional density of states. Generalizing the approach of the preceding section we define

an overcomplete set of fields $\Psi^\dagger(x,y)$ and $\Psi(x,y)$, and the Hamiltonian of the equivalent theory can be written as

$$\int dx\, dy \left\{ \Psi^\dagger(x,y) \, (\, -i\,\alpha_x \partial_x \, - \, i\,\alpha_y \partial_y \,) \, \Psi(x,y) \right\} \quad . \tag{4.4}$$

It follows that the Heisenberg equations of motion of the effective fields will, in the limit of large n, yield a good approximation to the usual relativistic equations of motion. Hence,we can use these fields to define localized packets, and by scattering these packets reconstruct the desired space-time.

In our original theory the sites are labelled by the indexing set $J_1 = \{ \, l : l = 1, \ldots, N = n^2 \, \}$ and link variables $X_{\langle ll' \rangle}$ connect every pair of sites l and l'. We would like to show that the 2+1–dimensional Hamiltonian (4.1) can be obtained as a subcase of this general theory. To begin, we identify the pairs $(p,q) \in J_2$ with single integers $l(p,q) = n(p-1) + q \in J_1$. This allows us to identify the fermion fields $\Psi(p,q)$ with $\Psi\,(\,l(p,q)\,)$. Under this identification, (4.1) corresponds to having the following ground-state expectation values for the link variables $X_{\langle ll' \rangle} \equiv X_{\langle l(p,q)\, l(p',q') \rangle}$:

$$C_{\langle l(p,q)\, l(p',q') \rangle} \;=\; \begin{cases} X_{\langle pp' \rangle} \cdot \sigma_x & \text{for} \quad q = q' \quad , \\ Y_{\langle qq' \rangle} \cdot \sigma_y & \text{for} \quad p = p' \quad , \\ 0 & \text{otherwise} \; . \end{cases} \tag{4.5}$$

Before constructing a Hamiltonian that leads to the above pattern of the link expectation values, we would like to point out those features of (4.5) which make it so different from the random N-simplices we discussed in the previous section.

1. Most of the links have vanishing ground-state expectation values. There are only $N^{3/2}$ non-vanishing link expectation values as opposed to the total of $\frac{1}{2}N^2$ link variables.

2. All links that do not vanish can be divided into two disjoint sets: the contribution of every non-vanishing link to the fermionic Hamiltonian is a two-site term that has either σ_x or σ_y sandwiched between fermion fields. To simplify the language we will refer to those links as x-links and y-links.

3. When only x-links are considered, the whole lattice breaks into $n = \sqrt{N}$ subsets of n points each. Every site in a subset is connected to every other site within the same subset, so we may call these subsets 'x-simplices'. However, no sites belonging to different x-simplices are connected to each other by an x-link.

4. Different x-simplices are connected to each other by y-links in such a way that any site in some x-simplex A is connected to one and only one site in any other x-simplex B.

5. When only y-links are considered, the lattice breaks into n y-simplices, that are connected to each other by x-links in the same way as x-simplices are connected to each other by y-links.

Suppose now that a pattern of link expectation values obeys conditions **1–5**. Label x-simplices with an index q and y-simplices with an index p. Relations between x- and y-simplices implied by **1–5** guarantee that every lattice site corresponds to one and only one pair (p, q) of indices and *vice versa*. By the very definition of p and q, only sites of equal q are connected by x-links, and only sites of equal p are connected by y-links. Conditions **1–5** thus imply a two-dimensional lattice structure, which we define to be a *generalized SLAC lattice*.

The only property of (4.5) not implied by **1–5** is the condition of q-independence of $X_{(pp')}$ and p-independence of $Y_{(qq')}$. If this condition does not hold,

i.e. $X_{\langle pp' \rangle}(q)$ and $Y_{\langle qq' \rangle}(p)$ are not constant functions of their arguments, then separately Fourier transforming p and q does not work. Analyzing the fermionic spectrum then becomes more difficult. Note, that this is the sort of situation one might expect to encounter when studying a system of fermions, moving in a fixed background gravitational field, *i.e.* non-flat space. We will return to the issue of whether this theory can admit weak gravity at the end of this section.

4.2 Generalized SLAC Lattice: the Construction.

We have seen that fermions indeed recognize the generalized SLAC lattice as a higher dimensional object. The question we have not yet answered is: *"Is there a Hamiltonian for which the ground-state expectation value of the link variables corresponds to a generalized SLAC lattice?"*. The purpose of this section is to provide an example of such a Hamiltonian. We will do so by selecting a particular type of Hamiltonian and tinker with its parameters until it yields the desired pattern of link VEVs. Obviously, a Hamiltonian found by such a method need not bear any relation to the universe we live in, but it will be sufficient for proving the existence theorem.

Let us consider a Hamiltonian that has the generic form

$$H = \tfrac{1}{2} \sum_{\substack{\langle lm \rangle \\ A}} \left(P^A_{\langle lm \rangle} \right)^2 + \mathbf{V}\left(\ldots, X^A_{\langle lm \rangle}, \ldots \right) - i \sum_{\substack{\langle lm \rangle \\ A}} \{ X^A_{\langle lm \rangle} \cdot \Psi^\dagger(l) M_A \Psi(m) - {X^A_{\langle lm \rangle}}^\dagger \cdot \Psi^\dagger(m) M_A \Psi(l) \} \quad . \tag{4.6}$$

Note that because the fermions are multi-component we have introduced a complete set $\{M_A\}$ of hermitian matrices to be sandwiched between fermion fields (so as to suppress the Dirac indices themselves). *Link fields* $X^A_{\langle lm \rangle}$ which multiply M_A

are chosen to be *independent quantum variables*; this allows for all possible *site-site* interactions. To simplify the discussion to follow we have also chosen these fields to be real, *i.e.* hermitian. The alternative choice (*i.e.* of complex link fields) would lead to residual gauge interactions of the sort discussed at the end of the section on 1+1–dimensional Hamiltonians. Interesting as these interactions are in the case of multi-component fermions, the question of dynamically generating a generalized SLAC lattice is more important at the moment. We introduce yet another simplification by restricting our attention to the case of two-component fermions; this will suffice for constructing 2+1 and 3+1–dimensional space-times. In this case we only need four different M matrices which we choose to be the Pauli σ-matrices for $M_{1,2,3}$ and the 2×2 unit matrix for M_0.

