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APPLICATION OF THE GREEN'S FUNCTION METHOD TO SOME NONLINEAR PROBLEMS OF AN ELECTRON STORAGE RING PART IV – STUDY OF A WEAK BEAM INTERACTION WITH A FLAT STRONG BEAM

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1. Introduction

Attempts to describe analytically the beam-beam interaction in a storage ring are as numerous as they are disappointing and frustrating. One can find in the literature many different theoretical approaches – single resonance models, trapping, linear approximations, resonance overlap models, applications of Lie algebra, etc.† None of them give a satisfactory explanation of the observed phenomena either quantitatively or even qualitatively.

At the same time the beam-beam instability is the most common limiting phenomenon for practically all storage rings. At least, it seems that all other observed instabilities are understood much better than the beam-beam instability and one or another cure has been found for them.

This justifies endless continuous attempts (including the present one) to describe theoretically the beam-beam phenomenon.

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†Here is not the place and the time to discuss all these models. More or less comprehensive description of different theoretical approaches can be found in several reviews[4–6] available on this subject.

It seems to me that the description of the beam-beam interaction presented here is a little bit more successful than the previous ones in the following ways. First, more than the other models this one allows all the important features of the beam-beam phenomenon to be taken into account: the nonlinear beam-beam force and its dependence on both transverse coordinates, damping of the oscillations, presence of noise in the particle motion, in particular the quantum noise in its synchrotron radiation, actual machine functions, layout of the interaction points, and to some extent imperfections present in the machine. Second, this model deals not with a separate particle, but with the beam as a whole using phase space distribution functions and the average (unperturbed and perturbed) characteristics of the bunch such as its emittances, space charge parameters, etc.

At the present stage of development of the theory presented in this note, the longitudinal particle motion is not yet implemented in the model. This constitutes a serious drawback. In particular, this makes difficult and unreliable the comparison of the obtained results to some of the more elaborate numerical simulations and of course to the experiments. Nevertheless, I believe that even such limited results are certainly of interest, mainly as a step toward a more extensive theory.

The calculations are done by a perturbation method[1] using the Green's function of the Fokker-Planck equation. This limits the applicability of the method in at least two ways: First, the current of the strong beam (or its space charge parameter) should not be too large. Evaluation of the magnitude of the space charge parameter ξ at which the approximation breaks down is very difficult. It is not clear what value of the parameter is actually small enough for approximation to be valid. As we shall see, the beam blowup is presented roughly speaking as a series in ratio $\xi/2\pi\tau$, where $2\pi\tau$ is the betatron phase advance between adjacent interaction points. This ratio is usually smaller than 1. Second, there are regions in the tune diagram where approximation breaks down even for small current (resonance regions). Due to the presence of damping, the calculation usually produces a finite result close to or even on the resonance. But in the vicinity of the resonances, especially inside the stop bands, the actual beam blowup curve may be quite different from the calculated one. Practically speaking, this does not restrict the method too much, since it is more important to find bad regions of the tune diagram rather than to be able to determine the behavior of the beam inside such a region. On the other hand the method treats all the resonances simultaneously. The blowup curve obtained in the limits of this work is a result of the action of an infinite number of resonances positioned at the same place. Such a treatment is considerably more sound than one which singles out one particular resonance and considers the motion in its vicinity.

There are several assumptions and limitations which are used in the course of the calculations. Not being as fundamental as the two restrictions mentioned previously (absence of energy oscillations and the perturbative treatment of the problem), they greatly simplify the calculations and make them possible in their presented form:

1. The first one is the limitation to the case of the weak beam – strong beam interaction. The particle distribution of the strong bunch is assumed to be unaffected by its interaction with the counter-rotating weak beam. That fixes the force acting on the particles of the weak beam. Otherwise the result of the interaction would depend on the distributions in both beams, demanding a self consistent solution of two coupled Fokker-Planck equations.

2. Next, the bunch is assumed to be short in comparison to the value of the beta function at the interaction point β. This allows me to consider the result of the passage of the weak bunch through the strong one as an instantaneous kick and to treat β itself as a constant. This last assumption fails for very small β values. This might become a serious restriction, especially when considering dynamic β (i.e., perturbed by the linear part of the beam-beam force).

3. The collisions are assumed to occur head-on. This assumption makes the beam-beam force antisymmetric, thus eliminating all odd order resonances. This restriction seems to be not too difficult to omit and is kept for the reason of the simplicity.

4. The aspect ratio of the strong beam is assumed to be very small (flat beam). This assumption is done in the very end of the calculation and allows me to present the result in its explicit form. Otherwise one needs to evaluate certain integrals numerically.

Under these assumptions the beam-beam interaction produces two effects in the motion of a weak beam particle. First, the linear part of the force with which the strong bunch acts on such a particle produces changes in the effective machine parameters for the weak beam. The tunes are shifted by amounts proportional to the strong beam current. Further, the values of the amplitude beta function at the interaction point, and consequently the values of space charge parameters and beam emittances are also changed. I will refer to the new (dynamic) tunes, beta functions, and emittances as perturbed machine parameters.

Second, from the rest portion of the force (i.e., its nonlinear part) a transverse component of the particle velocity experiences an instantaneous change ('kick'), the magnitude of which depends on the particle coordinates at the moment of interaction. Between the subsequent interactions a particle performed damped betatron oscillations in both lateral planes. The motion may be influenced by noise (such as the quantum noise in synchrotron radiation, for example) and this noise should be taken into account.*

The result of all the subsequent interactions should then be averaged over the particle distribution in the four dimensional phase space of coupled transverse motion. All these tasks are achieved here by using the Green's function method.

I restrict myself here to evaluation of the vertical emittance of the weak beam. Indeed, for the flat beam, the beam blowup is observed mainly in the vertical plane. The reason for this is of course that the vertical component of the interaction force is in this case much larger than the horizontal one for the vast majority of the particles. It will be instructive and not difficult to do the similar calculation for the horizontal emittance as well. The vertical beam blowup is presented here as a function of the tunes, the damping rates and the space charge parameters for both lateral planes of the perturbed (by the linear part of the beam-beam force) machine. It also depends on the number and distribution around the ring of the interaction points and the aspect ratio of the strong beam. The remarkable feature of the result is that for the machine staying away from the resonance, the vertical beam blowup is actually independent of the value of the damping rates.

Section 2 of this note contains expressions for the nonlinear beam-beam force as well as the first and second order corrections to the distribution function reproduced from Part I of this work. The first order calculation of the vertical emittance is performed in Section 3. Here one can find perturbed machine functions also, which are used in the next section to find the emittance in the second order approximation. In the last, Section 5, I discuss the results obtained and present a numerical illustration of the application of the derived formulae.

Several integrals relevant to the present calculations are evaluated and tabulated in the Appendices for the reader's convenience.

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* Strictly speaking, the particle motion is influenced also by other nonlinearities in the machine lattice (the most important of which are sextupole fields). I neglect here all nonlinear forces apart from the beam-beam force. The evaluation of the sextupole magnets influence on the particle motion is done in Part III of this work.[3] It appeared to be small enough to be neglected at the present stage of the work. It is interesting though to look into the problem of the beam-beam interaction in the presence of other nonlinearities.

2. Basic Relationships

To describe particle motion in a storage ring we use the Courant-Snyder variables[7] $u, \phi(u' \equiv du/d\phi)$ for the horizontal and $v, \theta(v' \equiv dv/d\theta)$ for the vertical planes, respectively. The sudden change in the particle velocity by a passage through a counter-rotating strong bunch ('kick') in these variables is connected to the kick in variables x and y by the following relationship [cf. expressions (5.9) and (5.10) of Part I]:

$$\tilde{F}_x(u,v) = \nu\sqrt{\beta_x}\hat{F}_x[x(u), y(v)] \quad , \tag{2.1}$$

and

$$\tilde{F}_y(u,v) = \tau\sqrt{\beta_y}\hat{F}_y[x(u), y(v)] \quad , \tag{2.2}$$

where β_x and β_y are the values of the horizontal and vertical beta functions at the interaction point. In order to avoid excessive indexing the tunes of the machine per interaction point are denoted ν and τ for horizontal and vertical planes respectively. In the same manner α and δ below mean the corresponding damping constants.

Let us assume for the sake of simplicity that apart from the nonlinear 'forces' (2.1) and (2.2) the storage ring is a linear machine. In other words we neglect the presence in a lattice of sextupole magnets and possible higher order magnetic fields in bending and quadrupole magnets. Under this assumption the unperturbed equilibrium distribution function of the strong beam is gaussian (cf. expression (5.16) of Part I):

$$\psi_0 = \frac{exp\{-\frac{u^2}{2\epsilon_x} - \frac{u'^2}{2\epsilon_x\nu^2} - \frac{v^2}{2\epsilon_y} - \frac{v'^2}{2\epsilon_y\tau^2}\}}{(2\pi)^2\epsilon_x\nu\epsilon_y\tau} \quad , \tag{2.3}$$

where ϵ_x, ϵ_y are horizontal and vertical emittances of the unperturbed (by the beam-beam perturbation) machine. We define then coupling as a ratio of these quantities

$$C = \frac{\epsilon_y}{\epsilon_x} \tag{2.4}$$

The nonlinear forces $\tilde{F}_x$ and $\tilde{F}_y$ for the gaussian distribution (2.3) have the following integral representation[8]:

$$\tilde{F}_x = -A_x u \int_0^1 dq \, \frac{exp(-Q)}{[p + q(1-p)]^{3/2}} \tag{2.5}$$

$$\tilde{F}_y = -A_y v \int_0^1 dq \, \frac{exp(-Q)}{[p+q(1-p)]^{1/2}} \quad , \tag{2.6}$$

where

$$A_x = \frac{\nu r_0 N_b \sqrt{p}}{\gamma \epsilon_x} \tag{2.7}$$

$$A_y = \frac{\pi r_0 N_b \sqrt{p}}{\gamma \epsilon_y} \tag{2.8}$$

$$p = \left(\frac{\sigma_y}{\sigma_x}\right)^2 \equiv \frac{\epsilon_y \beta_y}{\epsilon_x \beta_x} \quad , \tag{2.9}$$

r_0 is the classical electron radius, N_b is the number of particles per bunch, γ is the Lorentz factor, and at last

$$Q = \frac{qu^2}{2\epsilon_x[p+q(1-p)]} + \frac{qv^2}{2\epsilon_y} \quad . \tag{2.10}$$

