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PHYSICAL CHARACTERISTICS OF A SELF-DUAL GAUGE FIELD*

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ABSTRACT

Some of the intriguing physical characteristics of the self-dual field of Leutwyler are simply understood in terms of classical physics and some simple analogue quantum mechanical systems. The field's stability properties are clearly illustrated. Also, a surprising $O(4)$ symmetry exists at the classical level, but is broken quantum mechanically via the same physical mechanism as the Bohm-Aharonov effect.

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1. Introduction

Solutions to the classical equations of motion for non-Abelian gauge field theories are potentially of great interest for determining the structure of the physical vacuum. To be of relevance, these solutions should have lower energy density than the trivial perturbative ground state of vanishing field strength, and they should also be stable against quantum fluctuations corresponding to local deformations of the vacuum field. A particularly interesting field configuration which satisfies these criteria has been investigated by several authors [1] for an SU(2) gauge theory in Euclidean space. It consists of a constant self-dual Abelian gauge field given by the vector potential

$$A^a_\mu(x) = -\frac{1}{2}\, F_{\mu\nu}\, x_\nu\, \delta^{a3} \tag{1a}$$

$$F_{\mu\nu} = \frac{1}{2}\, \epsilon_{\mu\nu\alpha\beta}\, F_{\alpha\beta} \quad , \tag{1b}$$

with $F_{\mu\nu}$ a constant matrix. The condition of self-duality, eq. (1b), fixes the field strengths to consist of uniform parallel chromomagnetic and chromoelectric fields.

Some of the intriguing physical characteristics of this self-dual field can be very easily understood in terms of classical physics, with some straightforward comparisons to simple analogue quantum mechanical systems. In particular, it can be seen why the field configuration of parallel chromomagnetic and chromoelectric fields is stable, while either one alone is unstable. Also, this field will be shown to have a surprising $O(4)$ symmetry at the classical level, which is broken quantum mechanically via the same physical mechanism as the Bohm-Aharonov effect [2].

2. Classical Particle Motion in the Background Field

The Lagrangian density for the pure SU(2) theory in Euclidean space is given by

$$\mathcal{L} = -\frac{1}{4}\, F^a_{\mu\nu}\, F^a_{\mu\nu} \quad ,$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc} A^b_\mu A^c_\nu \quad . \tag{2}$$

The classical equations of motion generated from $\mathcal{L}$ are

$$D^{ab}_\mu\, F^b_{\mu\nu} = 0 \quad ,$$

$$D^{ab}_\mu = \delta^{ab}\,\partial_\mu - g\epsilon^{abc} A^c_\mu \quad . \tag{3}$$

As stated in the introduction, the background field configuration of interest that satisfies eq. (3) is given by

$$\bar{A}^a_\mu = -\frac{1}{2}\, \bar{F}_{\mu\nu}\ x_\nu\ \delta^{a3} \tag{4}$$

with $\bar{F}_{\mu\nu}$ a constant matrix.

Rather than impose the condition of self-duality on $\bar{F}_{\mu\nu}$ at this point, first the most general form for the field strength will be employed. When self-duality is later imposed, it can be clearly observed how the interesting properties previously mentioned arise. The most general constant field strength depends only upon the two invariants $\mathcal{F} = (1/4)\, \bar{F}_{\mu\nu}\, \bar{F}_{\mu\nu}$ and $\mathcal{G} = (1/8)\, \epsilon_{\mu\nu\alpha\beta}\, \bar{F}_{\mu\nu}\, \bar{F}_{\alpha\beta}$. An $O(4)$ rotation can always be made to transform the field strength tensor to the form

$$\bar{F}_{\mu\nu} = \sqrt{\mathcal{F}} \begin{pmatrix} 0 & 0 & 0 & \sqrt{1+\alpha} \\ 0 & 0 & \sqrt{1-\alpha} & 0 \\ 0 & -\sqrt{1-\alpha} & 0 & 0 \\ -\sqrt{1+\alpha} & 0 & 0 & 0 \end{pmatrix} \quad , \quad \alpha \equiv \sqrt{1 - \frac{\mathcal{F}^2}{\mathcal{G}^2}} \quad . \tag{5}$$

Note that $0 \leq \alpha \leq 1$, and the self-dual configuration has $\alpha = 0$.

