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CASIMIR ENERGY OF HIGGS FIELD CONFIGURATIONS IN A COLEMAN-WEINBERG-TYPE THEORY*

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ABSTRACT

The Casimir energy, the quantum correction part of the energy, of a class of space-dependent scalar field configurations is examined in a Coleman-Weinberg type $\lambda\phi^4$-theory in the d-dimensional regularization scheme. For the cases when the scalar field is dependent on only one space-coordinate and partially excludes its fluctuations from a region of space, we develop formulae effective for evaluating the full one-loop Casimir energy. As an application, we evaluate a simple case and find that the Casimir energy yields the familiar quantum correction to the volume energy, an extra surface energy whose coefficient is finite for $d \leq 4$, and a finite, exponentially small attractive term. It is shown that the divergences for $d \geq 5$ are due to sharp boundaries of the configuration.

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1. INTRODUCTION

This paper aims to examine the full one-loop Casimir energy for cases when a quantum field is partially excluded from a region of (infinite) space. The system consists of a massless real scalar field with a quadratic coupling (Coleman-Weinberg type). A background field provides a space-dependent mass to its fluctuation through the self-interaction.

This type of the Casimir energy calculation differs from most of the ones found in the literature. Earlier historical examples include the Van de Waals force between electrically neutral conducting objects[1] and Casimir's suggestion[2] for stabilizing the electron in the classical model, a charged conducting sphere. Some years later, Boyer[3] gave an explicit calculation of the latter case and showed that unfortunately the Casimir force provides a repulsive force and thus fails to stabilize the classical electron. A more recent application of the Casimir energy is found in the bag model of hadrons.[4,5] The zero-point energies of the gluon and quark fields perfectly confined in a bag are expected to explain a part of the potential (1/bag radius) necessary for phenomenology. There arises a problem of UV divergences. Although some regularization schemes[6] can be devised to remove the divergences, the real issue of how they should be dealt with physically remains unsolved.[7] This situation becomes somewhat clearer when one includes the confining (or excluding) force as a part of the dynamics, since our knowledge of the renormalization procedures in the local field theory should allow us to distinguish the divergences intrinsic to the theory from the divergences due to the special features of the configuration we deal with. An example of such a calculation is found in the two-dimensional soliton theory.[8] The classical solution gives space dependent mass to its fluctuations (it imperfectly confines the fluctuation). The Casimir energy, which gives a soliton mass correction, has been calculated and the UV divergences have been shown to be removed by the renormalization in the original theory. For more general configurations, investigations have been possible only at the lowest nontrivial order of the Feynman graph, or "multiple scattering" expansion.[9] In contrast, this paper deals with the full one-loop features of the Casimir energy, while having the "excluding force" as a part of the dynamics.

Besides the interests mentioned above, the type of investigation given in this paper may be of practical interest in the physics of the very early universe.[10] The current "new inflationary scenario" is based on the dynamics of Coleman-Weinberg type Higgs fields. The flatness of the effective potential around its origin supports the inflation in each fluctuation region, one of which then develops to our whole universe. The perturbations of the Higgs field in each fluctuation regions are the sources of our current inhomogeneity such as galaxies and clusters. People have calculated the spectrum of the perturbations using the effective potential in various models. For example, in the simple GUTs, the scale dependence is in agreement with the observation but the strength is not. One then naturally asks what is the validity of using only the quantum correction to the effective potential but not the momentum dependent part of the effective action. For a completely consistent analysis of the spectrum, the quantum analysis of the finite size objects in a Coleman-Weinberg type theory is necessary. Therefore we believe that although the investigation of the paper is not directly applicable it could be a small step towards it.

In the next section, we give some preliminaries including the method of the Casimir energy calculation using the phase shifts of the fluctuations. Section 3 gives an expansion of the phase shifts which is useful in evaluating the Casimir energy. Applying these techniques, we present a calculation for an Higgs field with sharp boundaries in Sec. 4. Section 5 gives the discussion of the nature of the divergences that appear in the Casimir energy calculated in the preceding section. The conclusions are given in Sec. 6. Even though the theory is of little physical interest unless the space-time is four-dimensional, we discuss the results in various dimensional space-times to clarify the divergence structures.