The linear part of the beam-beam force is easily obtained from expressions (2.5) and (2.6) by putting $Q = 0$ (integrals resulting from this can be found in Appendix C):

$$F_{x\ lin} = \frac{-2A_x u}{(1+\sqrt{p})} \quad , \tag{2.11}$$

$$F_{y\ lin} = \frac{-2A_y v}{(1+\sqrt{p})} \quad , \tag{2.12}$$

from which immediately follow expressions for the horizontal and vertical space charge parameters:

$$\xi_x = 2\pi\Delta\nu_x = \frac{r_0 N_b \beta_x}{\gamma \sigma_x(\sigma_x+\sigma_y)} \quad , \tag{2.13}$$

$$\xi_y = 2\pi\Delta\nu_y = \frac{r_0 N_b \beta_y}{\gamma \sigma_y(\sigma_x+\sigma_y)} \quad . \tag{2.14}$$

We rewrite now here formulae analogous to (3.8) and (3.9) of Part I for the first and the second order corrections for the distribution function:

$$\psi_1(V_1,s_k) = -\sum_{m<\ell}\int dV_0\; G(V_1,s_k,V_0,s_m)\Big(\tilde F_x\,\frac{\partial\psi_0}{\partial u'} + \tilde F_y\,\frac{\partial\psi_0}{\partial v'}\Big)_{V_{0,m}} \tag{2.15}$$

and

$$\begin{aligned}\psi_2(V,s_\ell) = \sum_{k<\ell}\int dV_1\; G(V,s_\ell,V_1,s_k)\Big(\tilde F_x\,\frac{\partial}{\partial u'} + \tilde F_y\,\frac{\partial}{\partial v'}\Big)_{V_{1,k}} \\ \times\sum_{m<k}\int dV_0\; G(V_1,s_k,V_0,s_m)\Big(\tilde F_x\,\frac{\partial\psi_0}{\partial u'} + \tilde F_y\,\frac{\partial\psi_0}{\partial v'}\Big)_{V_{0,m}}\end{aligned} \quad , \tag{2.16}$$

where $V = (u, u', v, v')$ and $V_0 = (u_0, u_0', v_0, v_0')$, are points in a four-dimensional phase space of the transverse motion, and

$$G(V,s_k,V_0,s_m) = G_u(u,u',\phi_k\,|u_0,u_0',\phi_m)\; G_v(v,v',\theta_k|\,v_0,v_0',\theta_m) \tag{2.17}$$

is the Green's function as it is discussed in Part I. The summations in expressions (2.15) and (2.16) for any given 'moment' s_k are performed over all the 'moments' $s_m < s_k$ at which a particle experiences a kick from the side of a nonlinear beam-beam force.

The perturbed distribution function $\psi = \psi_0 + \psi_1 + \psi_2$ allows us to calculate the perturbed beam emittances. As an example, I will perform the calculations for a vertical beam emittance E_y:

$$E_y = \frac{1}{2}\int dV\;\psi(V)\left(v^2 + \frac{v'^2}{\tau^2}\right) \quad . \tag{2.18}$$

Since the distribution function ψ is found in the form of a series expansion, the vertical beam emittance E_y is also an expansion. The zeroth order term of this series is the unperturbed beam emittance ϵ_y.

It is worth to mention also, that average value of both ψ_1 and ψ_2 over whole phase space is zero. This follows from the fact that there is no particle loss and the normalization of ψ should be 1. On the other hand the normalization of ψ_0 is also 1. Certainly the same can be found by a direct integration of ψ_1 over V_1 in expression (2.15) and ψ_2 over V in expression (2.16).

Tedious but straightforward calculations show that both v and v' are zeros by averaging over ψ_1 and ψ_2.

3. The First Order Approximation

The linear part of the beam-beam force (2.11), (2.12) produces a focusing effect in both planes: it increases the machine tunes, changes the beta functions (this effect is referred to as dynamic beta) and consequently the space charge parameters (due to change of the beta functions).

In effect all these changes produce also a change in the weak beam emittance. The last change will be found in Sec. 3.2 considering only the linear part of the beam-beam force as a perturbation. Then in Sec. 3.3 the beam emittance in the first approximation will be evaluated using the perturbed machine functions and only the nonlinear part of the beam-beam force as a perturbation.

The result thus obtained in the first order is of course the same as one obtained by using unperturbed machine parameters but at the same time using the full beam-beam force. The second order calculations though are more accurate if one uses the perturbed machine parameters. More on this subject can be found in Part II of the present work.

3.1 PERTURBED MACHINE PARAMETERS

In the presence of the beam-beam interaction machine parameters for electrons and positrons are generally different. In particular, for the weak-strong interaction the machine parameters for the strong beam remain unperturbed, while they change for the weak beam[9]:

$$cos2\pi\nu = cos2\pi\nu_0 - \xi_{x0}\ sin2\pi\nu_0 \tag{3.1}$$

$$cos2\pi\tau = cos2\pi\tau_0 - \xi_{y0}\ sin2\pi\tau_0 \tag{3.2}$$

$$\beta_x\ sin2\pi\nu = \beta_{x0}\ sin2\pi\nu_0 \tag{3.3}$$

$$\beta_y\ sin2\pi\tau = \beta_{y0}\ sin2\pi\tau_0 \tag{3.4}$$

Here ξ_{x0}, ξ_{y0} are unperturbed space charge parameters, $2\pi\nu_0, 2\pi\tau_0$ are unperturbed betatron phase advances between two adjacent interaction points, β_{x0}, β_{y0} are the values of the corresponding unperturbed beta functions at the interaction point. For considerably small value of ξ_{x0}, ξ_{y0} and for ν_0, τ_0 being far from any integer formulae (3.1-3.4) yield the following approximate results:

$$\nu = \nu_0\left(1 + \frac{\xi_{x0}}{2\pi\nu_0}\right) \tag{3.5}$$

$$\tau = \tau_0 \left(1 + \frac{\xi_{y0}}{2\pi\tau_0}\right) \tag{3.6}$$

$$\beta_x = \beta_{x0} \left(1 - \frac{\xi_{x0}}{tan(2\pi\nu_0)}\right) \tag{3.7}$$

$$\beta_y = \beta_{y0} \left(1 - \frac{\xi_{y0}}{tan(2\pi\tau_0)}\right) \tag{3.8}$$

Perturbed space charge parameters ξ_x, ξ_y are defined by the same formula (2.13), (2.14) in which one should substitute the perturbed values of the beta function (3.7), (3.8).

3.2 Emittance for the Perturbed Machine

The change in the vertical emittance ΔE_{lin} of the weak beam due to change of the machine parameters can be found evaluating integral (2.18) to the first order in ψ. For the perturbation force in (2.15) one should take the linear part of the beam-beam force (2.11), (2.12).

Let us define the first order correction as:

$$\Delta E^{(1)}_{y\ lin} = -\frac{1}{2}\sum_m \int dV_1 \left(v_1^2 + \frac{(v_1')^2}{\tau_{0m}^2}\right)$$
$$\int dV_0\ G(V_1, V_0, s_{km})\left(F_{x\ lin}\frac{\partial\psi_0}{\partial u'} + F_{y\ lin}\frac{\partial\psi_0}{\partial v'}\right)_{V_{0,m}} \quad . \tag{3.9}$$

It is simpler to perform the space integrations first over V_1 and then over V_0.

The integral of v_1^2 over V_1 is the second Green's function moment $P_2 = p_0 + p_1^2 v_0^2 + p_2^2 v_0'^2 + 2p_3 v_0 v_0'$, which has been evaluated in Appendix B of Part I[1] [see formulae (B.12) through (B.15) for coefficients $p_i(\theta)$]. The second Green's function moment (of $v_1'^2$) $Q_2 = q_0 + q_1^2 v_0^2 + q_2^2 v_0'^2 + 2q_3 v_0 v_0'$ is found in Appendix B of Part II[2] [see formulae (B.10) through (B.13) for coefficients $q_i(\theta)$]. Since neither P_2, nor Q_2 depend on u_0', only the term containing F_y contributes to the integral over V_0. In addition to this, only terms in P_2 and Q_2 which depend on v_0' contribute to the value of the integral. Moreover, since the unperturbed distribution function ψ_0 (2.3) is symmetric in v_0', only terms proportional to p_3 and q_3 contribute. We get:

$$\frac{\Delta E^{(1)}_{y\ lin}}{\epsilon_y} = -\sum_m \xi_{0ym}\left[\tau_{0m} p_3(\theta_m) + \frac{q_3(\theta_m)}{\tau_{0m}}\right] \quad , \tag{3.10}$$

where ξ_{0m}, θ_m are the unperturbed space charge parameter and the betatron phase advance from the first to the m-*th* sequential beam-beam kick (θ is defined in such a way that it is changed by 2π between interaction points, see Ref. 1 for definition). Using now the periodicity of the ring we rewrite the sum in (3.10) in the following way. We sum first over all homologous interaction kicks with the same ξ_{0ym}, τ_{0m}. There is infinite number of such kicks spaced in betatron phase one from another by $2\pi B$, where B is the number of interaction points on one turn. Then we sum over different interaction points:[3]

$$\frac{\Delta E^{(1)}_{y\ lin}}{\epsilon_y} = -2\sum_{i=1}^{B} \xi_{0yi} \sum_{m=0}^{\infty} \left[\tau_{0i} p_3(\theta_{im}) + \frac{q_3(\theta_{im})}{\tau_{0i}}\right] \quad , \tag{3.11}$$

where

$$\theta_{im} = \theta_{io} + 2\pi B \cdot m \quad . \tag{3.12}$$

The expressions for the coefficients p_3, q_3 and for the sum over m in (3.11) can be found in Appendix C of Part II. The result we get is

$$\frac{\Delta E^{(1)}_{y\ lin}}{\epsilon_y} = -\frac{2}{B}\sum_{i=1}^{B} \frac{\xi_{0yi}}{4\pi\tau_{0i}} \tag{3.13}$$

The minus sign in (3.13) is related to the focusing effect of the beam-beam force. The perturbed emittance of the weak beam $E_{y\ lin} = \epsilon_y + \Delta E^{(1)}_{lin}$ is now:

$$\frac{E_{y\ lin}}{\epsilon_y} = 1 - \frac{2}{B}\sum_{i=1}^{B} \frac{\xi_{0yi}}{4\pi\tau_{0i}} \quad . \tag{3.14}$$