Our problem is to determine a potential $\mathbf{V}$ which will make link expectation values in the ground state of (4.6) form a two-dimensional generalized SLAC lattice. As before, our analysis of this problem will be at the mean-link level primarily because we do not know any simple way to do better. We begin by noting that if all terms in $\mathbf{V}$ are positive-definite, then the universe should be fluctuation dominated. This is because the only negative contribution to the ground-state energy comes from the Dirac sea and there are more bosonic link variables than fermionic site variables. Following the approach introduced in our discussion of the 1+1–dimensional case we use generalized plaquette terms to drive link expectation values beyond the range of quantum fluctuations. The potential we consider has the form

$$\mathbf{V} = \sum_{\langle lm \rangle} \sum_{A=0}^{3} \left(X^A_{\langle lm \rangle} \right)^{2r} + \tfrac{1}{4} \sum_{j,k,l,m}{}' \sum_{A,\ldots,B=0}^{3} \mathcal{F}_{ABCD} \cdot \left(X^A_{\langle jk \rangle} X^B_{\langle kl \rangle} X^C_{\langle lm \rangle} X^D_{\langle jm \rangle} \right)^2 \quad , \tag{4.7}$$

where $\mathcal{F}_{ABCD}$ are symmetric under cyclic permutations of indices and *some of* $\mathcal{F}_{ABCD}$ *are negative.* Note that only *plaquette terms with four distinct vertices* j, k, l, m are included in (4.7); plaquette terms with other numbers of vertices will be discussed later in this section.

Consider for a moment the case where all $\mathcal{F}_{ABCD}$ are negative. In such a theory all $4^4 \times \frac{N^4}{24}$ of the plaquettes are welcome and thus every link variable $X^A_{\langle lm \rangle}$ *turns on, i.e.* develops a non-vanishing expectation value. The resulting lattice defines a fermionic Hamiltonian wherein the coefficients multiplying the fields form a randomly chosen quaternionic hermitian matrix. By the appropriate Dyson theorem this Hamiltonian has a linear, *i.e.* one-dimensional, spectrum. A slightly different lattice arises if only $\mathcal{F}_{AAAA} < 0$, $A = 0, \ldots, 3$. In this theory only "pure" plaquettes (*i.e.* made of links of the same kind) are welcome, and the resulting lattice "turns on" only one type of links — the most favored one. Note however, that every link of the favored type is present and so once again the resulting fermionic Hamiltonian is one-dimensional. In principle, since the number of independent parameters $\mathcal{F}_{ABCD}$ is finite, it should be possible to map the entire phase diagram of (4.6) with potential (4.7). At present we are quite far from the completion of *that* task. Nevertheless, we have found a range of values for $\mathcal{F}_{ABCD}$ which result in a generalized SLAC lattice; we will demonstrate this fact in the discussion to follow.

Consider the following arrangement of $\mathcal{F}_{ABCD}$:

$$\begin{aligned}
&(a) \qquad \mathcal{F}_{1212} \equiv -F < 0 \quad , \\
&(b) \qquad \mathcal{F}_{1111} = \mathcal{F}_{2222} \equiv f \geq 0 \quad , \\
&(c) \qquad \mathcal{F}_{1112} \, , \, \mathcal{F}_{1122} \, , \, \mathcal{F}_{1222} > F > 0 \quad , \\
&(d) \qquad \text{other } \mathcal{F}_{ABCD} > 0 \quad .
\end{aligned} \tag{4.8}$$

Let us begin the analysis of (4.8) by stating the obvious. First, due to condition (d), any link, which is not of x-type or y-type, contributes only positive terms to (4.7). Second, due to condition (c), for any link which has both an x and a y component the overall contribution to (4.7) coming from at least one of the components is positive. In terms of the link expectation values these two facts mean that every non-vanishing link prefers to be either a pure x-link or a pure y-link, which is exactly the *SLAC-lattice condition* **2.** It is also obvious that both x-links and y-links have to be present since only $xyxy$ plaquettes contribute negative terms to (4.7).

Showing that for an appropriate choice of f (4.8) yields a *generalized SLAC lattice* requires further discussion. Note, that due to condition (b), no large x or y simplices can form if f is *positive*. The $n \times n$ SLAC lattice has only $n^4/4$ plaquettes of the $xyxy$ type, but it contains $2n$ simplices having $n^4/8$ $xxxx$ or $yyyy$ plaquettes each. Hence, if all links of the SLAC lattice have equal strength, then in the large $n = \sqrt{N}$ limit the positive contribution of $xxxx$ and $yyyy$ plaquettes overwhelms the negative contribution of $xyxy$ plaquettes. There are two ways around this obstacle: first, f need not be strictly positive; second, link expectation values need not have equal strength. Since the second possibility cannot be discussed using the techniques we have introduced, we will arbitrarily assume $f = 0$.

Consider a pair of x-simplices, A and B. (It is obvious that for $f = 0$ *some* x-simplices have to form.) Any pair of y-links that connect A with B yields an $xyxy$ plaquette. Since a link can be shared by many plaquettes, k y-links between A and B together yield $k(k-1)/2$ $xyxy$ plaquettes. Therefore, k grows until y-links start to get into each other's way, which happens when two of them share the same vertex in A or B. Adding a y-link which shares a vertex with another y-link yields as many $xxyy$ plaquettes as $xyxy$ ones. Since $\mathcal{F}_{1122} + \mathcal{F}_{1212} > 0$ (condition (c)), turning such a y-link "on" yields positive net energy change. This argument tells us that the y-links connecting x-simplices A and B obey the *SLAC-lattice condition* **4**: any site in A is joined to a one and only one site in B. Now consider x-links joining sites in A with sites in B. Every such link yields k $xxxy$ plaquettes, where k is the number of y-links connecting A with B. Since the contribution of these plaquettes to the Hamiltonian is positive and large, it follows that A and B will not be connected by x-links unless $k = 0$. On the other hand, if $k = 0$ then every x-link between A and B will be "turned on", since there is nothing to forbid it. In this case A and B will be melded into a single x-simplex $A \cup B$. Since the two cases are complementary, any pair of x-simplices that are not completely melded together into a single x-simplex has to be disconnected, *i.e.* without a single x-link to connect the two simplices. This fact is equivalent to the *SLAC-lattice condition* **3**: the entire lattice can be partitioned into disconnected x-simplices.

The remaining *SLAC-lattice conditions* follow more easily. First, we observe that the same arguments used to establish the properties of x-simplices can be applied equally well to the case of y-simplices. This verifies condition **5**. Condition **1** is not really independent and can be easily derived from the other four.

Showing that the maximal number of $xyxy$ plaquettes is obtained when all the simplices have the same size is a minor problem, and we leave this as an exercise for the reader.