To investigate the classical dynamics generated by this constant field, the Lagrangian density in background gauge [3] is expanded about the solution of eq. (4) using $A_\mu^a = \bar{A}_\mu^a + b_\mu^a$, yielding

$$\mathcal{L} = -\frac{1}{4}\bar{F}_{\mu\nu}\,\bar{F}_{\mu\nu} + \frac{1}{2}\,b_\mu^a\Big(\delta_{\mu\nu}(\bar{D}_\sigma\,\bar{D}_\sigma)^{ac} - 2g\epsilon^{a3c}\,\bar{F}_{\mu\nu}\Big)b_\nu^c + O(b^3) \quad . \tag{6}$$

Using a technique analogous to Schwinger's proper time method [4], the quantum fluctuations, b_μ^a, will be temporarily viewed as "classical particles" with trajectories governed by the effective Lagrangian of eq. (6). In this way, physical intuition can be used to understand the dynamics of the quantum fluctuations which determine stability and symmetry properties of the classical background field, $\bar{F}_{\mu\nu}$. The proper time method corresponds to introducing a time parameter to go along with the four Euclidean "spatial" dimensions implicit in eq. (6). The operator from the quadratic term in eq. (6) is then the proper time Hamiltonian, and can be used to generate the proper time evolution for the coordinates of the "classical particle," b_μ^a. The Hamiltonian is defined by the equation

$$Hb_\mu = \Big(\delta_{\mu\nu}(\bar{D}_\sigma\,\bar{D}_\sigma) - 2ig(T_3)\,\bar{F}_{\mu\nu}\Big)b_\nu \tag{7}$$

where $(T_3)_{ac} = -i\epsilon_{3ac}$. For the background field given by eqs. (4) and (5) it is easily seen that b_μ^3 corresponds to a free particle. However, b_μ^1 and b_μ^2 satisfy the following equations for the coordinates, conjugate momenta $\pi_\mu = i\,\bar{D}_\mu$, and "color" coupling T_3:

$$\dot{x}_\mu = i[H, x_\mu] = 2\pi_\mu \quad , \tag{8a}$$

$$\dot{\pi}_\mu = i[H, \pi_\mu] = 2g(T_3)\,\bar{F}_{\mu\sigma}\,\pi_\sigma \quad , \tag{8b}$$

$$\dot{T}_3 = i[H, T_3] = 0 \quad . \tag{8c}$$

Note that a constant $\bar{F}_{\mu\nu}$ in this proper time formalism corresponds to a magnetostatic field in 4+1 dimensions, since $F_{5\mu} = 0$. This is evidenced by the simple generalization to 4+1 dimensions of the Lorentz force law for a particle moving in a magnetic field, as given by eq. (8b). Further note that, as expected, $d(\pi_\mu \pi_\mu)/d\tau = 0$.

For the arbitrary field strength of eq. (5), the solution of eq. (8) is immediate:

$$\begin{aligned} \pi_0 &= A\cos(\omega_+\tau + \delta_+) & \pi_1 &= B\cos(\omega_-\tau + \delta_-) \\ \pi_3 &= \mp A\sin(\omega_+\tau + \delta_+) & \pi_2 &= \mp B\sin(\omega_-\tau + \delta_-) \end{aligned} \tag{9}$$

$$\omega_\pm \equiv 2g\sqrt{\mathcal{F}(1 \pm \alpha)}$$

where A, B, $\delta_\pm$ are constants determined by the initial velocity of the particle, and the $(\mp)$ in front of π_2 and π_3 depends upon the color charge of the particle, $\langle T_3 \rangle = \pm 1$.

The physical picture of the dynamics is now very simple. There exist two dynamically independent two dimensional subspaces of the 4-d Euclidean space. Each of the two planes has an independent chromomagnetic field in an orthogonal direction, which gives uniform circular particle trajectories. For the above choice of $\bar{F}_{\mu\nu}$, the two independent planes are the 0-3 and 1-2 planes as in fig. 1.

3. Stability of the Field

The known stability of the self-dual field configuration [1] can now be understood in this physical picture. The condition for stability under local fluctuations at one loop level is that the eigenvalues of the quadratic operator in eq. (6) be ≥ 0, i.e. that all fluctuations change the energy by an amount ≥ 0 and are thus damped. This corresponds in our picture to the condition that the particle b^a_μ have total energy ≥ 0.

The particle Hamiltonian of eq. (7) has two parts – the first is proportional to $\bar{D}_\sigma \bar{D}_\sigma$ and generates the center-of-mass motion of the particle as explained in sect.