3.3 Emittance Change Due to Beam-Beam Interaction

We consider now the storage ring, parameters of which are modified by the linear part of the beam-beam force. The rest of this force, i.e., its nonlinear part, will also change the vertical emittance in the first order approximation. We use now the same formula (3.9) to find the change in the vertical emittance. The forces $F_{x\ lin}$ and $F_{y\ lin}$ in it should be replaced by the following expressions for the nonlinear part of the force:

$$F_{x\ n.l.} = -A_x u \int_0^1 dq \frac{(e^{-Q} - 1)}{[p + q(1-p)]^{3/2}} \tag{3.15}$$

$$F_{y\ n.l.} = -A_y v \int_0^1 dq \frac{(e^{-Q}-1)}{[p+q(1-p)]^{1/2}} \quad . \tag{3.16}$$

Repeating now the calculations done in Sec. 3.2 we get:

$$\frac{\Delta E_y^{(1)}}{E_{lin}} = -\frac{1}{B}\sum_{i=1}^{B} \frac{r_0 N_{bi}\beta_{yi}}{4\pi\gamma\sigma_x\sigma_y\tau_i} D(p) \quad , \tag{3.17}$$

where

$$D(p) = \int_0^1 \frac{dq}{[p+q(1-p)]^{1/2}} \left\{ \frac{1}{(q+1)^{3/2}\sqrt{\frac{q}{p+q(1-p)}+1}} - 1 \right\} \quad . \tag{3.18}$$

Integrals of this type are evaluated in Appendix C of this note. In this particular case we get:

$$D(p) = \left[\frac{2}{\sqrt{2-p}+\sqrt{p}} - \frac{1}{\sqrt{2-p}+1} - \frac{2}{1+\sqrt{p}} \right] \quad . \tag{3.19}$$

For $p \ll 1$

$$D(p) \simeq -1 + \sqrt{p} \quad . \tag{3.20}$$

Hence, in this case

$$\frac{\Delta E^{(1)}}{E_{lin}} \approx \frac{1}{B}\sum_{i=1}^{B} \frac{\xi_i}{4\pi\tau_i} \quad . \tag{3.21}$$

Substitute here $E_{y\ lin}$ by (3.14), neglect the higher order terms in ξ and obtain:

$$\frac{E_y^{(1)}}{\epsilon_y} \approx 1 - \frac{1}{B}\sum_{i=1}^{B} \frac{\xi_{0yi}}{4\pi\tau_{0i}} \quad . \tag{3.22}$$

This result can also be obtained directly from formula (3.9) if one uses unperturbed machine parameters and the full beam-beam force. Notice, that the nonlinear part of the force compensates for half of the emittance reduction due to the linear part of it (c.f. (3.22) and (3.14), respectively).

4. The Second Order Approximation

To obtain the second order (in the space charge parameter) correction to the vertical beam emittance one should use formula (2.16) for the second order correction to the distribution function:

$$\Delta E_y^{(2)} = \frac{1}{2}\sum_k \int dV\Big(v^2 + \frac{(v')^2}{\tau_k^2}\Big) \int dV_1\ G(V, V_1, s_k) \left(F_x \frac{\partial}{\partial u_1'} + F_y \frac{\partial}{\partial v_1'}\right)_{V_{1,k}}$$
$$\times \sum_m \int dV_0\ G(V_1, V_0, s_m) \left(F_x \frac{\partial \psi_0}{\partial u'} + F_y \frac{\partial \psi_0}{\partial v'}\right)_{V_{0,m}} \quad . \tag{4.1}$$

As it is discussed above I use the perturbed machine functions and the nonlinear part of the beam-beam force (3.15), (3.16) including the linear part of it (2.11), (2.12) into the machine lattice. Here and further on I omit the subscript $n.\ell.$ for the nonlinear force $F_{x,y}$.

In (4.1) again it is easier to perform integration starting from the outer integral and working in into the internal ones. The integration over dV yields:

$$\Delta E_y^{(2)} = \int_0^1 \frac{dq_1}{[p + q_1(1-p)]^{1/2}}$$
$$\times \sum_k A_{yk} \int dV_1 (p_2(\theta_k) v_1' + p_3(\theta_k) v_1) v_1 (e^{-Q} - 1) \tag{4.2}$$
$$\times \sum_m \int dV_0\ G(V_1, V_0, s_m) \left(F_x \frac{\partial \psi_0}{\partial u'} + F_y \frac{\partial \psi_0}{\partial v'}\right)_{V_{0,m}} \quad ,$$

where

$$Q = K_u u_1^2 + K_v v_1^2 \quad , \tag{4.3}$$

$$K_u = \frac{q_1}{2\epsilon_x [p + q_1(1-p)]} \quad , \tag{4.4}$$

$$K_v = \frac{q_1}{2\epsilon_y} \quad . \tag{4.5}$$

Denote

$$I_0(K_u) = \int du_1 du_1'\ e^{-K_u u_1^2}\ G_u(u_1, u_1', u_0, u_0', \phi_m) \tag{4.6}$$

$$I_{vv'}(K_v) = \int dv_1 dv'_1 v_1 v'_1 \; e^{-K_v v_1^2} \; G_v(v_1, v'_1, v_0, v'_0, \theta_m) \tag{4.7}$$

$$I_{v^2}(K_v) = \int dv_1 dv'_1 v_1^2 \; e^{-K_v v_1^2} \; G_v(v_1, v'_1, v_0, v'_0, \theta_m) \tag{4.8}$$

then

$$\Delta E_y^{(2)} = \int_0^1 \frac{dq_1}{[p + q_1(1-p)]^{1/2}}$$
$$\times \int dV_0 \sum_k A_{yk} \Big[p_2(\theta_k) \sum_m (I_0(K_u) I_{vv'}(K_v) - I_0(0) I_{vv'}(0)) \tag{4.9}$$
$$+ p_3(\theta_k) \sum_m (I_0(K_u) I_{v^2}(K_v) - I_0(0) I_{v^2}(0)) \Big] \left(F_x \frac{\partial \psi_0}{\partial u'} + F_y \frac{\partial \psi_0}{\partial v'} \right)_{V_{0,m}}$$

Integrals I_0, $I_{vv'}$ and I_{v^2} to be found in Appendix A.

Substitute now expressions for F_x, F_y and ψ_0, then:

$$\Delta E_y^{(2)} = \int_0^1 \frac{dq_1}{[p + q_1(1-p)]^{1/2}} \sum_k \Big[p_2(\theta_k) \sum_m (g_{1k} + g_{3k}) + p_3(\theta_k) \sum_m (g_{2k} + g_{4k}) \Big] \; , \tag{4.10}$$

where

$$g_{1k} = A_1 \int_0^1 \frac{dq_2}{[p + q_2(1-p)]^{3/2}} \Big[\Phi_1(q_1, q_2) - \Phi_1(0, q_2) - \Phi_1(q_1, 0) + \Phi_1(0,0) \Big] \tag{4.11}$$

$$g_{2k} = A_2 \int_0^1 \frac{dq_2}{[p + q_2(1-p)]^{3/2}} \Big[\Phi_2(q_1, q_2) - \Phi_2(0, q_2) - \Phi_2(q_1, 0) + \Phi_2(0,0) \Big] \tag{4.12}$$

$$g_{3k} = A_3 \int_0^1 \frac{dq_2}{[p + q_2(1-p)]^{1/2}} \Big[\Phi_3(q_1, q_2) - \Phi_3(0, q_2) - \Phi_3(q_1, 0) + \Phi_3(0,0) \Big] \tag{4.13}$$

$$g_{4k} = A_4 \int_0^1 \frac{dq_2}{[p + q_2(1-p)]^{1/2}} \Big[\Phi_4(q_1, q_2) - \Phi_4(0, q_2) - \Phi_4(q_1, 0) + \Phi_4(0,0) \Big] \tag{4.14}$$

In these expressions I used the following notations:

$$A_1 = -\frac{r_0 N_b \tau \epsilon_y}{4\epsilon_x} \sqrt{\frac{\beta_y}{\beta_x}} \sqrt{p} \quad , \tag{4.15}$$

$$A_2 = -\frac{r_0 N_b \epsilon_y}{2\epsilon_x} \sqrt{\frac{\beta_y}{\beta_x}} \sqrt{p} \quad , \tag{4.16}$$

$$A_3 = \frac{r_0 N_b \tau \beta_y}{\beta_x} \sqrt{\frac{\epsilon_y}{\epsilon_x}} \quad , \tag{4.17}$$

$$A_4 = \frac{r_0 N_b \beta_y}{2\beta_x} \sqrt{\frac{\epsilon_y}{\epsilon_x}} \quad , \tag{4.18}$$

$$\Phi_1(q_1, q_2) = \frac{1}{\pi^2} \int dx dx' dy dy' \; I_0 I_{vv'} x x' \; exp(-(L_u+1)x^2 - x'^2 - (L_v+1)y^2 - y'^2) \; , \tag{4.19}$$

$$\Phi_2(q_1, q_2) = \frac{1}{\pi^2} \int dx dx' dy dy' \; I_0 I_{v^2} x x' \; exp(-(L_u+1)x^2 - x'^2 - (L_v+1)y^2 - y'^2) \; , \tag{4.20}$$

$$\Phi_3(q_1, q_2) = \frac{1}{\pi^2} \int dx dx' dy dy' \; I_0 I_{vv'} y y' \; exp(-(L_u+1)x^2 - x'^2 - (L_v+1)y^2 - y'^2) \; , \tag{4.21}$$

$$\Phi_4(q_1, q_2) = \frac{1}{\pi^2} \int dx dx' dy dy' \; I_0 I_{v^2} y y' \; exp(-(L_u+1)x^2 - x'^2 - (L_v+1)y^2 - y'^2) \; , \tag{4.22}$$

where

$$L_u = \frac{q_2}{p + q_2(1-p)} \quad , \tag{4.23}$$

$$L_v = q_2 \quad . \tag{4.24}$$

Integrals $\Phi_{1,2,3,4}$ can be expressed as combinations of integrals f_{mn} which are tabulated in Appendix B.