Having presented the argument which says that at the level of vacuum expectation values we expect a generalized SLAC lattice to form for a specific range of parameters in (4.8), we now turn to a discussion of the effect of quantum fluctuations. We will argue that in the limit of large N link fluctuations become small on the scale of the expectation values. Once again we will use the approximation in which we assign an independent wave-function to each link. Since every link is either "on" or "off" we will generalize our previous approach and represent each of these states in terms of variational wavefunctions $\Psi_{on}(X) \propto \exp\left(\frac{\gamma}{2}(X-C)^2\right)$ and $\Psi_{off}(X) \propto \exp\left(\frac{\gamma'}{2}X^2\right)$. Ignoring for a moment link variables which are turned "off", we write the ground state energy as

$$\begin{aligned} \mathcal{E}_{vac} \;=\; N^{3/2}\cdot\gamma \;+\; N^{3/2}\,C^{2r}\left(1+\frac{r(2r-1)}{C^2\gamma}+\cdots\right) \\ -\,\tfrac{1}{4}\,N^2\,F\,C^8\left(1+\frac{28}{C^2\gamma}+\cdots\right)+\cdots \quad , \end{aligned} \tag{4.9}$$

where we have ignored terms with non-leading powers of N (including fermionic contributions). Minimizing (4.9) with respect to C and γ we obtain

$$\begin{aligned} C \;=\; \left(\tfrac{F^2}{r^2}\cdot N\right)^{1/4r-16} \quad , \quad \gamma \;=\; \sqrt{2r(r-4)}\,\left(\tfrac{F^2}{r^2}\cdot N\right)^{1/4r-16} \quad , \\ \mathcal{E}_{vac} \;=\; \left(1-\tfrac{r}{4}\right)\,(F/r)^{r/r-4}\cdot N^{2r-6/r-4} \quad . \end{aligned} \tag{4.10}$$

Thus the relative fluctuation size $(C^2\gamma)^{-1}$, behaves like $O\left(N^{-3/4r-16}\right)$ and becomes negligible in the large N limit. Accounting for fluctuations in the "off"

links is more complicated due to existence of several types of "phantom" plaquettes, *i.e.* plaquettes which appear in the Hamiltonian, but include links that are "turned off". Keeping the contribution of "phantom" plaquettes in mind it is a straightforward exercise to show that the gaussian width of the "off" links decreases in the large N limit (in fact, it decreases even faster than the width of the "on" links). Since the expectation value of the "on" links grows with N, the concept of the lattice of link expectation values is indeed a meaningful approximation.

4.3 Three-Dimensional SLAC Lattices.

So far we have constructed a 2+1–dimensional space-time. This was done by introducing two-component fermions and ensuring that ground-state expectation values of the link variables form a two-dimensional generalized SLAC lattice. We argued that this two-dimensional structure was a phase of a very general Hamiltonian which was selected by imposing certain constraints on various coupling parameters. The natural question to ask at this juncture is: *"Can one construct a 3+1–dimensional space-time using the same techniques?"*. We will show that the answer to this question is in the affirmative.

As in the 2+1–dimensional case we begin by exhibiting an example of a 3+1–dimensional fermionic Hamiltonian. In this case we label lattice sites by triples of integers belonging to an indexing set $J_3 = \{(p,q,r) : p,q,r = 1,\ldots,n\}$. The fermion fields are $\Psi(p,q,r)$ and the Hamiltonian has the form

$$H = \sum_{p,q,r} \Big[-i \sum_{p'} X_{pp'}(q,r) \cdot \Psi^\dagger(p',q,r)\, \sigma_x\, \Psi(p,q,r) - i \sum_{q'} Y_{qq'}(p,r) \cdot \Psi^\dagger(p,q',r)\, \sigma_y\, \Psi(p,q,r) - i \sum_{r'} Z_{rr'}(p,q) \cdot \Psi^\dagger(p,q,r')\, \sigma_z\, \Psi(p,q,r) \Big] \quad . \tag{4.11}$$

Once again *flat* three-dimensional space corresponds to the case in which $X(q,r)$, $Y(p,r)$ and $Z(p,q)$ are constant functions of their arguments. In this case one has only to diagonalize three random antisymmetric matrices. If this is done, H becomes

$$H = \sum_{k_x,k_y,k_z} \Psi^\dagger(k_x,k_y,k_z)\, \{\, k_x\sigma_x + k_y\sigma_y + k_z\sigma_z \,\}\, \Psi(k_x,k_y,k_z) \quad . \tag{4.12}$$

The eigenvalues of (4.12) are given by

$$\mathcal{E} = \pm\sqrt{k_x^2 + k_y^2 + k_z^2} \quad ,$$

so H indeed can be mapped onto a relativistic 3+1–dimensional theory.

Similarly to the 2+1–dimensional case, we will use the name *"generalized three-dimensional SLAC lattice"* for a lattice of link expectation values of the form (4.11). To shorten our discussion of this case, we will not discuss the analogue of conditions **1**–**5**, but will instead begin by writing down an appropriate Hamiltonian for the link fields. We will then show it indeed leads to a three-dimensional SLAC lattice. The link Hamiltonian we wish to consider has the

form

$$H = \sum_{\langle lm \rangle} \sum_{A=0}^{3} \left[\left(P^A_{\langle lm \rangle} \right)^2 + \left(X^A_{\langle lm \rangle} \right)^{2r} \right]$$

$$+ \frac{1}{12} {\sum_{l_1,\ldots,l_6}}' \sum_{A,\ldots,F=0}^{3} \mathcal{T}_{ABCDEF} \cdot \Big(\; X^A_{\langle l_1 l_2 \rangle} X^B_{\langle l_2 l_3 \rangle} X^C_{\langle l_3 l_4 \rangle} \times \tag{4.13}$$

$$X^D_{\langle l_4 l_5 \rangle} X^E_{\langle l_5 l_6 \rangle} X^F_{\langle l_6 l_1 \rangle} \Big)^2 \quad ,$$

where $\mathcal{T}_{A\ldots F}$ are symmetric under cyclic permutation of indices or reversal of their order. The only plaquettes included in (4.13) are *six-vertex plaquettes where all vertices are distinct.*

Let us recall for a moment the conditions (4.8) which led to the formation of the two-dimensional SLAC lattice. Negative $\mathcal{F}_{1212}$ favored the formation of $xyxy$ plaquettes characteristic of the two-dimensional SLAC lattice. On the other hand, all plaquette types which do not occur on a SLAC lattice were forbidden by choosing sufficiently large positive values for the corresponding $\mathcal{F}_{ABCD}$'s. Plaquettes of the types $xxxx$ and $yyyy$ which abound on the SLAC lattice required special care since they are even more abundant on the global simplex. Therefore we had to set their coefficients to zero. The desired range of couplings for the three-dimensional case can be obtained from the obvious generalization of these guidelines. To be precise, we are interested in the following range of $\mathcal{T}_{A\ldots F}$:

- $\mathcal{T}_{0\ldots} > 0$

  This condition excludes 0-links from the lattice.