2. THE PRELIMINARIES

In this paper, we study the simplest Coleman-Weinberg type theory within the d-dimensional regularization scheme: We have a real scalar Higgs field with a tree-level potential

$$V(\phi) = \frac{\lambda}{4!}\ \phi^4 \tag{2.1}$$

The unrenormalized one-loop effective potential is given as follows (in Euclidean notation),

$$\Delta V(\phi_c) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d}\ \ell n\Big(\sum_{i=1}^{d} p_i^2 + \frac{\lambda}{2}\ \phi_c^2\Big) = -\frac{1}{2}\ \frac{\Gamma(-d/2)}{(4\pi)^{d/2}} \left(\frac{\lambda \phi_c^2}{2}\right)^{d/2} \quad . \tag{2.2}$$

The above calculation is essentially a summation over all the zeroth-order zero-point energies of the excitation modes $\tilde{\phi} = \phi - \phi_c$ on the constant Higgs field background ϕ_c. In fact, by performing the integration over d_1-dimensional "time" components and letting $d_1 \to 1$, we obtain the following,

$$\begin{aligned} \Delta V(\phi_c) &= \frac{1}{2}\Big(-\frac{\Gamma(-d_1/2)}{(4\pi)^{d_1/2}}\Big) \int \frac{d^{d-d_1} p}{(2\pi)^{d-d_1}} \Big(\sum_{i=1}^{d-d_1} p_i^2 + \frac{\lambda}{2}\ \phi_c^2\Big)^{d_1/2} \\ &\underset{d_1 \to 1}{\longrightarrow} \frac{1}{2} \int \frac{d^{d-1} p}{(2\pi)^{d-1}} \sqrt{p^2 + \frac{\lambda}{2}\ \phi_c^2} \quad , \end{aligned} \tag{2.3}$$

where $p^2 \equiv \sum_{i=1}^{d-1} p_i^2$. Since with the potential (2.1) the fluctuation $\tilde{\phi}$ has $(\text{mass})^2 = \lambda \phi_c^2/2$, (2.3) gives the first order quantum correction to the energy as,

$$\Delta E = (d-1\ \text{ dimensional volume}) \cdot \Delta V(\phi_c) \quad . \tag{2.4}$$

The Casimir energy is the sum of the zero point energies of the excitation modes in a space-dependent background field $\phi_c(x)$. The calculation is simplified when ϕ_c is "membrane-like," that is, ϕ_c depends on only one space coordinate, say x_1, and not on other $d-1$ coordinates. In such a case, the excitation modes are labeled by the "momenta" $p_1 \sim p_{d-1}$. The Casimir energy is expressed as

$$E_c[\phi_c(x_1)] = L_p^{d-2} \int \frac{dp_p^{d-2}}{(2\pi)^{d-2}} \int dp_1 \, \rho_{[\phi_c]}(p_1) \cdot \frac{\sqrt{p^2}}{2} \quad , \tag{2.5}$$

where L_ℓ and p_ℓ denote the size of the space and the momenta in the directions $2 \sim d-1$ parallel to the membrane and $\rho_{[\phi_c]}(p_1)$ the level density for the perpendicular momentum p_1.[11] Generally, $\rho_{[\phi_c]}(p_1)$ consists of a discrete part and a continuous part,

$$\rho_{[\phi_c]}(p_1) = \sum_n 2\pi\delta(p_1 - p_{(n)}) + \rho_c(p_1)\, \theta(p_1 - p_{min}) \quad . \tag{2.6}$$

The continuous part ρ_c is known to be related to the phase shifts of the excited states.[8] For completeness, we describe the method below for the cases when $\phi_c(-x) = \phi_c(x)$ and $\lim_{|x_1|\to\infty} \phi_c(x_1) = 0$, while extentions to other cases are straightforward. First, we note that there are no bound states and $p_{min} = 0$ in such cases. We need to calculate the eigenfunctions $f_p(x_1)$ that satisfy the following equation,

$$\left(-\Delta + \frac{\lambda\phi_c^2(x_1)}{2}\right) f_{p_1}(x_1) = p_1^2 \, f_{p_1}(x_1) \qquad (p_1 > 0) \tag{2.7}$$

and have definite parities under $x_1 \to -x_1$. They can be written as follows in the asymptotic regions, $x_1 \to \pm\infty$,

$$f_{p_1}(x) \simeq \begin{cases} \cos(p_1 x_1 \pm \delta_e(p_1)/2) & \text{for even } f_p \quad , \\ \sin(p_1 x_2 \pm \delta_o(p_1)/2) & \text{for odd } f_p \quad . \end{cases} \tag{2.8}$$