The result of tedious but straightforward calculations is:

$$\Phi_1(q_1,q_2) = [p+q_1(1-p)]^{1/2}[p+q_2(1-p)]^{3/2} \times \frac{q_1 q_2\ e^{-2\alpha\phi-2\delta\theta}\ sin2\nu\phi\ sin2\tau\theta}{S_1^{3/2}S_2^{3/2}} \tag{4.25}$$

$$\Phi_2(q_1,q_2) = [p+q_1(1-p)]^{1/2}[p+q_2(1-p)]^{3/2} \times \frac{q_1(1+q_2-q_2\ e^{-2\delta\theta}\ cos^2\tau\theta)\ e^{-2\alpha\phi}\ sin2\nu\phi}{S_1^{3/2}S_2^{3/2}} \tag{4.26}$$

$$\Phi_3(q_1,q_2) = [p+q_1(1-p)]^{1/2}[p+q_2(1-p)]^{1/2} \times \frac{\left\{(1+q_1)(1+q_2)\ cos2\tau\theta - q_1q_2\ e^{-2\delta\theta}\ cos^2\tau\theta(1+sin^2\tau\theta)\right\}e^{-2\delta\theta}}{S_1^{1/2}S_2^{5/2}} \tag{4.27}$$

$$\Phi_4(q_1,q_2) = [p+q_1(1-p)]^{1/2}[p+q_2(1-p)]^{1/2} \times \frac{\left\{(2-q_1)(1+q_2)+q_1q_2\ e^{-2\delta\theta}\ cos^2\tau\theta\right\}e^{-2\delta\theta}\ sin2\tau\theta}{S_1^{1/2}S_2^{5/2}} , \tag{4.28}$$

where

$$S_1 = [p+q_1(2-p)]\ [p+q_2(2-p)] - q_1q_2\ e^{-2\alpha\phi}\ cos^2\nu\phi \tag{4.29}$$

$$S_2 = (1+q_1)(1+q_2) - q_1q_2\ e^{-2\delta\theta}\ cos^2\tau\theta \tag{4.30}$$

In all above formulae for the sake of abbreviation, ϕ and θ denote $\phi_k - \phi_m$ and $\theta_k - \theta_m$.

From now on I will restrict the calculations to the case of a flat beam, aspect ratio of which is small:

$$\sqrt{p} = \frac{\sigma_y}{\sigma_x} \ll 1 \ . \tag{4.31}$$

This case is the most common one for large storage rings and allows one to simplify substantially the integrations in (4.10). Since there is no exponential factors involved anymore, one should not expect the loss of accuracy while expanding the expressions in (4.10) in the power series in parameter $\sqrt{p}$. The corresponding integrals can be found in Appendix C.

The final result for the vertical beam blowup, i.e., the ratio of the rms value for the perturbed beam Σ_y to the rms value of the unperturbed beam σ_y is:

$$\frac{\Sigma_y}{\sigma_y} = \sqrt{\frac{E_y}{\epsilon_y}} \cdot \sqrt{\frac{\beta_y}{\beta_{y0}}} \tag{4.32}$$

where

$$\begin{aligned}\frac{E_y}{\epsilon_y} =& 1 - \frac{1}{B}\sum_{j=1}^{B}\frac{\hat{\xi}_{yj}}{4\pi\tau_j}(1-\frac{\sigma_y}{\sigma_x}) \\ &+ \sum_{j=1}^{B}\hat{\xi}_{yj}\Big\{\frac{1}{4\pi\delta B}\sum_{i=1}^{B}\Big(\hat{\xi}_{yj+i-1}W_{1j,j+i-1} + \hat{\xi}_{xj+i-1}W_{2j,j+i-1}\Big) \\ &+ \frac{1}{4\pi B\tau_j}\sum_{i=1}^{B}\Big(\hat{\xi}_{yj+i-1}W_{3j,j+i-1} + \hat{\xi}_{xj+i-1}W_{4j,j+i-1}\Big)\Big\} ,\end{aligned} \tag{4.33}$$

where

$$\hat{\xi}_x = \frac{r_0 N_b \beta_x}{\gamma\sigma_x^2} \tag{4.34}$$

$$\hat{\xi}_y = \frac{r_0 N_b \beta_y}{\gamma\sigma_x\sigma_y} , \tag{4.35}$$

are the zeroth order terms of the expansion of the space charge parameters of the perturbed machine in the power series in $\sqrt{p}$. The factors W have the following meaning

$$W_{1j,j+i-1} = \sum_{\ell=0}^{\infty}{}' e^{-2\delta\theta}\Big\{\frac{2}{\sqrt{z_1}}\Big[z_3(1+cos2\tau_i\theta) - \frac{1-cos2\tau_i\theta}{\sqrt{3+z_2}}\Big] - \frac{4}{z_1}\sqrt{p}\ cos2\tau\theta_i\Big\} \tag{4.36}$$

$$W_{2j,j+i-1} = -\sqrt{p}\sum_{\ell=0}^{\infty} e^{-2\delta\theta-2\alpha\phi}\ \frac{z_3}{z_1^{3/2}}\ sin2\tau_i\theta\ sin2\nu_i\phi \tag{4.37}$$

$$W_{3j,j+i-1} = \sum_{\ell=0}^{\infty} e^{-2\delta\theta}\ sin2\tau_i\theta\Big\{\frac{2z_3\sqrt{3+z_2}+1}{\sqrt{z_1}\sqrt{3+z_2}} + \frac{1}{2} - \frac{3}{z_1}\sqrt{p} - \frac{1}{2}\sqrt{p}\Big\} \tag{4.38}$$

$$W_{4j,j+i-1} = \sum_{\ell=0}^{\infty} e^{-2\alpha\phi}\ sin2\nu_i\phi \left\{ \frac{1}{2} - \frac{1}{z_1} + \sqrt{p}\left[\frac{2z_3(1-z_2)+\sqrt{3+z_2}}{z_1^{3/2}} - \frac{3}{2} \right] \right\} \ . \quad (4.39)$$

Here

$$z_1 = 4 - e^{-2\alpha\phi}\ cos^2\nu_i\phi \ , \quad (4.40)$$

$$z_2 = 1 - e^{-2\delta\theta}\ cos^2\tau_i\theta \ , \quad (4.41)$$

$$z_3 = \frac{1}{\sqrt{1-z_2}}\ arctan\sqrt{\frac{1-z_2}{3+z_2}} \ , \quad (4.42)$$

and $(i, j = 1, 2 \ldots B)$

$$\theta \equiv \theta_{0j,j+i-1} + 2\pi B\ell \ , \quad (4.43)$$

$$\phi \equiv \phi_{0j,j+i-1} + 2\pi B\ell \ , \quad (4.44)$$

where $\theta_{0j,j+i-1}$ and $\phi_{0j,j+i-1}$ are initial vertical and horizontal betatron phases of the i-th interaction point if the j-th interaction point is considered to be a starting one. The prime on the sum sign in (4.36) means that the value of the zeroth term in it should be taken with the weight 1/2.

The summation procedure leading from formula (4.10) to formula (4.33) is developed in Part III[3] for summing the sextupole perturbations. I refer reader to work[3] for more details.

5. Discussion and Numerical Illustration

There are several interesting points which are worth being mentioned here.

1. Apart from damping each term in the infinite sums (4.36) through (4.39) depends on θ and ϕ only through the functions $sin2\tau\theta$ or $cos2\tau\theta$ and $sin2\nu\phi$ or $cos2\nu\phi$ correspondingly. This is a consequence of the antisymmetric beam-beam force, which is assumed in the present work.

2. The nonlinear character of the dependence of the terms in the sums W on sines and/or cosines, produces all kinds of the nonlinear resonance enhancements in the beam blowup. The condition for nonlinear resonance of the $(m+k)$th order for an imperfect ring is as follows:

$$2\nu Bm + 2\tau Bk = \ell \tag{5.1}$$

   where m, k, and ℓ are any positive or negative integers. An infinite number of these resonances are positioned on each of the resonance lines (5.1). The sums W represent the result of simultaneous action of all such resonances.

3. An ideal symmetric lattice with B identical superperiods and B interaction points does not differ from a lattice built out of one superperiod and with only one interaction point. Hence for the symmetric ring without errors formula (4.33) should be (and indeed is) invariant under the following transformation:

$$\frac{\Sigma_y}{\sigma_y}(\nu, \alpha, \tau, \delta, B) = \frac{\Sigma_y}{\sigma_y}(B\nu, B\alpha, B\tau, B\delta, 1) \ . \tag{5.2}$$

4. Due to the damping of the oscillations, the blowup appears to be finite even when the perturbed tunes ν and τ are exactly on one of the resonance lines (5.1). Still the magnitude of the blowup at such a point should not be considered to be strictly correct since here breaks the validity of the perturbation theory.

5. Formula (4.33) explicitly depends on the damping rates α and δ, but its construction (especially the form of the sums W_1 and W_2) is such that the result for tunes away from any resonance (at least for an ideal ring) does not depend on the values of α and δ separately (but only on their ratio α/δ). The reason for this is the following. Summations in formulae (4.36) through (4.39) result effectively in the appearance of resonant denominators, in which the damping constants enter as quadratic terms. Away from any resonance such terms are negligibly small in comparison to the term depending on the distance to the resonance, since we assume that $\delta \ll \tau$ and $\alpha \ll \nu$. These inequalities are usually well fulfilled.

Since the zeroth term in (4.36) is taken with the weighting factor 1/2, the sums W_1 and W_2 are proportional to δ. A more detailed discussion of such behavior of the sums like (4.36) can be found in Part II of this work.[2] In regions around the resonance lines (5.1), where the approximation is not valid anyway, the magnitude of the blowup does depend on the damping rates.

The dependence of the beam blowup, not on the damping rates α and δ separately but only on their ratio, can be very effectively used to decrease the computation time[10] needed to perform summations in (4.36) through (4.39).

6. Formula (5.1) once more implies the importance of the machine imperfections in the problem of the beam-beam instability — the fact understood as the result of computational studies.[11] Indeed, the resonant structure of the beam-beam blowup is richer for the machine with imperfections. Expression (4.33) allows one to take into account several causes of the breaking of the symmetry of the storage ring: differences in betatron phase advances per superperiod, β-asymmetries and asymmetries in bunch currents.

To give an idea of the results which might be obtained by application of the derived formula, I present here (as an illustration only) the results of calculation for the current PEP configuration. Since no imperfections of the machine are included, the results cannot be compared to an experiment.

Table I

The unperturbed nominal PEP parameters used in the numerical example, presented in Figs. 1-7.