- For the sake of cubic symmetry we assume that $\mathcal{T}_{A\ldots F}$ are symmetric under interchanges $x \leftrightarrow y \leftrightarrow z$. (It is not clear whether this is necessary.)

- $\mathcal{T}_{123123} = \mathcal{T}_{123213} = \mathcal{T}_{121323} = \mathcal{T}_{123132} \equiv -T < 0$

  These are the only negative coefficients (up to permutations).

- $\mathcal{T}_{111111} = \cdots \equiv t_1 \geq 0$

  $\mathcal{T}_{112112} = \mathcal{T}_{111212} = \cdots = \mathcal{T}_{112212} = \cdots \equiv t_2 \geq 0$

  We will arbitrarily assume $t_1 = t_2 = 0$ until we learn how to explore lattices with unequal link strengths.

- All other $\mathcal{T}_{A\ldots F}$ should be positive and large (*i.e.* larger than $O(T)$).

We will now prove that under these conditions link expectation values form a generalized three-dimensional SLAC lattice. Once again we begin by stating the obvious, namely that every link which is turned on should be a pure x, y or z link, and that all three types of links have to be present. Next, we observe that for $t_1 = 0$ formation of homogenous simplices (*i.e.* x, y or z simplices) is not suppressed. The next step is to show that as far as only links of one particular kind are concerned, the entire lattice breaks into disconnected maximal simplices. This can be done by trivial modifications of the arguments we have used in the two-dimensional case; we see no point of reproducing them here. The rest of the proof goes as follows: first, we consider two simplices of the same kind and investigate links which join their sites; second, we demonstrate that as far as only x-links and y-links are concerned, the lattice breaks into many replicas of the two-dimensional SLAC lattice; and finally, we show that these two-dimensional sublattices are joined by z-links to form the three-dimensional generalized SLAC lattice. The reader who is not interested in technical details may skip the rest of this subsection.

As we promised we begin by considering a pair of x-simplices, A and B,

having $O(n)$ points each. Suppose there is a y-link joining A with B and a z-link as well. Then there have to be $O(n^2)$ six-site plaquettes of the energetically suppressed types $xxyxxz$ and $xxxyxz$. We see, that links of different kind joining the same pair of simplices interfere with one another. Now suppose there are links of only one kind, say y, but two of these links share a common site in A or B. Then there have to be $O(n^3)$ plaquettes of the type $xxxxyy$, which is also energetically suppressed; therefore two y or z links which share a common site also interfere with one another. On the other hand, links of the same kind that join the sites of A with the sites of B *in one-to-one fashion* do not interfere with one another, so they can be turned "on" simultaneously. The proof that all such links are turned "on" in the ground-state configuration constitutes a simple exercise which we leave for the reader to solve. At this stage we may conclude that *if two x-simplices are joined to each other by y or z links; then all sites of these simplices are joined in one-to-one fashion; and the links which join them are all of the same kind.* Of course, *similar statements hold with respect to y-simplices and z-simplices.*

Let us forget for a moment about z-links and focus attention on the way x-simplices and y-simplices enmesh with each other. Let A_0 be an x-simplex and let us consider all x-simplices $A_1, A_2, \ldots, A_n$ which are connected to A_0 by y-links. Let l_0 be a site in A_0. Then in any of the A_q , $q = 1, \ldots, n$, there is one and only one site l_q which is joined to l_0 by a y-link. Now recall that as far as y-links are concerned, the entire lattice breaks into disconnected maximal y-simplices. Hence, one of these y-simplices has to consist of the sites $l_0, l_1, \ldots, l_n$. Since there is nothing special about the site $l_0 \in A$, *every site in A_0 belongs to some y-simplex B_p which intersects every one of the x-simplices $A_0, A_1, \ldots, A_n$*

at one and only one site. If we now define S to be

$$S = \bigcup_{q=0}^{n} A_q = \bigcup_{p=0}^{n'} B_p \quad , \tag{4.14}$$

then every site in S can be uniquely identified by its coordinates p and q. Moreover, two sites in S are joined by an x-link if and only if their q coordinates are equal; a similar statements holds for y-links. Therefore, S *is a two-dimensional generalized SLAC lattice.*

Now consider the entire lattice. We have started from an arbitrary x-simplex A_0 and have shown that A_0 is a part of a two-dimensional SLAC-like lattice "xy-slice" S. It is easy to see that two such xy-slices, S and S', cannot intersect one another; therefore the entire lattice can be described as sum of disjoint two-dimensional SLAC-like slices S_r, $r = 0, 1, \ldots, n'$. At this stage we return to considering the z-links. Observe that if two sites of the same xy-slice are joined by a z-link, then two x-simplices already joined by n y-links are also joined by a z-link. Since such a configuration would include $O(n^4)$ energetically suppressed plaquettes of types $xxxxyz$, $xxyxxz$, *etc.* (actually more, since y-simplices are also involved), it is clear that xy-slices have no internal z-links.

On the other hand, links which joins sites of different xy-slices yield multitudes of energetically favored plaquettes, so we should expect the lattice to contain as many z-links as one can put in without causing interference. Our question is, therefore, when do z-links interfere with one another? First, consider two z-links which join the same lattice site with two sites in the same xy-slice. In this case there have to be $O(n^3)$ plaquettes of energetically suppressed types $xxxyzz$, $xyxyzz$, *etc.* Since the usual yield of a z-link is $O(n^2)$ energetically favored plaquettes, this is a clear case of interference. To avoid this kind of interference, *the*

sites of any pair of xy-slices should be joined by z-links in one-to-one fashion. Next, consider two sites, l_1 and l_2, in one xy-slice and two more sites, l_3 and l_4, in another xy-slice, such that l_1 and l_2 belong to the same x-simplex but l_3 and l_4 do not. What happens if z-links join l_1 with l_3 and l_2 with l_4? In this case these two z-links participate in n' energetically favored $zxzyxy$ plaquettes, but they also participate in $2n'$ energetically suppressed plaquettes of types $zxzyyx$. Once again we have interference between z-links.