2, and the second is an interaction energy between the spin chromomagnetic moment of the particle and the background field proportional to $2g\bar{F}_{\mu\nu}$. The spin interaction has the following energy eigenvalues

$$E_{spin} = \pm 2g\sqrt{\mathcal{F}(1+\alpha)} \ , \ \pm 2g\sqrt{\mathcal{F}(1-\alpha)} \quad . \tag{10}$$

Obviously, the spin projection corresponding to an energy eigenvalue of $-2g\sqrt{\mathcal{F}(1+\alpha)}$ has the most destabilizing influence and must somehow be compensated by the contribution from the first term in H which generates the center-of-mass motion. This contribution can be simply determined from our physical picture. The components of the trajectory of the particle in the 0-3 plane respond to an orthogonal chromomagnetic field of strength $2g\sqrt{\mathcal{F}(1+\alpha)}$, generating a circular path. As shown in any elementary quantum mechanics text [5], a particle in a magnetic field has a zero point energy, which in this case would be $g\sqrt{\mathcal{F}(1+\alpha)}$. (The zero point energy arises from localizing the particle trajectories in circular orbits.) Similarly for the 1-2 plane, a zero point enegy of $g\sqrt{\mathcal{F}(1-\alpha)}$ is generated. The sum of the stabilizing zero point energies is compared with the destabilizing spin interaction energy in fig. 2.

It is clear that only the configuration with $\alpha = \sqrt{1-(\mathcal{G}^2/\mathcal{F}^2)} = 0$ has non-negative energy and is stable, which just corresponds to the self-dual case. The apparent paradox of why the configuration with both "chromomagnetic" ($F_{12} \neq 0$) and "chromoelectric" ($F_{03} \neq 0$) fields is stable while either alone is not can now be easily explained. Consider the situation where only $F_{12} \neq 0$. The destabilizing spin interaction exists, as does a zero point energy from localization in only the 1-2 plane. However, the field F_{03} in the dynamically decoupled 0-3 plane can be turned on up to the strength of F_{12} without affecting the magnitude of the spin interaction. This adds a contribution to the zero point energy from localization in the 0-3 plane. In the optimum situation of $F_{03} = F_{12}$, the zero point energy is just enough to cancel the negative spin energy.

4. Classical $O(4)$ Symmetry

Any candidate vacuum field should have the property of rotational symmetry. At the classical level, this means that it should not be possible to single out any direction in space as unique by observing classical particle trajectories. For example, in a region of space with zero field strength, a particle with an initial velocity will continue in a straight line regardless of direction, and all trajectories are connected by a simple rotation – this is a rotationally symmetric situation. An example of a non-symmetric situation is an ordinary magnetic field in three-space where a circular particle trajectory in the transverse plane is not connected by a simple rotation to an unbounded particle trajectory along the magnetic field.

The classical trajectories of particles moving in the self-dual field configuration of sect. 2 exhibit an $O(4)$ symmetry. This is easily illustrated by the coordinate trajectories which come from the equations of notion (8a,b,c) and eq. (9):

$$\begin{aligned} x_0 &= \frac{2A}{\omega}\ \sin(\omega\tau + \delta_+) \qquad & x_1 &= \frac{2B}{\omega}\ \sin(\omega\tau + \delta_-) \\ x_3 &= \pm\frac{2A}{\omega}\ \cos(\omega\tau + \delta_+) \qquad & x_2 &= \pm\frac{2B}{\omega}\ \cos(\omega\tau + \delta_-) \quad , \end{aligned} \tag{11}$$

where an irrelevant coordinate origin is suppressed, $(A, B, \delta_\pm)$ are constants determined by the initial velocity in the 4+1 dimensions, and $\omega = 2g\sqrt{\mathcal{F}}$. A global $O(4)$ rotation can be made on this general trajectory, bringing it to the form

$$\begin{aligned} x_0 &= r\ \sin(\omega\tau) \qquad & x_1 &= 0 \\ x_3 &= \pm r\ \cos(\omega\tau) \qquad & x_2 &= 0 \end{aligned} \tag{12}$$

where $r = 2\sqrt{A^2 + B^2}/\omega$ is just the magnitude of the initial velocity. Thus, two particles with initial velocities of the same magnitude but arbitrarily different directions follow trajectories which are $O(4)$ rotations of one another. Stated another way, a classical particle with an initial velocity in any direction will follow a circular path with a radius dependent only upon the magnitude of the velocity, and period $2\pi/\omega$. This is a most remarkable property for the self-dual field configuration. Now it must be determined if this rotational symmetry remains beyond the classical level.