Particle energy	14.5	GeV
Full beam current (strong beam)	24.0	ma
Value of the amplitude function at interaction point		
horizontal plane	3.0	m
vertical plane	0.11	m
Coupling factor (the ratio of vertical to horizontal emittances)	0.01	
Number of interaction points	6	
Horizontal tune per superperiod	3.545	
Vertical tune per superperiod	3.032	

Let us first look on the tune dependence at given values of all other machine parameters, in particular of the strong beam current. Figures 1 through 4 present the beam blowup (i.e., the ratio of the vertical rms size of the bunch perturbed by the beam-beam interaction to the unperturbed rms size) in function of the unperturbed vertical tune for several different values of the unperturbed horizontal tune. One can clearly see the resonant regions where the blowup actually occurs. Several main resonances are identified with the help of the horizontal and vertical perturbed tunes, dependence of which on the unperturbed values are presented in Figs. 5 and 6, correspondingly. Remember though, that the beam blowup at each point is the result of the simultaneous action of the infinite number of resonances which appear at the same place.

Next, Fig. 7 illustrates the dependence of the beam blowup on the beam current for one particular point of the tune diagram. The rising branch of the curve is a natural one and it is easy to understand. The falling branch needs an explanation since it is never observed in real life. One can attribute absence of the the decrease in the blowup magnitude with the current increase to several reasons. The most obvious one is the negligence of the coherent beam-beam instability.[12–14] It produces two main effects: a) creates additional unstable regions for the tune values, depending in particular on the number of bunches and b) offsets the bunches at the interaction points breaking thus the assumption of the head-on collision. The machine imperfections neglected here should produce much more dense mash of the resonances, especially close to the half-integer to where the tune is shifted with the increase of the current. That also can eliminate the falling branch of the blowup.

At last it is not excluded that the decrease in the blowup might be connected to the failure of the perturbation treatment used in present work. Indeed, both space charge parameters grow with the increasing current. This point deserves serious attention and additional study.

In conclusion it might be said that the derived approach presents the first quantitative treatment of the beam-beam interaction, at least for the case of not too large current.

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References

1. S. Kheifets, Application of the Green's Function Method to Some Nonlinear Problems of an Electron Storage Ring. Part I – The Green's Function for the Fokker-Planck Equation, SLAC-PEP Note 372, July 1982.
2. S. Kheifets, Application of the Green's Function Method to Some Nonlinear Problems of an Electron Storage Ring. Part II – Checking the Method by a Quadrupole Perturbation, SLAC-PEP Note 377, October 1982.
3. S. Kheifets, Application of the Green's Function Method to Some Nonlinear Problems of an Electron Storage Ring. Part III – Beam Size Enhancement Due to the Presence of Nonlinear Magnets in a Ring, SLAC/AP-2, January 1983.
4. A. Chao, A Summary of Some Beam-Beam Models, Nonlinear Dynamics and the Beam-Beam Interaction, American Institute of Physics Conference Proceedings, Vol. 57, eds., M. Month and J. Herrera, New York, 1979, p. 42.
5. S. Kheifets, Experimental Observations and Theoretical Models for Beam-Beam Phenomena. Long-time Predictions in Dynamics, eds., C. W. Horton, Jr., L. E. Reichl and A. G. Szebehely, John Wiley and Sons, 1982, p. 397.
6. J. F. Schonfeld, The Effects of Beam-Beam Collisions on Storage Ring Performance – A Pedagogical Review, Fermi National Accelerator Laboratory, Fermilab-Conf-83/17-THY, January 1983.
7. E. Courant and H. Snyder, Ann. Phys. 3, 1-48 (1958).
8. B. W. Montague, Calculation of Luminosity and Beam-Beam Detuning in Coasting-Beam Interaction Regions, CERN/ISR-GS/75-36 (1975). For the beam-beam force representation in the form used here, see S. Kheifets, Ref. 5, p. 398.
9. See A. Chao, Ref. 4, and the references on this subject therein.
10. A. D. Thrall advised me this technique, see also Ch. N. Moore, Summable Series and Convergence Factors, Dover, New York, 1966.
11. A. Piwinski, Computer Simulation of the Beam-Beam Interaction DESY 80/131, December 1980; A. Piwinski, Dependence of the Luminosity on Various Machine Parameters and Their Optimization at PETRA, DESY 83-028, April 1983.
12. A. W. Chao and E. Keil, CERN-ISR-TH/79-31 (1979).
13. E. Keil, Nucl. Instr. Meth., 188, 9 (1981).
14. E. Keil, CERN-LEP-TH/83-19 (1983).

15. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, 1980 (especially, formula 2.261, p. 81).

APPENDIX A

Integrals of the Green's Function Weighted by a Gaussian Exponent

In this Appendix I present the results of evaluation for the integrals of the type:

$$I_{mn}(q,u,u',\theta) = \iint_{-\infty}^{\infty} dv dv' v^m (v')^n \; e^{-qv^2} \; G(v,v',u,u',\theta) \quad , \tag{A.1}$$

where m, n are integers, q is a constant and $G(v, v', u, u', \theta)$ is the Green's function of the Fokker-Planck equation,[1] representation of which one can find in Appendix A of Part I. Since G is proportional to exponent of a polinomial of the second order (both in v, v' and u, u'), the integral (A.1) is easy to evaluate for any m and n. I tabulate here in terms of Green's function coefficients $A_i(i = 1, \ldots, 10)$ three integrals of the type of (A.1) which are used in the present note. Let us define:

$$I_{mn}(q,u,u',\theta) = \frac{exp(-R_1u^2 - R_2(u')^2 - 2R_3uu')}{\sqrt{1+q/\bar{A}_1}} \times (\delta_0 + \delta_1 u^2 + \delta_2 (u')^2 + 2\delta_3 uu') \quad , \tag{A.2}$$

where

$$R_1 = \frac{q}{1+q/\bar{A}_1} \frac{\bar{A}_7^2}{4\bar{A}_1^2} \quad , \tag{A.3}$$

$$R_2 = \frac{q}{1+q/\bar{A}_1} \frac{\bar{A}_8^2}{4\bar{A}_1^2} \quad , \tag{A.4}$$

$$R_3 = \frac{q}{1+q/\bar{A}_1} \frac{\bar{A}_7 \bar{A}_8}{4\bar{A}_1^2} \quad , \tag{A.5}$$

here

$$\bar{A}_1 = A_1 - \frac{A_3^2}{4A_2} \quad , \tag{A.6}$$

$$\bar{A}_7 = A_7 - \frac{2A_3A_9}{4A_2} \quad , \tag{A.7}$$

$$\bar{A}_8 = A_8 - \frac{2A_3A_{10}}{4A_2} \quad . \tag{A.8}$$

Expressions for $A_i (i = 1, \ldots, 10)$ to be found in Appendix A of Part I. Expressions for $\delta_i (i = 0, \ldots, 3)$ are tabulated in Table A.

Table A

Coefficients δ in the integral I_{mn}.

$m = 0$ $n = 0$	$\delta_0 = 1, \quad \delta_1 = \delta_2 = \delta_3 = 0$
$m = 2$ $n = 0$	$\delta_0 = 1/2\bar{A}_1(1 + q/\bar{A}_1)$ $\delta_1 = \bar{A}_7^2/4\bar{A}_1^2(1 + q/\bar{A}_1)^2$ $\delta_2 = \bar{A}_8^2/4\bar{A}_1^2(1 + q/\bar{A}_1)^2$ $\delta_3 = \bar{A}_7\bar{A}_8/4\bar{A}_1^2(1 + q/\bar{A}_1)^2$
$m = 1$ $n = 1$	$\delta_0 = -A_3/4A_2\bar{A}_1(1 + q/\bar{A}_1)$ $\delta_1 = A_9\bar{A}_7/4A_2\bar{A}_1(1 + q/\bar{A}_1) - A_3\bar{A}_7^2/8A_2\bar{A}_1^2(1 + q/\bar{A}_1)^2$ $\delta_2 = A_{10}\bar{A}_8/4A_2\bar{A}_1(1 + q/\bar{A}_1) - A_3\bar{A}_8^2/8A_2\bar{A}_1^2(1 + q/\bar{A}_1)^2$ $\delta_3 = A_9\bar{A}_8/8A_2\bar{A}_1(1 + q/\bar{A}_1) + A_{10}\bar{A}_7/8A_2\bar{A}_1(1 + q/\bar{A}_1)$ $-A_3\bar{A}_7\bar{A}_8/8A_2\bar{A}_1^2(1 + q/\bar{A}_1)^2$

APPENDIX B

Table of Several Useful Integrals

Here are tabulated some integrals which are used to evaluate integrals $\Phi_i(i = 1,\ldots,4)$ (see Sec. 4). Let us define:

$$f_{mn}(\alpha,\beta,\gamma) = \iint_{-\infty}^{\infty} dudu'u^m(u')^n\, exp(-\alpha u^2 - \beta(u')^2 - 2\gamma uu') \qquad (B.1)$$

and

$$\Delta = \sqrt{\alpha\beta - \gamma^2}\ . \qquad (B.2)$$

Table B

Integrals $f_{mn}(\alpha,\beta,\gamma)$

$m \backslash n$	0	1	2	3
0	π/Δ	0	$\pi\alpha/2\Delta^3$	0
1	0	$-\pi\gamma/2\Delta^3$	0	$-3\pi\alpha\gamma/4\Delta^5$
2	$\pi\beta/2\Delta^3$	0	$\pi(\alpha\beta+2\gamma^2)/4\Delta^5$	0
3	0	$-3\pi\beta\gamma/4\Delta^5$	0	$-3\pi\alpha(\alpha\beta+4\gamma^2)/8\Delta^7$
4	$3\pi\beta^2/4\Delta^5$	0	$-3\pi\beta(\alpha\beta+4\gamma^2)/8\Delta^7$	0

APPENDIX C

Evaluation of Some Integrals With Algebraic Functions in the Limit $\sqrt{p} \ll 1$

The calculation of the weak beam blowup by a strong gaussian beam involves several integrals with algebraic functions with $\sqrt{p} = \sigma_y/\sigma_x$ as a parameter (see Sec. 4). Here I collect relevant integrals.