Now consider two xy-slices which whose sites are joined in one-to-one fashion by z-links. This connection between the two slices defines a mapping from one slice onto another. We have just seen that if this mapping does not respect the partitioning of the slices into x-simplices this causes z-links to interfere with one another. Naturally, failure to respect the partitioning of the slices into y-simplices causes the same effect. On the other hand, a mapping that respects the x-simplex and the y-simplex structures of the slices causes no interference. Next observe, that if x and y simplices of one xy-slice are mapped onto appropriate simplices of the other slice, then the coordinates (p, q) of a two-dimensional SLAC lattice can be properly mapped from the first slice onto the second. Moreover, this mapping of coordinates from one xy-slice onto another is transitive, *i.e.* if one maps coordinates from S_r onto $S_{r'}$ and then from $S_{r'}$ onto a third slice $S_{r''}$, the resulting coordinate system on $S_{r''}$ is the same as the one obtained by direct mapping from S_r.

We are now ready to introduce the global coordinate system (p, q, r) on the entire lattice. The r-coordinate of some site l is the index of the xy-slice l happens to belong to, while the coordinates p and q are obtained by mapping S_r onto some fixed reference slice S_{r_0}. The sum total of our arguments implies that

this coordinate system has the following properties: *two sites with respective coordinates* (p,q,r) *and* (p',q',r') *are joined by an* x*-link if and only if* $q=q'$ *and* $r=r'$*, joined by a* y*-link if and only if* $p=p'$ *and* $r=r'$*, and joined by a* z*-link if and only if* $p=p'$ *and* $q=q'$. Therefore, the fermionic Hamiltonian has the form (4.11) and *we have constructed the three-dimensional generalized SLAC lattice. Q. E. D.*

4.4 Dimension Versus Internal Symmetries.

We have now exhibited Hamiltonians which, for specified ranges of couplings, dynamically generate $(1+1)$, $(2+1)$ and $(3+1)$ dimensional relativistic space-times. It is important to emphasize that we used different Hamiltonians to discuss each of these cases for pedagogical reasons alone. In fact we consistently pointed out that the 1+1–dimensional theory can be obtained as a trivial subcase of each Hamiltonian, and it is not difficult to show that for negative values of parameter t_2 Hamiltonian (4.13) yields a 2+1–dimensional theory. Thus, although we did not emphasize the point, at the same time that we demonstrated the existence of Hamiltonians which generate $(d+1)$ dimensional space-times, we also established that all of these space-times can exist as different "phases" of the same theory. For this class of theories the spatial dimension d plays the role of an integer valued *"order parameter"*. This is of course just the result which we promised in the introduction. The existence of these phases implies the possibility that finite-temperature effects can cause dimension-changing phase transitions. It is interesting to speculate as to the significance of this result within the context of discussion of the early universe, but we will refrain from doing so at this time.

Having established that there exist Hamiltonians for which the dimension of

the effective low-energy space-time is tunable, we would now like to consider what determines the maximum dimension of the system. As we have seen in specific examples, the maximal dimension of the lattice is determined by the structure of the terms in the link-field Hamiltonian. In particular, higher dimensional generalized SLAC lattices can be obtained only if generalized plaquette terms involving a sufficiently large number of links are present. However, we have seen that the *perceived dimensions* of an effective theory can be different from the lattice connectivity. In particular, a generalized d-dimensional SLAC lattice is not perceived as such by fermions which have too few components. Our previous discussion suggests that in order to see a system as having d-spatial dimensions the fermions must form a representation of the Clifford algebra for that number of dimensions; the lowest representation of this algebra needs 2^p–component fermions for $d = 2p$ or $d = 2p + 1$. For this reason, one component fermions can perceive at most 1+1 dimensions.

Consider now what happens when fermions have more components than required for the dimension which is defined by the lattice they live upon. It appears that in this case the resulting theory acquires a gauged internal flavor symmetry; at least this is what happens in the simple example of quaternionic matrices. The details of the way in which this works, and the flavor decomposition of the fermionic multiplets seems to depend on the detailed structure of the plaquette terms in the Hamiltonian, and we have no general results to present at this time. However, it is clear that if one adopts the simplest scheme, then one appears to trade off dimensions against $SU(n)$ flavor symmetries in a manner reminiscent of Kaluza-Klein theories; except that there seem to be amusing algebraic constraints on the way in which this can happen.

For example, if we have four component fermions then we can arrange for the Hamiltonian to produce a 1+1–dimensional space-time with an $SU(4)$ internal symmetry; or 1+3–dimensional space-time with a doublet of chiral fermions; or, if mass terms are included in the Hamiltonian, a single massive fermion and no flavor symmetry. If we have eight component fermions, then we can have a 1+1–dimensional theory with $SU(8)$ symmetry; or a 1+3–dimensional theory with $SU(2)$-symmetry; or a 1+5–dimensional theory with no internal symmetry (for the purposes of this discussion we will always assume mass terms so as not to get confused by chiral versus non-chiral fermions). Thus, we see that for a given choice of Hamiltonian parameters and number of fermion components only certain specific combinations of space-time dimension and internal symmetry are possible. Amusingly, if we have twenty component fermions then the biggest dimension the effective theory can have is four space-time dimensions with an $SU(5)$ internal symmetry, since the dimensions of the Clifford algebras for massive fermions are $2, 4, 8, 16, 32, \ldots$. Clearly, this interplay of internal symmetry, structure of the Hamiltonian and the dimension of space-time is fascinating and should be explored further.

4.5 Is There a Room for Gravity?: a Question of Scales.

The infinite N limit of the Hamiltonian (4.11) defines a 3+1–dimensional relativistic continuum theory. In the case of large, but finite N, one obtains an effective theory which has finite volume L^3 and finite resolution a. A simple counting of degrees of freedom makes it obvious that $L/a = n \equiv \sqrt[3]{N}$. What is not obvious is the way L or a are related to physical quantities. We will argue that a is comparable in magnitude to the Planck length M_{Planck}^{-1} while L may

become arbitrarily large.

We begin by multiplying the generic Hamiltonian (4.6) by an arbitrary scale parameter Λ. Let us assume that the link-field Hamiltonian yields a three-dimensional SLAC lattice with all non-vanishing link expectation values having equal strength χ; then each of the matrices $X_{pp'}$, $Y_{qq'}$ and $Z_{rr'}$ that appear in (4.11) is an antisymmetric $n \times n$ matrix whose elements are $\pm\Lambda\chi$. Diagonalizing these matrices, as we did in the one-dimensional case, we find an energy spectrum

$$\mathcal{E} = \sqrt{k_x^2 + k_y^2 + k_z^2} \times \left(1 + O\left(\frac{k^2}{n\Lambda^2}\right)\right) \quad , \tag{4.15}$$

where

$$k_{x,y,z} = \frac{2\pi}{L} \cdot j_{x,y,z} \quad , \qquad j_x, j_y, j_z = -\frac{n}{2}, \cdots, \frac{n}{2}$$

$$\text{and} \quad L = \frac{2\sqrt{n}}{\Lambda\chi} \quad .$$

This spectrum is characteristic of a theory defined on an equivalent lattice with volume L^3 and lattice spacing $a = \pi/k_{max} = L/n$.