5. Quantum Mechanics and the Breaking of $O(4)$ Symmetry

The symmetry properties of the self-dual field are investigated beyond the classical level by computing correlation functions of color singlet field operators using the quantum propagators. To facilitate the illustration of the physics of how the classical $O(4)$ invariance is broken quantum mechanically, the spin degrees of freedom of the "particle" b^a_μ will be suppressed, and the color degrees of freedom will be restricted to the fundamental representation of SU(2). The propagator for b^a becomes [6]

$$\left\langle T\left(b^a(x)\, b^c(y)\right)\right\rangle = \left\langle x \left| -\frac{1}{\bar{D}^2} \right| y \right\rangle$$
$$= f\left((x-y)^2, \mathcal{F}^{1/2}\right) exp\left(\frac{i\sigma_3\, g \bar{F}_{\alpha\beta}\; x_\alpha\, y_\beta}{4}\right)_{ac} \quad , \qquad (13)$$

$$\text{with} \qquad f\left((x-y)^2, \mathcal{F}^{1/2}\right) = \frac{exp[-g\mathcal{F}^{1/2}(x-y)^2/8]}{4\pi^2(x-y)^2} \quad .$$

Note that the propagator is rotationally symmetric except for the phase factor which depends upon $\bar{F}_{\alpha\beta}\; x_\alpha\, y_\beta$. The phase factor is essentially a gauge string which originates from the phase information the particle gathers as it passes through the background field. It is apparent that the $O(4)$ rotational symmetry will be broken if there exist color singlet correlation functions sensitive to this phase information.

We first investigate the correlation function of the simple operator $: b^a(x)\, b^a(x) :$ giving

$$\left\langle T\left(: b^a(x)\, b^a(x) : : b^c(y)\, b^c(y) :\right)\right\rangle \sim \left\langle T\left(b^a(x)\, b^c(y)\right)\right\rangle \left\langle T\left(b^c(y)\, b^a(x)\right)\right\rangle$$
$$\sim \left[f\left((x-y)^2, \mathcal{F}^{1/2}\right)\right]^2 \quad . \qquad (14)$$

This is $O(4)$ invariant. The origin of the rotational invariance here is the same as that for the classical trajectory calculations of the previous section, i.e. neither is sensitive to the quantum mechanical phase information. The physical process which corresponds to the correlation function of eq. (14) is illustrated in fig. 3(a). A point source emits particles which pass through a magnetic field and are recorded on a screen. Although each particle's wave function, ψ, carries a phase factor of $exp(i \int_P A \cdot d\ell)$, this information is lost when the probability $\psi^* \psi$ is computed.

In order to find a correlation function that could sample this phase information and break the classical $O(4)$ symmetry, an appeal can be made to a simple well-known physical system. This system is illustrated in fig. 3(b) where two separated coherent sources (a "double slit" configuration) emit particles which pass through a magnetic field and are recorded on a screen. The wave function at the screen has the form

$$\psi \sim \frac{1}{2}\, exp\Big(i \int_{\Delta+P_1} A \cdot d\ell\Big) + \frac{1}{2}\, exp\Big(i \int_{P_2} A \cdot d\ell\Big) \quad . \tag{15a}$$

For small Δ_μ, the probability function is

$$\text{Prob.} = \psi^* \psi \sim 1 - \frac{(B\, dA)^2}{4} \tag{15b}$$

where B is the strength of the magnetic field and dA is the transverse area enclosed by $(\Delta + P_1 + P_2)$. In this case, quantum mechanical phase information makes a physical contribution to the interference pattern, via the Bohm-Aharonov effect [2]. This physical effect suggests the types of analogue correlation functions that could be expected to sample phase information, and thus break the $O(4)$ symmetry. Rather than the bilinear function $: b^a(x)\, b^a(x) :$ investigated previously, the source must be point-split (recall the two slit experiment) in a gauge invariant way. This is easily done by inserting the operator of infinitesimal translations, $\bar{D}_\mu$, into the bilinear function.