The level density is obtained by restricting the whole system to a large "box," $|x_1| \leq L_1/2$, which induces the periodic boundary condition, and then by taking the limit $L_1 \to \infty$: For each of the odd and even sectors, the boundary condition yields

$$p_1 L_1 + \delta(p_1) = 2n\pi \qquad (n = 0, 1, 2, \ldots) \quad , \tag{2.9}$$

which translates into the following level density,

$$dn = \frac{1}{2\pi}\left(L_1 + \frac{d\delta(p_1)}{dp_1}\right) dp_1 \quad . \tag{2.10}$$

Thus, the total level density is given by,

$$\rho_c(p_1) = \frac{1}{2\pi}\left[2L_1 + \frac{d}{dp_1}\left(\delta_e(p_1) + \delta_o(p_1)\right)\right] \quad . \tag{2.11}$$

3. THE PSEUDO-REFLECTION AMPLITUDE EXPANSION

As presented in the previous section, a Casimir energy calculation consists of (i) solving (2.7) to obtain δ's and (ii) evaluating the integral (2.5). Unfortunately, unless one keeps to numerical calculations, there are very few cases which allow (i). Even when δ's are given, we usually encounter difficulties in the procedure (ii). In this section, we give formulae that would help us to deal with both procedures (i) and (ii) for a class of $\phi_c(x)$.

Let us take a symmetric $\phi_c(x)$ that is almost constant ϕ_c for $|x| < (L/2) - (1/\mu)$ and is zero for $|x| > (L/2) + (1/\mu)$; the transition regions near $x = \pm L/2$ have a width $1/\mu \ll L$. We first solve the following equation,

$$(-\partial^2 + m^2(x))\, g_p(x) = p^2\, g_p(x) \quad , \tag{3.1}$$

where the mass term is given by the following,

$$m^2(x) = \begin{cases} m^2 \equiv \lambda\phi_c^2/2 & \text{for } x < -L/2 \quad , \\ \lambda\phi_c^2(x + L/2)/2 & \text{for } x > -L/2 \quad . \end{cases} \tag{3.2}$$

(Hereafter, we omit the subscripts 1 of p_1 when there could be no confusion.) A typical $m^2(x)$ is sketched in Fig. 1. The solutions of (3.1) are L-independent. For $p > m$, we need g_p that has the following asymptotic behavior,

$$g_p(x) \to e^{ip'x} \quad \text{for } x \ll -\frac{1}{\mu} \quad , \quad A_p e^{ipx} + B_p e^{-ipx} \quad \text{for } x \gg \frac{1}{\mu} \quad , \tag{3.3}$$

where $p' \equiv \sqrt{p^2 - m^2}$. When translated by $L/2$, g_p yields a solution of (2.7) for $x > 0$. A solution for $x < 0$ is obtained by reflection, translation by $-L/2$, and complex conjugation. By connecting the resulting solutions at $x = 0$, we obtain a solution of (2.7),

$$exp\Big(i\ \frac{p'L}{2}\Big)\ g\Big(x_1 - \frac{L}{2}\Big)\ \theta(x_1) + exp\Big(-i\ \frac{p'L}{2}\Big)\ g^*\Big(-x_1 - \frac{L}{2}\Big)\ \theta(-x_1)\quad . \tag{3.4}$$

Since the above is $e^{ip'x}$ for $x \sim 0$, the even and odd f_p's are obtained by taking the real and the imaginary parts of (3.4). The phase shifts are found by comparing (3.4) and (2.8),

$$\tan\frac{\delta_e}{2} = \frac{b-h}{a+g}\quad ,\quad \tan\frac{\delta_o}{2} = \frac{b+h}{a-g}\quad , \tag{3.5}$$

where a, b, g and h are real quantities defined as follows,

$$a+ib \equiv A_p\ exp\Big(i\ \frac{p'-p}{2}\ L\Big)\ (\equiv A')\quad ,\quad g+ih = B_p\ exp\Big(i\ \frac{p'+p}{2}\ L\Big)\ (\equiv B')\quad . \tag{3.6}$$

It is now straightforward to show that

$$\tan\frac{\delta_e + \delta_o}{2} = \frac{Im\ (A'^2 - B^{*\prime 2})}{Re\ (A'^2 - B^{*\prime 2})}\quad , \tag{3.7}$$

or

$$\delta_e + \delta_o = -2pL + 2p'L + 2\ arg(A_p^2 - B_p^{*2} e^{-2ip'L})\quad . \tag{3.8}$$