Having computed L and a in terms of the parameters of the defining Hamiltonian we now wish to relate those parameters to some dimensionful physical parameters. The only such parameter one can reasonably hope to compute without detailed analysis of effective gauge theories, let alone GUTs, Technicolor, *etc.*, is Newton's constant. Of course, Newton's constant has meaning if and only if the effective low-energy theory has some sort of Newton-like gravity. To verify whether this is the case in our theory one has to demonstrate the following points:

1. Show that the lowest energy configuration corresponds to flat space.
2. Identify the low-lying excitations of the lattice which violate the flatness and compute their energies.

3. Show that these excitations interact with fermions in an approximately local way. Here locality refers to the effective space-time coordinates rather than to lattice sites.

4. Derive the effective gravity theory and show that is has a proper Newtonian limit.

Unfortunately, to this point in time, we have no rigorous results pertaining to the first two points and no results whatsoever pertaining to the last two. However, we have some interesting numerical results concerning the behavior of the fermions on small two-dimensional generalized SLAC lattices. Assuming that these results can be extrapolated to large three-dimensional lattices, and also assuming that points **3** and **4** are true, we will argue that Newton's constant should be $G \sim a^{-2}$.

Recall that the link Hamiltonian is symmetric with respect to independent changes of signs of the link fields. Therefore, an eigenstate of the link Hamiltonian which has b links with non-zero expectation values is 2^b-fold degenerate. In particular, if the ground state is a three-dimensional generalized SLAC lattice its degeneracy is $2^{3n^4/2}$, and most of these degenerate configurations are not perceived by fermions as flat three-dimensional spaces. On the other hand, since different configurations lead to different fermionic spectra, the total zero-point energy of the fermions is also different, thus lifting the degeneracy between non-equivalent configurations. Hence the true ground state is a configuration for which the sum of the negative fermionic eigenvalues is most negative. Our first goal is to show that this configuration is flat — this will solve the problem of fine-tuning the cosmological constant.

As we have already mentioned, we have no analytic results for the case of fermions on generalized SLAC lattices which are not flat. The results we will

present here were obtained by numerical diagonalization of fermionic Hamiltonians for rather small lattices. To be precise, the lattices which we considered were two-dimensional generalized SLAC lattices of sizes up to 9×9. The absolute values of all links were taken equal, but signs of the links formed different patterns from lattice to lattice. For every such lattice we have diagonalized the fermionic Hamiltonian and computed the net negative energy of the Dirac sea. To our relief, we have found that the most negative zero-point fermionic energies were indeed obtained for lattices which yielded flat spaces. If this results persists for large three-dimensional SLAC lattices, then point **1** is true.

Having found that the true ground state seems to be flat, we considered non-flat excited states. We compared plots of the fermionic spectra for many non-flat SLAC lattices and found that the more "frustrated" the lattice was, the less two-dimensional the fermionic spectrum looked. (We define "frustration" of a lattice as a fraction of its four-link Wilson loops which are negative.) More frustrated lattices yielded linear fermionic spectra, and at the same time less negative zero-point fermion energies than flat lattices. This reduction in negative energy of the Dirac sea reaches a maximum of about 15%. Moreover, the relative energy cost of randomizing the signs of links seems to be independent of the size of the lattice. Inspired by these facts, we denote by $-\mathcal{E}_n$ the energy of the Dirac sea of the flat $n \times n \times n$ generalized SLAC lattice and by $-\mathcal{E}'_n$ the Dirac sea energy of the totally frustrated lattice of the same size. Making a wild extrapolation, we assume that in the large n limit the difference between $-\mathcal{E}'_n$ and $-\mathcal{E}_n$ remains finite fraction of $\mathcal{E}_n$.

Unfortunately, all we know about effective space-times which correspond to frustrated lattices is that they are not flat. If we assume that highly frustrated

lattices correspond to *some* effective 3+1–dimensional space-times, we might expect such space-times to have curvature of the order of magnitude $R \sim a^{-2}$. Obviously we cannot even tell whether this curvature is positive or negative, let alone answer more detailed questions. Our goal is to relate Newton's constant to the resolution a of the effective theory and to the number N of degrees of freedom. In order to do so we make two unsubstantiated assumptions about the effective gravity in our theory: first, we assume the effective gravity to have Einstein-like behavior at large distances; second, we assume that even at curvatures comparable to the inverse square of the shortest distance in the effective theory, the dominant contribution to the gravitational energy is given by

$$\mathcal{E}_{grav} \;=\; G \cdot \int d^3\vec{x}\; R(\vec{x}) \quad . \tag{4.16}$$

Highly frustrated lattices have $R \sim a^{-2}$ all over the space, so their gravitational energy should be $\mathcal{E}_g \sim GL^3/a^2$. Equating this gravitational energy with the change in the Dirac sea energy $\mathcal{E}_n - \mathcal{E}'_n \sim \mathcal{E}_n$, we can estimate Newton's constant

$$G \;\sim\; \frac{a^2}{L^3} \cdot \mathcal{E}_n \;= \frac{1}{a} \cdot \frac{\mathcal{E}_n}{N} \qquad (N \equiv n^3)\,. \tag{4.17}$$

It remains to estimate the Dirac sea energy $\mathcal{E}_n$. This can be easily done by observing that the Dirac sea energy per fermion $\mathcal{E}_n/N$ has the same order of magnitude as the maximal fermion energy ϵ_{max}. The latter is given in a relativistic theory by $k_{max} = \pi/a$. Combining this result with (4.17) we obtain

$$G \sim \frac{1}{a} \cdot \frac{\mathcal{E}_n}{N} \;\sim\; \frac{1}{a^2} \quad , \tag{4.18}$$

or in other words $a = 1/M_{\text{Planck}}$. *Q. E. D.*

The case of massive Dirac fermions is similar to that of massless Weyl fermions. In this case the fermionic Hamiltonian has the form

$$
\begin{aligned}
H \;=\; \sum_{p,q,r} \Big[& \; m\; \Psi^\dagger(p,q,r)\,\beta\,\Psi(p,q,r) \\
& - i\Lambda \sum_{p'} X_{pp'}(q,r)\cdot \Psi^\dagger(p',q,r)\,\alpha_x\,\Psi(p,q,r) \\
& - i\Lambda \sum_{q'} Y_{qq'}(p,r)\cdot \Psi^\dagger(p,q',r)\,\alpha_y\,\Psi(p,q,r) \\
& - i\Lambda \sum_{r'} Z_{rr'}(p,q)\cdot \Psi^\dagger(p,q,r')\,\alpha_z\,\Psi(p,q,r) \Big] \quad ,
\end{aligned}
$$

where $\alpha_{x,y,z}$ and β are Dirac matrices. The spectrum of this Hamiltonian is given by

$$
\mathcal{E} \;=\; \sqrt{m^2 + k_x^2 + k_y^2 + k_z^2} \;+\; O\left(\frac{k^3}{n\Lambda^2}\right) \quad ,
$$

where $k_{x,y,z}$ are the same as in (4.15). We see that the effective continuum fermion fields have mass m. To understand the physical consequences of this we need to express m in physical units.