Let us define:

$$I(A, B, a, b) = \int_0^1 \frac{dq_2}{(A + Bq_2)^{1/2}(a + bq_2)^{1/2}} \ , \tag{C.1}$$

where

$$A = p^2 + p(2 - p)q_1 \tag{C.2}$$

$$B = p(2 - p) + (2 - p)^2 \ q_1 - q_1 \ e^{-2\alpha\phi} \ cos^2\nu\phi \tag{C.3}$$

$$a = 1 + q_1 \tag{C.4}$$

$$b = 1 + q_1(1 - e^{-2\delta\theta} \ cos^2\tau\theta) \ . \tag{C.5}$$

Integral I is tabulated in[15]:

$$I(A, B, a, b) = \frac{2}{\sqrt{bB}} \ \ell n \ \frac{\sqrt{B(a + b)} + \sqrt{b(A + B)}}{\sqrt{aB} + \sqrt{bA}} \ . \tag{C.6}$$

Differentiation by a yields:

$$\frac{\partial I}{\partial a} = \frac{1}{\sqrt{b}} \left\{ \frac{1}{\sqrt{a + b}(\sqrt{b(A + B)} + \sqrt{B(a + b)})} - \frac{1}{\sqrt{a}\,(\sqrt{aB} + \sqrt{bA})} \right\} \tag{C.7}$$

Differentiate this equation once more by b and expand in the power series in $\sqrt{p}$. The lowest order term is:

$$\frac{\partial^2 I}{\partial b \partial a} \simeq \frac{1}{2z^{1/2}q_1^{1/2}a(a + b)^{3/2}} \ , \tag{C.8}$$

where

$$z_1 = 4 - e^{-2\delta\theta} \cos^2 \tau\theta \ . \tag{C.9}$$

Expansion of $\partial I/\partial a$ itself yields:

$$\frac{\partial I}{\partial a} \simeq - \frac{1}{z_1^{1/2} q_1^{1/2} a} \left[\frac{1}{\sqrt{a+b}} - \sqrt{\frac{2p}{az_1}} \right] \ . \tag{C.10}$$

Similarly,

$$\frac{\partial^2 I}{\partial B \partial a} \simeq \frac{1}{2 q_1^{3/2} z_1^{3/2} a \sqrt{a+b}} \ , \tag{C.11}$$

$$\begin{aligned} \frac{\partial^2 I}{\partial A \partial a} \simeq & \frac{1}{2 z_1 q_1^{3/2} a^{3/2} \sqrt{2p}} \left(1 - 2 \sqrt{\frac{2pb}{az_1}} \right) \\ & - \frac{1}{2 z_1^{3/2} q_1^{3/2} \sqrt{a+b} (\sqrt{b} + \sqrt{a+b})^2} \ , \end{aligned} \tag{C.12}$$

and

$$\frac{\partial^2 I}{\partial a^2} \simeq \frac{1}{2 z_1^{1/2} q_1^{1/2} \sqrt{b}} \left[\frac{2}{a^2} - \frac{3}{a^2} \sqrt{\frac{2bp}{az_1}} - \frac{\sqrt{b} + 2\sqrt{a+b}}{(a+b)^{3/2} (\sqrt{b} + \sqrt{a+b})^2} \right] \ . \tag{C.13}$$

The second integration over dq_1 in all the terms is reduced to the integrals collected in Table C. The following identity is helpful at certain places of calculation:

$$\begin{aligned} \frac{1}{2\sqrt{a+b} (\sqrt{b} + \sqrt{a+b})^2} = & \frac{\sqrt{a+b}}{a^2} - \frac{\sqrt{b}}{2a(a+b)} \\ & - \frac{\sqrt{b}}{a^2} - \frac{1}{2(a+b)(\sqrt{b} + \sqrt{a+b})} \ . \end{aligned} \tag{C.14}$$

Table C

Some integrals with algebraic functions
$I(x) = \int_0^1 f(x,q)dq.$

$f(x,q)$	$I(x)$
$(q+x)^{-1/2}$	$2\left[\sqrt{x+1}-\sqrt{x}\right]$
$(q+x)^{-3/2}$	$2\left[\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x+1}}\right]$
$q^{-1/2}(q+x)^{-3/2}$	$\frac{2}{x\sqrt{x+1}}$
$(q+x)^{-5/2}$	$(2/3)\left[\frac{1}{x^{3/2}}-\frac{1}{(x+1)^{3/2}}\right]$
$q^{-1/2}(q+x)^{-5/2}$	$\frac{(4/3)}{x^2\sqrt{x+1}}+\frac{(2/3)}{x(x+1)^{3/2}}$
$q^{-1/2}(1+q)^{-1}(q+x)^{-1/2}$	$\frac{2}{\sqrt{x-1}}\ arctan\sqrt{\frac{x-1}{x+1}}$
$q^{-1/2}(1+q)^{-1}(q+x)^{-3/2}$	$\frac{2}{(x-1)^{3/2}}\ arctan\sqrt{\frac{x-1}{x+1}}-\frac{2}{(x-1)x(x+1)}$
$q^{-1/2}(1+q)^{-2}(q+x)^{-1/2}$	$\frac{(x-2)}{(x-1)^{3/2}}\ arctan\sqrt{\frac{x-1}{x+1}}+\frac{\sqrt{x+1}}{2(x-1)}$

Figure Captions

1. Ratio of the perturbed rms vertical size of the bunch Σ_y to the unperturbed one σ_y versus the unperturbed vertical tune τ_0 per one superperiod of PEP. The unperturbed tune is 3.528 (= 21.17/6). The strongest resonances are identified by the lowest order integers for perturbed vertical τ and horizontal ν tunes (for example, $2\nu - 2\tau = 1$ represents all resonances $m(2\nu - 2\tau) = m$, where m is any integer). The widths of the resonance curves represent the estimate for the upper boundary, i.e., the step size in the increment of the independent viariable. The actual resonance curve might be narrower.
2. The same as Fig. 1, but the unperturbed horizontal tune in 3.537 (= 21.22/6).
3. The same as Fig. 1, but the unperturbed horizontal tune is 3.545 (= 21.27/6).
4. The same as Fig. 1, but the unperturbed horizontal tune is 3.553 (= 21.32/6).
5. The dependence of the perturbed tune (curve 1) and the space charge parameter (curve 2) on the unperturbed tune for the horizontal plane.
6. The dependence of the perturbed tune (curve 1) and space charge parameter (curve 2) on the unperturbed tune for the vertical plane.
7. The beam blowup Σ_y/σ_y as a function of the full strong beam current I in amp.