For $p < m$, we need to have two g_p's which behave as follows,

$$g_p(x) \to \begin{cases} e^{-\rho x} & \\ & (x \ll -1/\mu) \\ e^{\rho x} & \end{cases} \quad \begin{matrix} ,\quad A_{+,p}\ e^{ipx} + A^*_{+,p}\ e^{-ipx} \\ ,\quad A_{-,p}\ e^{ipx} + A^*_{-,p}\ e^{-ipx} \end{matrix} \quad (x \gg 1/\mu)$$

where $\rho \equiv \sqrt{m^2 - p^2}$. In this case, similar procedures involving linear combinations of the above two yield the desired definite parity f_p's. The sum of the phase shifts is found to be,

$$\delta_e + \delta_o = -2pL + 2\ arg\{i(-A^2_{+,p}\ e^{-\rho L} + A^2_{-,p}\ e^{\rho L}\}\quad . \tag{3.9}$$

The branches of the arguments in (3.8) and (3.9) should be chosen such that $\delta_e + \delta_o$ is continuous for $0 < p < \infty$. The remaining overall phase is of no physical interest.

Since A's and B's are L-independent, (3.8) and (3.9) are useful for deriving large L formulae. It is easy to see that the *arg* part of (3.8) and (3.9) is $O(L^o)$ for large L. For (3.8), as we increase L, the value of $A_p^2 - B_p^{*2} e^{-2ip'L}$ traces a circle in the complex plane with an angle velocity $-2p'$. Since B/A is a reflection amplitude in the "potential" $m^2(x)$ (Fig. 2), we have

$$\left|\frac{B}{A}\right|^2 < 1 \quad . \tag{3.10}$$

Thus the circle does not include the origin (Fig. 3). Therefore, the *arg* term simply oscillates as L increases. For (3.9), as L increases, only the A-term survives which yield a constant value. This situation is evident in the following identities, which are obtained by using $arg\ z = (i/2)\ \ell n(z/z^*)$ and then expanding the logarithm,

$$arg\left(A^2 - B^{*2}\ e^{-2ip'L}\right) = -i\ \ell n\ \frac{A}{A^*}$$
$$-\frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n}\left\{\left(\frac{B}{A^*}\right)^{2n} e^{2inp'L} - \left(\frac{B^*}{A}\right)^{2n} e^{-2inp'L}\right\}, \tag{3.11a}$$

$$arg\left(i(-A_+^2 e^{-\rho L} + A_-^2 e^{\rho L})\right) = -\frac{i}{2}\ \ell n\left(-\frac{A_+^2}{A_-^{*2}}\right)$$
$$-\frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n}\left\{\left(\frac{A_+^*}{A_-^*}\right)^{2n} - \left(\frac{A_+}{A_-}\right)^{2n}\right\} e^{-2n\rho L} \quad , \tag{3.11b}$$

where the subscript p is omitted for brevity. The convergence of the above series is guaranteed by (3.10). We call (3.11a,b) "pseudo reflection amplitude" expansion, since the expansion coefficients of (3.11a) resembles the reflection amplitude, B/A, of the Schrödinger problem (3.1) (see Fig. 2). The same is true for (3.11b).

The leading L^1 terms in (3.8) and (3.9) have a trivial physical interpretation. They are the major part of the volume energy terms; after substituting (3.8) and (3.9) into

(2.11) and (2.5), we find that

$$\frac{E_c}{L_p^{d-2}} = \int \frac{dp_p^{d-2}}{(2\pi)^{d-2}} \left[(L_1 - L) \int_0^\infty \frac{dp_1}{\pi} + L \int_0^\infty \frac{dp'}{\pi} \right] \frac{\sqrt{p_1^2 + p_p^2}}{2} + e_c$$
$$= (L_1 - L)\, \Delta V(0) + L \Delta V(\phi_c) + e_c \quad , \tag{3.12}$$

where e_c denotes the part that comes from the *arg* terms.[12]

4. THE MEMBRANE WITH SHARP BOUNDARY

As an example of formalism developed in the previous section, we investigate the Casimir energy of the following configuration,

$$\phi_c(x_1) = \phi_c\, \theta\Big(\frac{L}{2} - |x_1|\Big) \quad , \tag{4.1}$$

when the quantized scalar field is massive ($m^2 = \lambda\phi_c^2/2$) in the membrane ($|x_1| < L/2$) and is massless outside. In this case, it is possible to solve (2.7) directly and obtain the following phase shifts,

$$\delta_e(p) = -pL + 2\tan^{-1}\left(\frac{p'}{p}\ \tan\ \frac{p'L}{2}\right) \quad ,$$
$$\delta_o(p) = -pL + 2\tan^{-1}\left(\frac{p}{p'}\ \tan\ \frac{p'L}{2}\right) \quad , \tag{4.2}$$