We begin by observing that if the original Hamiltonian has only one defining scale Λ, then $m = O(\Lambda)$. Next, we recall that $a^{-1} = O(\Lambda\chi\sqrt{n})$ and that $L/a = n$. Finally, we generalize (4.9) and (4.10) to the 3+1–dimensional case and find that $\chi = O(n^{1/r-6})$. After that, deriving

$$
m \;\sim\; a^{-1}\cdot(L/a)^{-\frac{1}{2}-\frac{1}{r-6}}
$$

becomes straightforward. We have already established the physical value of $a^{-1} \sim M_{\rm Planck}$. As far as L is concerned, we know only the lower bound $L \gtrsim 10^{10} l.y.$, so

we can give only the upper limit on m, which comes out as

$$m \lesssim 0.005 \text{ ev} \cdot 10^{-61/r-6} \quad . \tag{4.19}$$

Therefore, the only known fermions whose masses can be due to the mass term in the original Hamiltonian are neutrinos; the masses of quarks and leptons must have other explanation.

5. CONCLUSION

5.1 What Have We Got?

At this point we have some specific examples under our belt and so the time has come to summarize just what we believe we have accomplished. Our primary goal was to establish a framework for discussing the question of where space-time comes from, and what controls the dimension of a given theory. It is our belief that this paper makes a step in that direction.

To be more precise, we have shown that one can define a class of theories and a computational scheme within which one can meaningfully discuss dynamically generating an effective space-time. Moreover, we have shown that in this class of theories dimension can be thought of as an integer valued "order parameter" which characterizes distinct phases of a single dynamical system. It is interesting to notice that when a theory does develop an effective space-time it is almost certain to be a relativistic one.

In addition to establishing a framework for discussing space-time as an illusion of low energy physics, we noted some remarkable facts which made this class of theories even more interesting. First, there are residual interactions among

the low-energy degrees of freedom which have the structure of a gauge theory. Second, our computer calculations indicate that fermionic effects provide room for an effective gravitational theory of some sort, and that the true physical vacuum has a flat metric. Finally, if this effective gravity is similar to Einstein's theory, then Newton's constant is a calculable quantity. Specifically, we have shown that if our computer results can be extrapolated to large numbers of degrees of freedom, then the cutoff imposed by the condition that gravitational fluctuations should stay small coincides with the shortest distance of the effective theory. This remarkable result is intimately related to the fact that it is the fermionic degrees of freedom which cause the flattening of the space-time. It is worth noting that if the system was dominated by bosonic rather than fermionic fields then the space-time would curl up instead of flattening.

5.2 Caveats

Having listed these wonderfully looking results, honesty forces us to list potential problems. First, we have used a variant of mean-field theory to obtain all of our results and it could be a seriously flawed tool. To the best of our knowledge this method has not been used on a problem of this sort and it is possible it just doesn't work. This question deserves and will get further attention. Moreover, to simplify our analysis we have limited the discussion to cases when all non-vanishing vacuum expectation values of the link fields are presumed to have equal magnitudes; obviously, this is a strong restriction. It is possible that the requirement of zero coefficients for certain terms in the Hamiltonian in order to obtain a higher-dimensional space-time are artifacts of our calculational scheme. It may even be possible to find a Hamiltonian whose ground state is a SLAC lat-

tice in the strong sense, rather than a generalized one. In this case the residual gauge interactions can be shown to be local with respect to the x_μ's (at least in the abelian case). Analyzing this kind of problems will require the application of more sophisticated techniques (*e.g* real-space Hamiltonian renormalization group methods) to the problem, or the development of new techniques.

Another question arises due to the fact that we are dealing with systems which have very large but finite numbers of degrees of freedom. It is well known that such systems never undergo true phase transitions, and from our point of view this means that states with different patterns of link expectation values must mix if their energies coincide. In the case of apparently different generalized SLAC lattices which correspond to different coordinatizations of the same system, their mixing would be analogous to what happens in gauge theories and would provide the explanation of why in general relativity physics is independent of coordinate system. In other cases the key question relates to the rate at which the mixing occurs, and if it involves changes over a few or many degrees of freedom. If the mixing involves changes in a relatively small number of degrees of freedom at a time, then it presumably can occur rapidly and perhaps invalidate our link mean-field theory results. If, on the other hand, mixings involve many degrees of freedom at once, then the fact that different configurations must mix is presumably completely harmless, because the mixing times involved will be enormous. In this case we would expect that the dimension and effective geometry of the theory will be well defined, even if the way in which they are *coordinatized* changes.

The most important caveat relates to the locality of the residual *gauge* and *gravitational* interactions. Clearly, if these interactions do not come out approx-

imately local in terms of the coordinates x_μ, then the space-time constructed by carrying out scattering processes need not look like that determined by our analysis of the motion of the fermions in the background field of the links. Such an occurance would be a disaster for this approach. At present we know that for certain patterns of link-field expectation values the residual gauge interactions are local, but we do not know whether there exist Hamiltonians for which these patterns of expectation values occurs naturally.

5.3 Irresponsible Remarks

If the reader has not found the material in the body of this paper speculative enough, we would like to close with a few remarks which are tantalizing but have not been carefully explored to date. First, we would like to observe that even if we do not take the dynamical picture of the way in which the link variables achieve vacuum expectation values seriously, the unusual way in which fermions *see* lattice structures provides a new way of introducing unusual dynamics by hand. For example, if one thinks about replacing each of the vertices of a generalized SLAC-lattice by an N_0-simplex, one obtains a *lumpy* system for which, because of the fact that all N-simplices yield one-dimensional systems, we have *string-like* dynamical degrees of freedom connected together in a peculiar way. We have no real experience with these type of theories, but we have played with them numerically and found they have very amusing properties. In particular, if N_0 is odd, each of these *vertices* has zero-modes which interact through the connecting links provided by the remaining "SLAC lattice". Depending upon the relative strengths of the links within vertices and the links joining distinct vertices, these zero modes can behave as zero-mass particles with short distance

structure. The question of whether or not dynamics of this sort can be exploited to produce *preon* theories with properties vastly different from those considered to date is entirely open.