A short calculation yields the form

$$\left\langle T\Big(: b^a(x)\, \bar{D}^{ac}_\mu\, b^c(x) : \; : b^d(y)\, \bar{D}^{de}_\nu\, b^e(y) : \Big) \right\rangle \sim (x-y)_\mu (x-y)_\nu\, f_1\Big((x-y)^2, \mathcal{F}^{1/2}\Big)$$
$$+ \bar{F}_{\mu\alpha}(x-y)_\alpha\, \bar{F}_{\nu\beta}(x-y)_\beta\, f_2\Big((x-y)^2, \mathcal{F}^{1/2}\Big) \quad . \tag{16}$$

The first term is obviously $O(4)$ invariant, while for the oriented field $\bar{F}_{\mu\alpha}$, the second is not. To complete the connection with the physical process of fig. 3(b), if eq. (16) is contracted with an infinitesimal displacement vector Δ_μ, the $O(4)$-breaking second term is $\sim (B\,dA)^2$ where dA is the transverse surface area formed by Δ_μ and $(x-y)_\nu$.

Thus, while the self-dual field $\bar{F}_{\mu\nu}$ has an intriguing and (initially) surprising classical $O(4)$ symmetry, the full quantum mechanical theory is not $O(4)$ invariant. It originates in the fact that the rotational non-invariance resides in a phase factor in the quantum propagator, and this phase information is only relevant for true quantum processes.

6. Summary and Conclusions

The interesting properties of the self-dual field described in sect. 1 have been understood using a simple physical picture. First, the classical equations of motion for the quantum fluctuations showed explicitly how stability for the background field arises. The fluctuations are localized in all four Euclidean dimensions giving a zero point energy large enough to cancel the destabilizing coupling of the spin chromomagnetic moment to the background field. Secondly, it is shown that there exists a surprising $O(4)$ symmetry for classical particle motion in the background field. This symmetry however is broken at the quantum mechanical level via gauge field phase information that does not share in the $O(4)$ symmetry.

This analysis allows an intuitive understanding of the relevant dynamics of the self-dual field. While illustrating the interesting features, it also explicitly shows why this configuration in its simple form must be rejected as a candidate physical vacuum field due to its lack of $O(4)$ symmetry.

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FIGURE CAPTIONS

1. Dynamically independent 0-3 and 1-2 planes with orthogonal chromomagnetic fields.
2. —— zero point energy in units of $2g\sqrt{\mathcal{F}}$. – – – destabilizing spin energy in units of $-2g\sqrt{\mathcal{F}}$.
3. (a) Particle traversing transverse B-field and being recorded. (b) Coherent sources producing particles traversing B-field and being recorded.

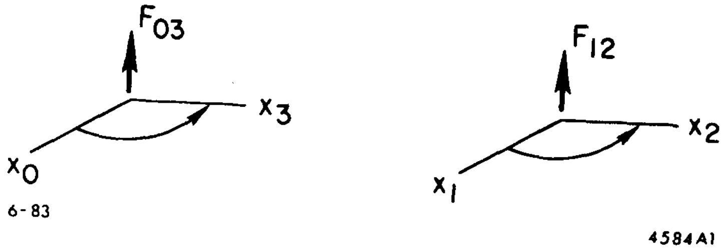


Fig. 1

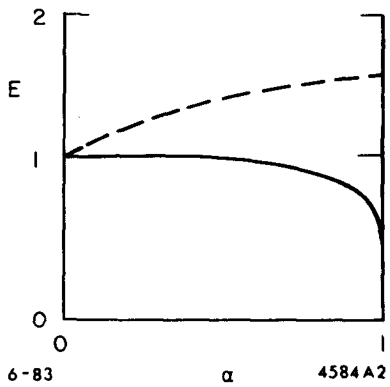
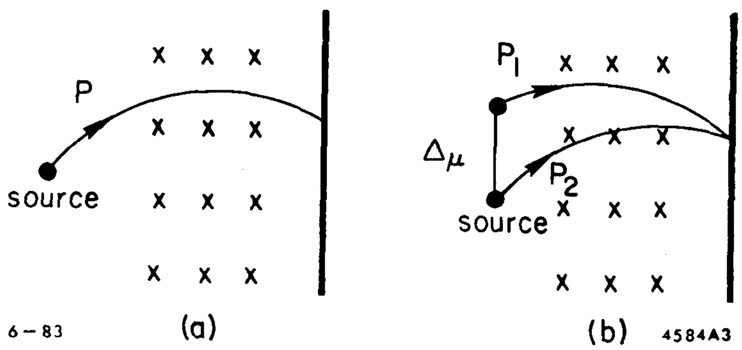


Fig. 2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6–83
(a)
P1
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Δμ
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P2
source
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(b)
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Fig. 3