where for $p < m$, p' should be understood as $i\rho$ (or $-i\rho$, equivalently). The above leads to the following expression for the nonleading term e_c of (3.13),

$$e_c = -\frac{1}{2} \int \frac{dp_p^{d-2}}{(2\pi)^{d-2}} \int_0^\infty \frac{dp_1}{\pi} \frac{p_1}{\sqrt{p_1^2 + p_\ell^2}} \Big\{ f_1(p_1)\, \theta(p_1 - m) + f_2(p_1)\, \theta(m - p_1) \Big\} \ , \tag{4.3}$$

$$f_1(p_1) \equiv \tan^{-1}\left[\frac{1}{2}\left(\frac{p_1}{p'} + \frac{p'}{p_1}\right) \tan\ p'L\right] - p'L \quad ,$$
$$f_2(p_1) \equiv \tan^{-1}\left[\frac{1}{2}\left(\frac{p_1}{\rho} - \frac{\rho}{p_1}\right) \tanh\ \rho L\right] \quad , \tag{4.4}$$

where we have partially integrated p_1 (no boundary terms appear due to the dim. regularization scheme).

So far, it has not been necessary to use formulae given in the previous section. The evaluation of (4.3), however, needs the help of the "pseudo reflection amplitude" expansion. Solving (3.1) for the configuration (4.1), we obtain

$$A = \frac{p-p'}{2p} \quad , \quad B = \frac{p+p'}{2p} \quad , \quad A_\pm = \frac{p \pm i\rho}{2p} \quad . \tag{4.5}$$

Consequently, the f's in (4.4), which are halves of the *arg* terms in (3.8) and (3.9), are expanded as follows,

$$f_1(p) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{p-p'}{m}\right)^{4n} \sin\, 2np'L \quad , \tag{4.6a}$$

$$f_2(p) = \tan^{-1}\left[\frac{1}{2}\left(\frac{p}{\rho} - \frac{\rho}{p}\right)\right] + \sum_{n=1}^{\infty} \frac{1}{n}\, Im \left(\frac{p-i\rho}{m}\right)^{4n} e^{-2n\rho L} \quad , \tag{4.6b}$$

where $\tan^{-1}$ term should take its principal value ($-\pi/2 \sim \pi/2$) for the continuity of the phase shifts. Note that f_1 does not have an L^o term of (3.11a) due to A being real. The p_1-integrations of the n*th* terms in (4.6a) and (4.6b) are combined into a complex integral,

$$\int_C \frac{dp_1}{2\pi i} \frac{p_1}{\sqrt{p_1^2 + p_p^2}} \frac{1}{n} \left(\frac{p_1 - p'}{m}\right)^{4n} e^{2nip'L} \quad ,$$

where the contour C is given in Fig. 3. After closing the contour in the upper half-pane and integrating p_p first, we find that

$$(e_c)_{nth} = -\frac{1}{(4\pi)^{(d-1)/2}\Gamma((d-1)/2)n} \int_0^\infty dx\, x^{d-2} \left(\frac{x - \sqrt{x^2+m^2}}{m}\right)^{4n} \times exp(-2nL\sqrt{x^2+m^2}) \quad . \tag{4.7}$$

When resummed over n, the above leads to a compact expression of e_c. Instead, we choose to evaluate e_c for $mL \gg 1$ (note that mL is the only dimensionless parameter for E_c). The above yields

$$(e_c)_{nth} \underset{Lm\to\infty}{\longrightarrow} -\frac{1}{2n}\left(\frac{m}{4\pi nL}\right)^{(d-1)/2} e^{-2nmL}\left[1+O\left(\frac{1}{\sqrt{mL}}\right)\right] \quad .$$

The L^o term or the "zero-reflection" term, in (4.6b) allows the analytical integration for e_c as follows,

$$\begin{aligned}(e_c)_{0th} &= \alpha(d)\int_0^m dp\, p^{d-2}\tan^{-1}\left[\frac{1}{2}\left(\frac{p}{\rho}-\frac{\rho}{p}\right)\right] \quad , \\ &= \alpha(d)\,\frac{1}{d-1}\left[\frac{\pi}{2}-B\left(\frac{1}{2},\frac{d}{2}\right)\right]m^{d-1} \quad ,\end{aligned} \tag{4.8}$$

where $\alpha(d)$ comes from the p_p-integration,

$$\alpha(d) = -\frac{4}{(4\pi)^{(d+1)/2}}\,\Gamma\left(\frac{3-d}{2}\right) \quad .$$