Along the same lines, it is clear that one can by hand produce systems wherein strongly interacting fermions cannot break their chiral symmetries by the mechanism discussed by Drell *et. al.*[10] This is because in the case of N-simplices the effective strong-coupling theory becomes a Heisenberg antiferromagnet on a triangular lattice; just about the only thing known about that system is that it does not spontaneously magnetize.

Although we could go on and on in this vein, we will conclude with the following observation. Aside from the deeper questions relating to origins of space-time, we have also discovered that there are many poorly understood facts about the dynamics of fermions, which could perhaps be exploited in more conventional contexts. This subject reminds us of the story of an elephant presented to blind men: the more you poke it, the more unusual and seemingly contradictory aspects it chooses to present. Nevertheless, even if we are not sure what kind of beast we are dealing with, it is certain to be a remarkable one.

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APPENDIX

The purpose of this appendix is to compute the density of eigenvalues for random imaginary antisymmetric matrices. The derivation presented here is due to Brezin, Itzykson and Zuber. As noted in section 3, we define a random imaginary antisymmetric matrix as a member of a gaussian ensemble of hermitian $N \times N$ matrices whose matrix elements are constrained to be imaginary. For any function $F(A)$ of such matrices the average value of F is defined as

$$\langle F \rangle \;=\; \int\!\!\prod \left(dA_{ij}\, e^{-|A|^2/2a^2} \right) \cdot F(A) \quad , \tag{A.1}$$

where the product is over independent matrix elements of A. The quantity which interests us is the average spectral density of matrices belonging to our ensemble.

We begin by computing the average square of the resolvent of A:

$$\begin{aligned} \left\langle \mathrm{Tr}^2(\lambda - A)^{-1} \right\rangle \;&=\; \sum_{kl} \left\langle (\lambda - A)^{-1}_{kk} \cdot (\lambda - A)^{-1}_{ll} \right\rangle \\ &= \sum_{kl} \left\langle \frac{\partial}{\partial A_{kl}} (\lambda - A)^{-1}_{kl} \right\rangle \quad . \end{aligned}$$

Now we use the explicit definition (A.1) and see that

$$\left\langle \frac{\partial}{\partial A_{kl}} F(A) \right\rangle \;=\; \int\!\!\prod \left(dA_{ij}\, e^{-|A|^2/2a^2} \right) \cdot \frac{\partial}{\partial A_{kl}} F(A)$$

(integrating by parts)

$$\begin{aligned} &= \int\!\!\prod \left(dA_{ij}\, e^{-|A|^2/2a^2} \right) \cdot \frac{A^*_{kl}}{a^2} F(A) \\ &= a^{-2} \left\langle A^*_{kl}\, F(A) \right\rangle \;=\; a^{-2} \left\langle A_{lk}\, F(A) \right\rangle \quad . \end{aligned}$$

Therefore,

$$\begin{aligned}\left\langle \mathrm{Tr}^2(\lambda - A)^{-1} \right\rangle &= a^{-2} \sum_{kl} \left\langle A_{lk} (\lambda - A)^{-1}_{kl} \right\rangle \\ &= a^{-2} \left\langle \mathrm{Tr}\left(A(\lambda - A)^{-1} \right) \right\rangle = a^{-2} \left\langle \mathrm{Tr}\left(\lambda(\lambda - A)^{-1} - 1 \right) \right\rangle \\ &= \frac{\lambda}{a^2} \cdot \left\langle \mathrm{Tr}(\lambda - a)^{-1} \right\rangle - \frac{N}{a^2} \quad . \end{aligned} \tag{A.2}$$

In the "thermodynamical" limit of large N one can use the approximate equality $\left\langle \mathrm{Tr}^2 F(A) \right\rangle \approx \left\langle \mathrm{Tr}\, F(A) \right\rangle^2$ which holds up to terms of relative order of magnitude $O(1/N)$. This allows us to rewrite (A.2) as

$$G^2(\lambda) - \frac{\lambda}{a^2} G(\lambda) + \frac{N}{a^2} = 0 \quad , \tag{A.3}$$

where

$$G(\lambda) \equiv \left\langle \mathrm{Tr}(\lambda - A)^{-1} \right\rangle$$

is the average resolvent of A. Solving (A.3) with respect to $G(\lambda)$ we find

$$G(\lambda) = \frac{\lambda}{2a^2} \pm \sqrt{\frac{\lambda^2}{4a^4} - \frac{N}{a^2}} \quad . \tag{A.4}$$

Finally, we notice that the relation

$$\rho(E) = \frac{1}{2\pi i} \Big[G(E + i0) - G(E - i0) \Big]$$

between the spectral density of a hermitian matrix and its resolvent is linear; therefore it also holds for the average spectral density and the average resolvent.

Combining this fact with (A.4) we find that for the ensemble under consideration

$$\rho(E) \;=\; \frac{1}{2\pi a^2}\sqrt{4a^2N - E^2} \quad \text{for} \quad |E| \le 2a\sqrt{N} \quad , \qquad \text{(A.5)}$$

and $\rho(E) = 0$ otherwise. Note, that (A.5) obeys the normalization condition

$$\int_{E_{\min}}^{E_{\max}} dE\ \rho(E) \;=\; N \quad .$$

It remains to fix the value of a. Fermionic Hamiltonians we are actually considering are $(2n+1)\times(2n+1)$ hermitian matrices, all of whose entries are $\pm i$. Any such matrix A has $\mathrm{Tr}\,A^2 = 2n(2n+1)$. On the other hand, in our gaussian ensemble

$$\begin{aligned}
\left\langle \mathrm{Tr}\,A^2 \right\rangle \;&=\; \int_{E_{\min}}^{E_{\max}} dE\,\rho(E)\cdot E^2 \\
&= \frac{1}{2\pi a^2}\int_{-2a\sqrt{2n+1}}^{2a\sqrt{2n+1}} dE\ E^2\sqrt{4a^2(2n+1) - E^2} \\
&= a^2\cdot(2n+1)^2 \quad .
\end{aligned}$$

Hence, equalizing the average $\left\langle \mathrm{Tr}\,A^2 \right\rangle$ for both cases requires $a = 2n/2n+1$. This gives us the following expression for the average density of states for fermions on a simplex:

$$\rho(E) \;=\; \frac{2n+1}{4\pi n}\cdot\sqrt{8n - E^2} \quad .$$

Q. E. D.

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