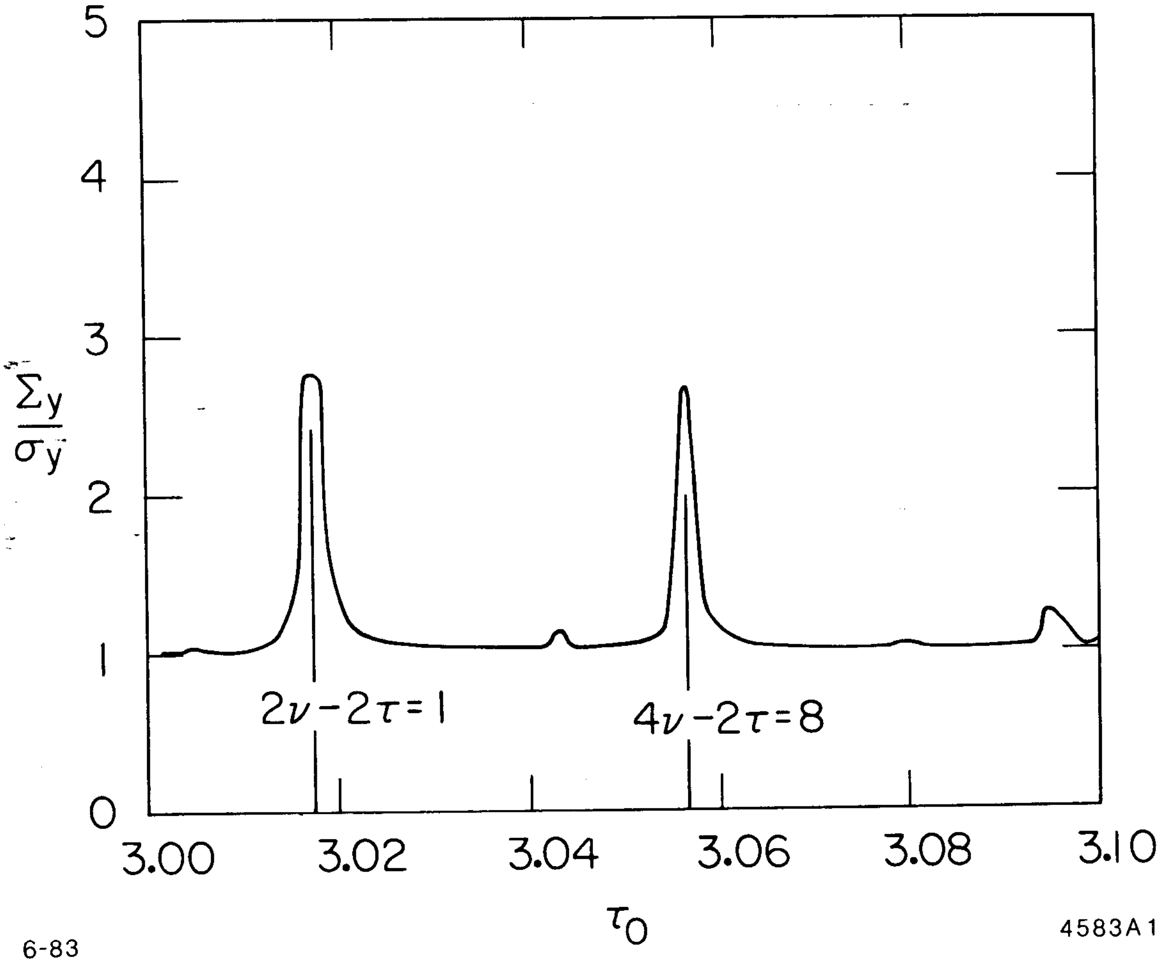
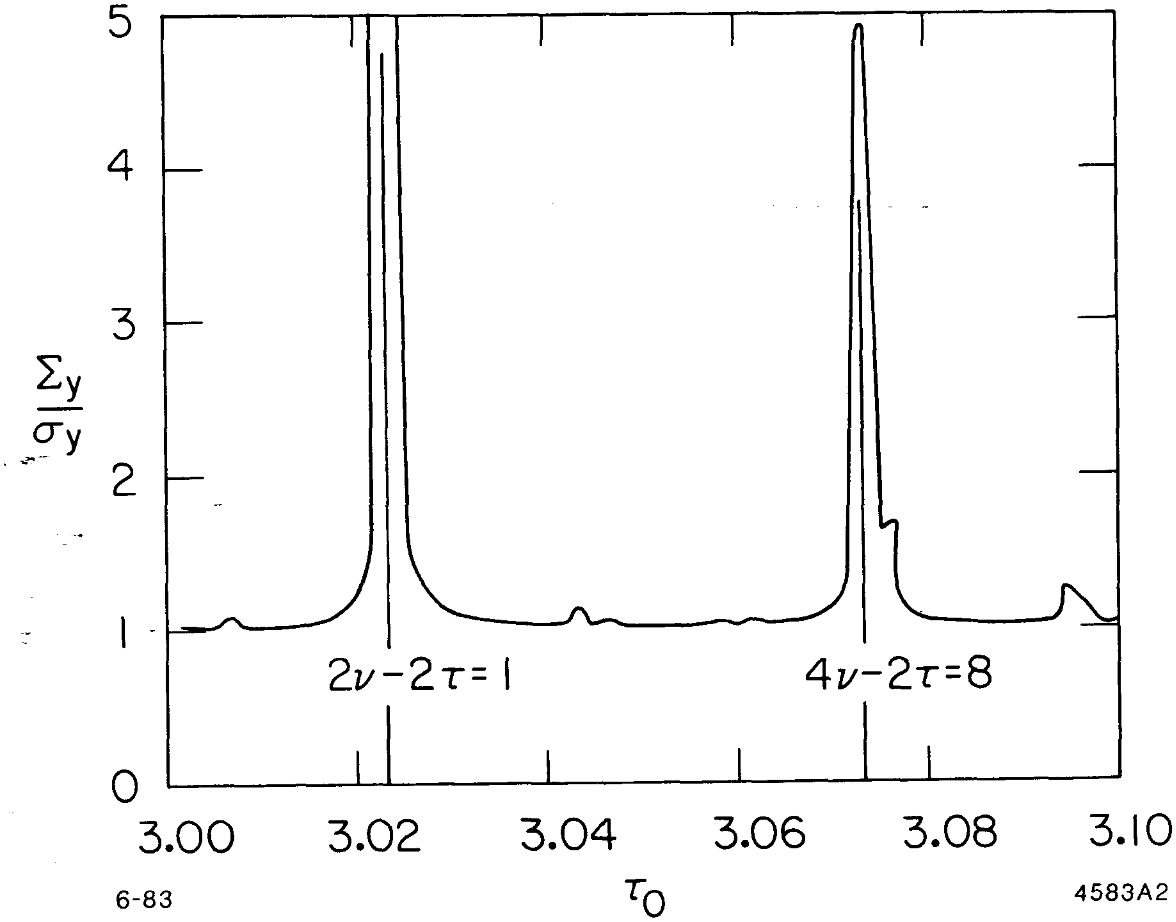
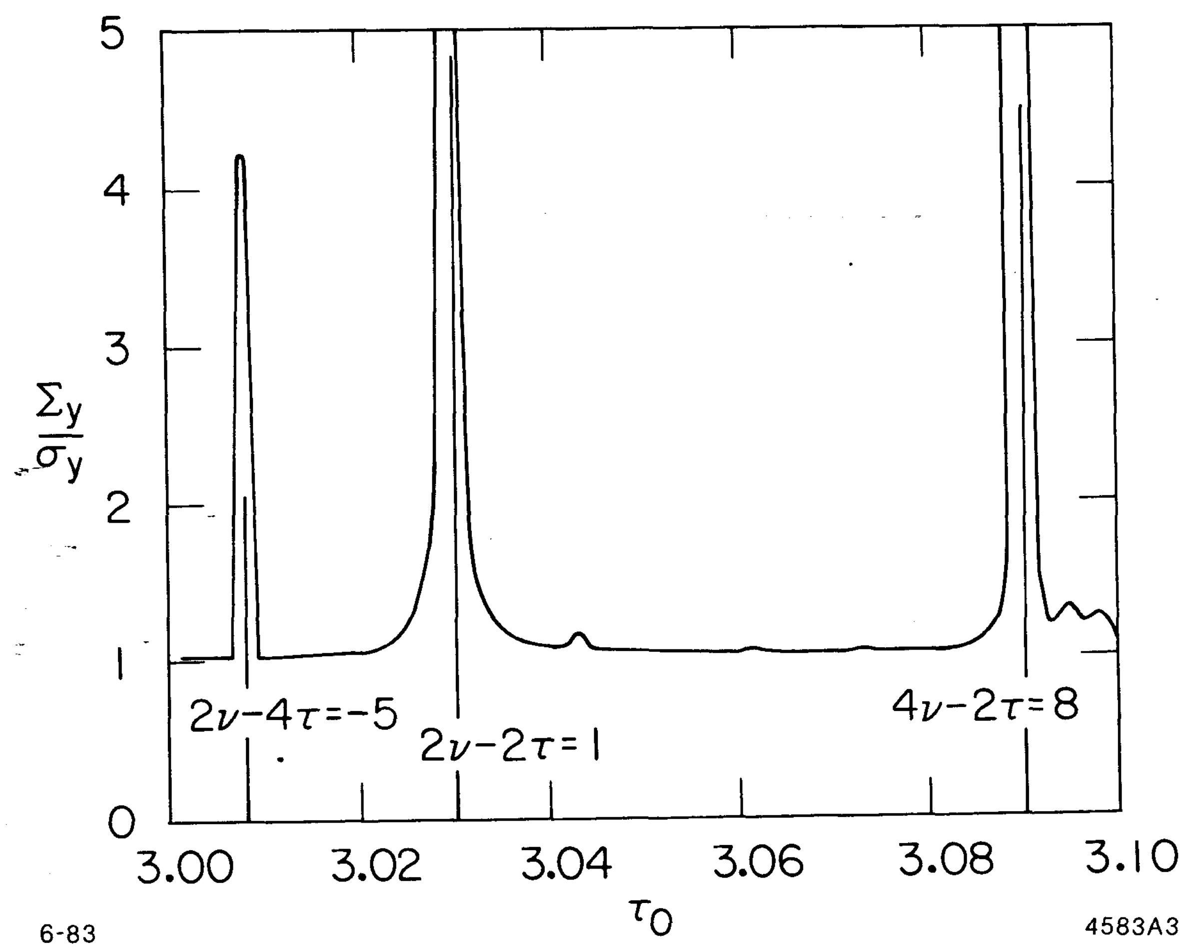


Fig. 1



5
4
3
2
1
0
$\frac{\Sigma_y}{\sigma_y}$
$2\nu-2\tau=1$
$4\nu-2\tau=8$
3.00
3.02
3.04
3.06
3.08
3.10
τ_0
6-83
4583A2

Fig. 2



5
4
3
2
1
0
$\frac{\Sigma_y}{\sigma_y}$
$2\nu-4\tau=-5$
$2\nu-2\tau=1$
$4\nu-2\tau=8$
3.00
3.02
3.04
3.06
3.08
3.10
τ_0
6-83
4583A3