Therefore, we obtain the following,[14]

$$e_c = c(d)\,m^{d-1} - \frac{1}{2}\left(\frac{m}{4\pi L}\right)^{(d-1)/2} e^{-2mL} + \ldots \quad , \tag{4.9}$$

where the L^o-term is

$$c(d) = \frac{2}{(4\pi)^{(d+1)/2}}\,\Gamma\left(\frac{1-d}{2}\right)\left[\frac{\pi}{2}-B\left(\frac{1}{2}\,,\,\frac{d}{2}\right)\right] \quad . \tag{4.10}$$

At first sight, it is a little surprising that the nonleading term in (4.9) is small exponentially rather than in powers of $1/mL$. Therefore, we elaborate on the mathematics of e_c in the rest of this section. (The discussion on the result (4.9), (4.10) is given in the next section.) To assure the correctness of (4.9) and (4.10), we evaluate the two pieces of e_c, one from f_1 and one from f_2, separately. Here again the pseudo reflection amplitude expansion is a great help. First, we do the p_ℓ-integration as in

(4.8). For the f_1 part, we follow the steps; (i) change the integration variable from p to p'; (ii) expand the coefficients of sin $2np'L$ in powers of p'; (iii) integrate each terms; and (iv) sum over n. Since the integration of a $(p')^N$ term yield zero for odd N and is $\propto 1/(nmL)^N$ otherwise, we obtain a $1/mL$ power expansion as follows,

$$e_c(f_1) = \alpha(d)\, m^{d-1} \sum_{n=1}^{\infty} \frac{\alpha_n}{(mL)^{2n+1}} \quad , \tag{4.11}$$

The steps (iii) and (iv) do not yield any divergences. Therefore all the coefficients α_n are finite. In particular, the first two expansion coefficients are given by,

$$\alpha_1 = \varsigma(3) \quad , \quad \alpha_2 = -\frac{3}{2}\,\varsigma(5)d - 8\varsigma(3) + 5\varsigma(5) \quad . \tag{4.12}$$

We have done numerical p_1-integrations without using the expansion (4.6a) for several values of the parameters in the range $d = 2 \sim 5$, $mL = 5 \sim 1000$. The results is in good agreement with the leading A_1-term. For the piece of e_c that comes from f_2, similar procedures (small ρ expansion instead of p' expansion, etc.) yield the following large mL expansion,

$$e_c(f_2) = c(d)\ m^{d-1} + \alpha(d)\, m^{d-1} \sum_{n=1}^{\infty} \frac{\beta_n}{(mL)^{2n+1}} \quad . \tag{4.13}$$

The explicit calculations of the first two coefficients yield

$$\beta_1 = -\alpha_1 \quad , \quad \beta_2 = -\alpha_2 \quad , \tag{4.14}$$

which is consistent with (4.9): We expect $\alpha_n = -\beta_n$ for all n and as a result, the total e_c consists of a L^o-term and the terms that decay faster than any power of mL. We have also done numerical p_1-integrations for (4.13) without using (4.6b) and the results confirmed the β_1 and β_2 terms.

5. DIVERGENCES

The Casimir energy obtained in the previous section contains some divergences. Besides the divergence in the volume energy term, which can be renormalized away, they are in the first nonleading term $c(d)\,m^{d-1}$. From (4.10), we see that $c(d)$ is finite for $d = 2 \sim 4, 6, 8, \ldots$, and is divergent for $d = 5, 7, \ldots.$ due to the p_p-integration. (For $d = 3$, the divergence in the p_p-integration is removed by the vanishing p_1-integration.) These divergences are not intrinsic to the theory but are due to the special feature of the configuration (4.1) we have taken; the vanishing thickness of the boundary at $x_1 = \pm L/2$. They cannot be intrinsic to the theory, because (i) These divergences cannot be renormalized away by any parameters in the Lagrangian. (ii) The $\lambda\phi^4$ theory in the dimensional regularization scheme is finite for $d = 5, 7, \ldots.$ We can also observe that these divergences are associated with the boundary from the fact that they are in the "surface" energy term, i.e., are proportional to the "area" of the surface L_p^{d-2}.

This nature of the divergences is seen in the Feynman graph calculation. We illustrate it here for $d = 5$. Symbolically, the Feynman graphs are generated by the expansion,

$$E_c = \frac{1}{T}\,\frac{1}{2}\;Tr\;\ell n\Big[\partial^2 + m^2(x)\Big] = \frac{1}{T}\left\{\frac{1}{2}\;Tr\;\ell n\;\partial^2 + Tr\;\frac{1}{\partial^2}\;m^2 - \frac{1}{2}\;Tr\;\frac{1}{\partial^2}\;m^2\;\frac{1}{\partial^2}\;m^2 + \ldots\right\}\;. \tag{5.1}$$

The first two terms of (5.1) are zero in the dim. regularization scheme. For the configuration (4.1), the second term yields,

$$E_c^{(2)} = L_\ell^{d-2}\;F(d)\;\int_0^\infty\;dk\;k^{d-6}\;\sin^2\frac{kL}{2}\quad, \tag{5.2}$$

where $F(d)$ comes from the loop-momentum integration and is finite for odd d. The k-integrand receives k^{d-4} from the loop integral and the rest from the absolute square of the Fourier transform of $m^2(x)$. The k-integral has logarithmic UV divergence for $d = 5$.

$$E_c^{(2)} \propto -\frac{1}{2(d-5)} + \frac{1}{2}\,\ell n\, L + \text{constants} + O(d-5) \quad . \tag{5.3}$$

This divergence is clearly due to the sharp boundary. Imagine that the boundary is smoothed to the thickness of order $1/\mu$ ($\ll L$). In such a case, the Fourier transform of $m^2(x)$ decays more rapidly for $k/\mu \to \infty$. Then the pole term is replaced by $(\ell n \mu)/2$ to yield a finite result,[14]

$$E_c^{(2)} \propto \frac{1}{2}\,\ell n\, \mu L + \ldots \quad . \tag{5.4}$$

Unfortunately, the Feynman graph expansion (5.1) corresponds to the small mL expansion and most of the terms are infected by the mass singularities. Therefore, it is not possible to extend above argument systematically and prove the fake nature of the divergence in $c(d)$. The above observation, however, strongly supports our argument on $c(d)$. Furthermore, the identification of the pole with the logarithmic singularity of the thickness leads to the following finite expression,

$$\begin{aligned} c(d)\, m^{d-1} \underset{d\sim 5}{\sim} & -\frac{1}{512\pi^2}\, m^4 \left(\frac{2}{d-5} + \ell n\, m^2 + \ldots\right) \\ \to & -\frac{1}{512\pi^2}\, m^4 \left(\ell n\, \frac{m^2}{\mu^2} + \ldots\right) \quad , \end{aligned} \tag{5.5}$$

for the surface term.

For $d = 7, 9, \ldots$, similar procedure gives the ℓn terms. However, the highest order terms of μ/m are hidden by the dim. regularization scheme, which picks up only the ℓn divergent terms. Our result that the ℓn term first appears for $d = 5$ and then for odd d's, while the degree of divergence is expected to monotonically increase as d increases, suggests that the real leading terms of the surface energy for $d > 5$ is as follows,

$$m^{d-1}\left[*\left(\frac{\mu}{m}\right)^{d-5}+*\left(\frac{\mu}{m}\right)^{d-7}+\ldots\right] \quad , \tag{5.6}$$

where $*$'s denote unknown coefficients. This can be seen by using the heat-kernel method, or proper-time formulation as follows. Our Casimir energy is, to within an additive constant,

$$E_c=-\frac{1}{2}\int_0^\infty \frac{ds}{s}\, Tr\, exp\left[-s\left(\Box+\frac{\lambda}{2}\,\phi_c^2(x)\right)\right] \quad . \tag{5.7}$$

The logarithmic divergence at $s=0$ is subtracted by the unwritten constant, which however affects only the volume energy term and thus is irrelevant for our discussion of the surface energy term. Using the p_1 level density ρ, the above is,

$$\begin{aligned} E_c &= -L_\ell^{d-2}\,\frac{1}{2}\int_0^\infty \frac{ds}{s}\int \prod_{i=2}^{d}\frac{dp_i}{2\pi}\int_0^\infty dp_1\,\rho(p_1)\,exp(-sp^2) \\ &= -L_\ell^{d-2}\,\frac{1}{2(4\pi)^{(d-1)/2}}\int_0^\infty \frac{ds}{s^{(d+1)/2}}\int_0^\infty dp_1\,\rho(p_1)\,exp(-sp_1^2) \quad . \end{aligned}$$