Fig. 3

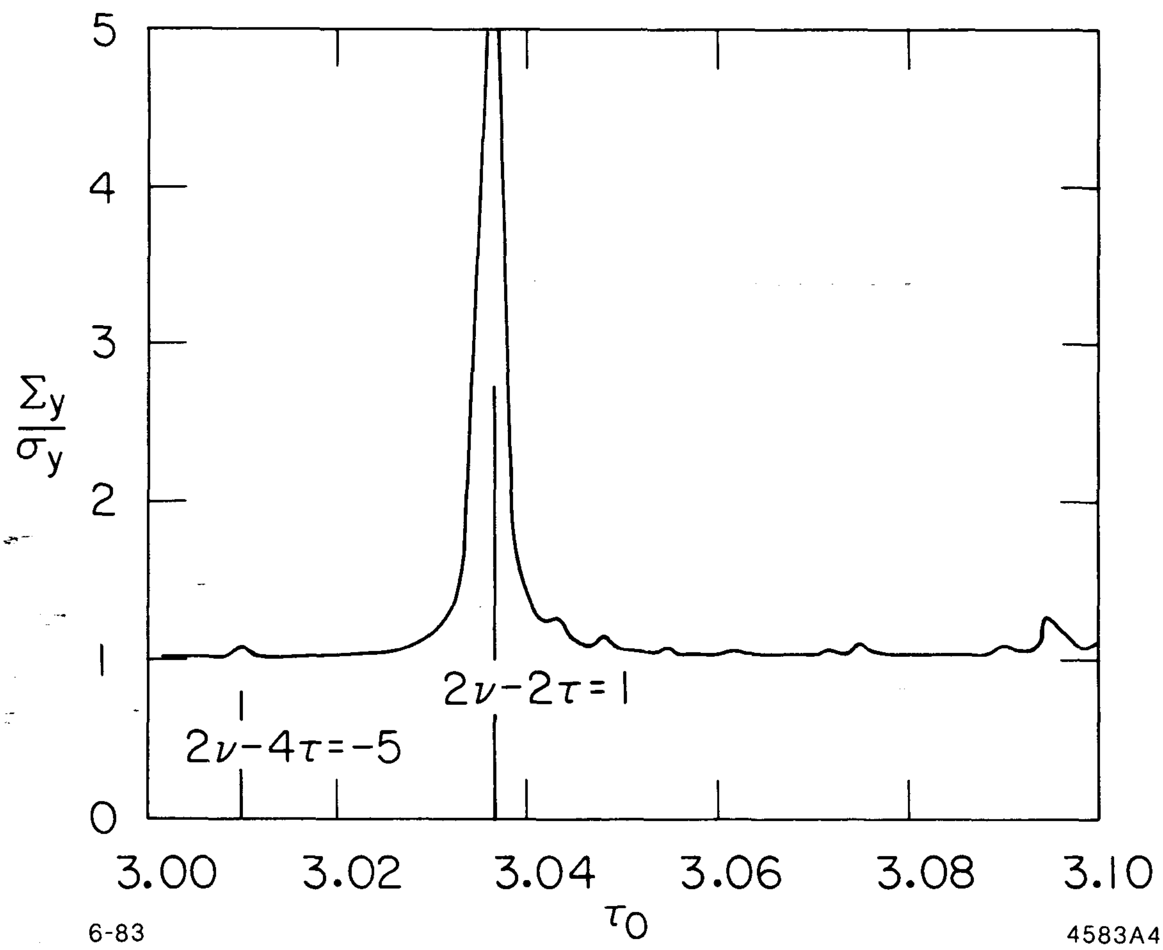
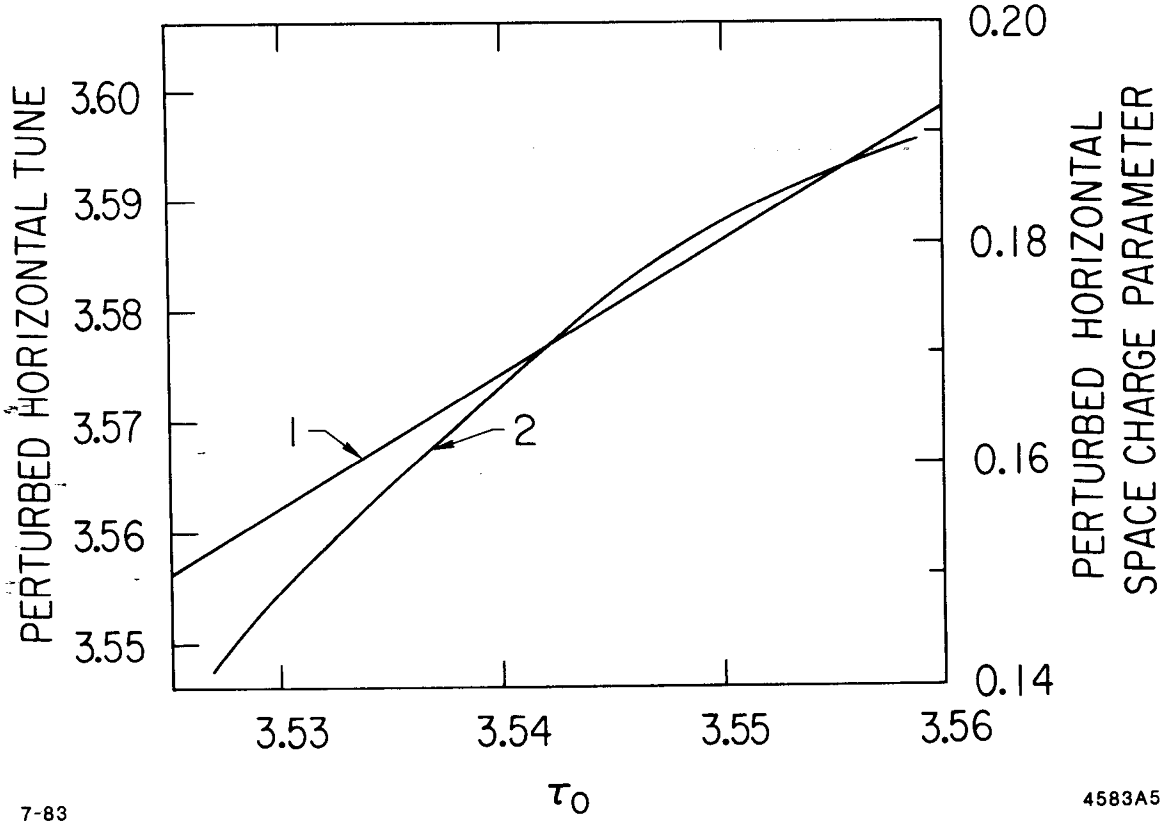


Fig. 4



PERTURBED HORIZONTAL TUNE
3.60
3.59
3.58
3.57
3.56
3.55
PERTURBED HORIZONTAL SPACE CHARGE PARAMETER
0.20
0.18
0.16
0.14
1
2
3.53
3.54
3.55
3.56
τ_0
7-83
4583A5

Fig. 5

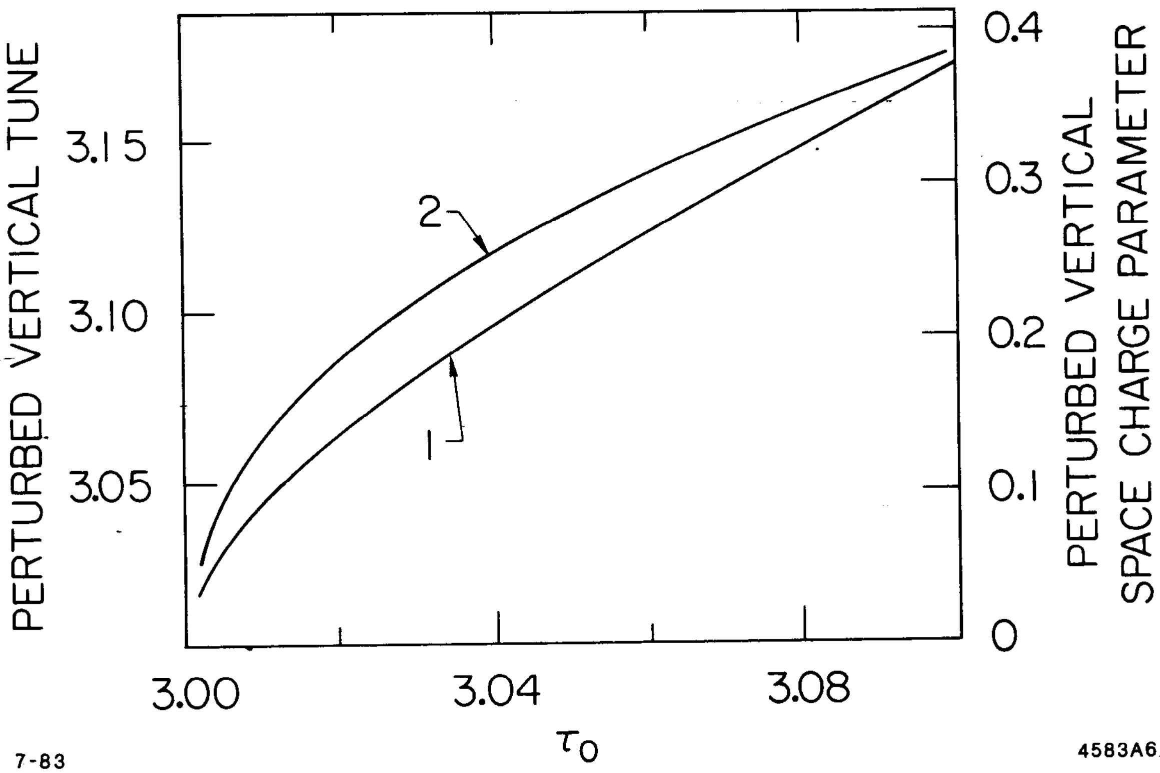

PERTURBED VERTICAL TUNE
3.15
3.10
3.05
0.4
0.3
0.2
0.1
0
PERTURBED VERTICAL
SPACE CHARGE PARAMETER
2
1
3.00
3.04
3.08
τ_0
7-83
4583A6

Fig. 6

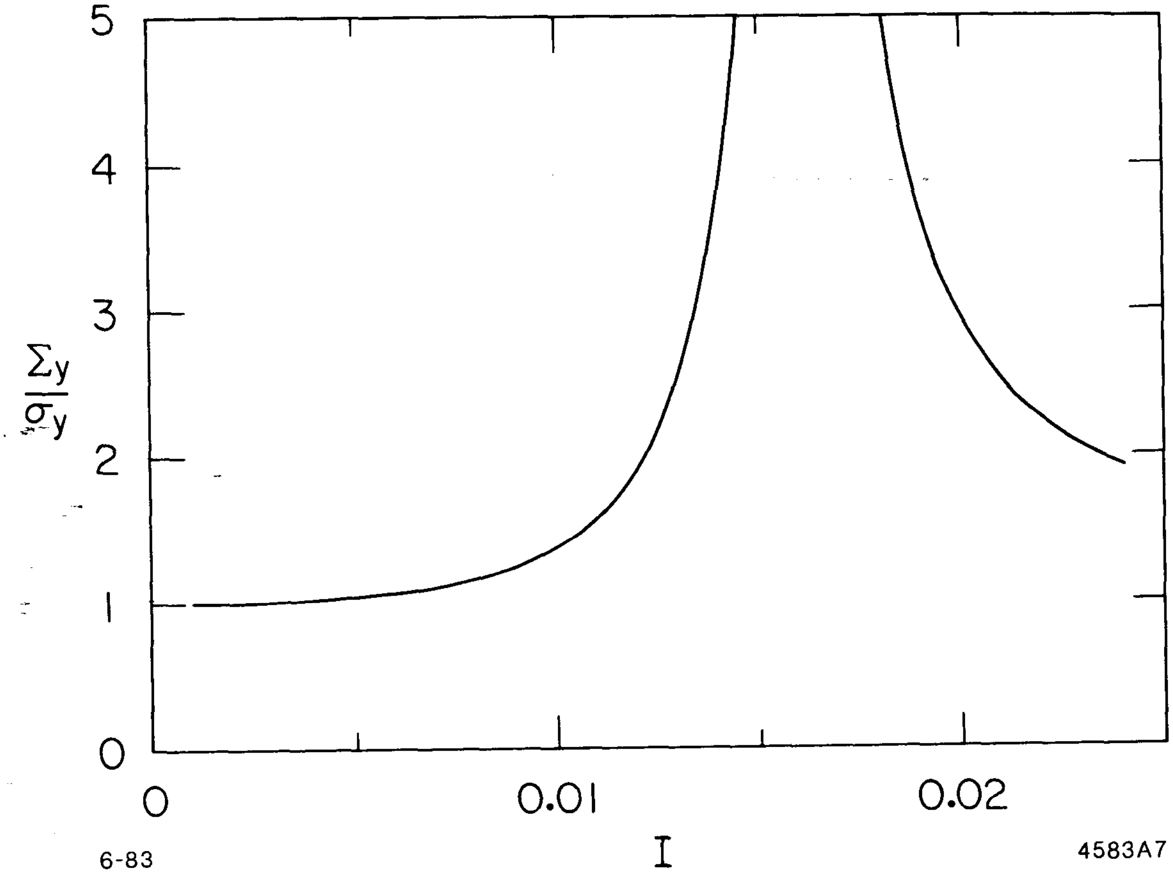


Fig. 7