Our surface energy came from the integration of the phase shifts for $p_1 < m$ [see (4.8)]. In the proper-time formalism, it is proportional to the following,

$$\int_0^\infty \frac{ds}{s^{(d+1)/2}}\int_0^m dp_1\,\tan\left[\frac{1}{2}\left(\frac{p_1}{\rho}-\frac{\rho}{p_1}\right)\right](-2sp_1)\,exp(-sp_1^2) \quad . \tag{5.8}$$

For small s, the p_1 integration is of $O(s)$, corresponding to the fact that the p-integral in (4.8) vanishes at $d=3$. The above (5.8) is

$$\int_0^\infty \frac{ds}{s^{(d+1)/2}}\,(-2s)\cdot\left(-\frac{\pi}{32}\,sm^4+O(s^2m^6)\right)=\int_0^\infty \frac{ds}{s^{(d-3)/2}}\left(\frac{\pi}{16}\,m^4+O(sm^6)\right) \quad . \tag{5.9}$$

This is divergent for $d\geq 5$ in agreement with the result of our dimensional calculation, (4.10). We also see that these divergences are due to the sharp boundary, since

$s \sim 0$ corresponds to short distances in space-time as can be seen in the following free, proper-time propagator,

$$\langle y|e^{-s\Box}|x\rangle = \frac{1}{(4\pi s)^{d/2}}\ exp\Big(-\frac{(x-y)^2}{2s}\Big) \quad .$$

The integral (5.9) can be regulated by a small cut-off ϵ. (Remember that d is a mere integer here, not a variable for analytical continuation.) Then, the leading divergences in (5.9) are, for $d > 5$,

$$\frac{\pi}{32}\ \frac{m^4}{\epsilon^{(d-5)/2}} + *\ \frac{m^6}{\epsilon^{(d-7)/2}} + \cdots \quad .$$

When the boundaries are smooth and have thickness $1/\mu$, we expect $\epsilon \propto 1/\mu^2$. Therefore (5.7) is reproduced.

6. CONCLUSIONS

We have developed a formalism suitable for the analysis of the Casimir energy of Higgs field configurations which are (i) dependent only on one space coordinate ("membrane" type); (ii) zero in the asymptotic region of space; and (iii) constant (ϕ_c) in the central region. We obtained an expansion of the Casimir energy which coefficient functions are closely related to the reflection amplitude of the Higgs fluctuations outside the membrane. The expansion automatically separates the volume energy term, thereby locating major UV divergences that allows usual renormalization for the effective potential.

We have used these techniques for the analysis of a configuration of thickness L with sharp boundaries, (4.1), when the quantized field is massive inside the membrane and is massless outside. We have found that in the d-dimensional space-time the total energy, including the classical part, is given by the following for $mL \gg 1$ (remember $m^2 \equiv \lambda\phi_c^2/2$),

$$E = E_{kinetic} + L_\ell^{d-2}\ L\ V_{eff}(\phi_c) + L_\ell^{d-2}\ c(d)\ m^{d-1} + \ldots \quad . \tag{6.1}$$

where we have adjusted the energy of the vacuum $\langle\phi\rangle = 0$ to zero. In the above, we find that the Casimir energy yields the usual quantum correction to the effective potential, which gives the volume energy, and a new "surface" energy (for $d = 4$, L_ℓ^2 is the total area of the membrane). The usual renormalization for the effective potential $V_{eff}(\phi_c)$ replaces the bare coupling constant λ in this surface energy term by the renormalized one, since the difference is of higher order.

The surface energy is defined uniquely and is physically relevant. Our Casimir energy E_c is defined to vanish for $L = 0$. This is evident from the fact that the phase shifts vanish in that limit. We have directly calculated the zero-point energy sum and renormalized the theory according to the usual renormalization procedure. This renormalization affects only to the volume energy term. Since no additional subtractions were done for the surface energy, our results satisty this definition.[15] As long as one keeps to this definition of the Casimir energy, the surface energy is uniquely defined. As a result, Any different methods of calculation should yield the same result. (For example, if one employs the Green's function method,[16] one may suspect that the surface energy may be ambiguous due to the contact terms. But it is not so.) The coefficient $c(d)$ is given in (4.10) for general d. In particular,

$$c(d) = \begin{cases} \frac{1}{\pi} - \frac{1}{4} \simeq 0.0683 & \text{for } d = 2 \quad , \\ \frac{\ell n\, 4 - 1}{16\pi} \simeq 0.00769 & \text{for } d = 3 \quad , \\ \frac{1}{9\pi^2}\left(\frac{3\pi}{8} - 1\right) \simeq 0.00201 & \text{for } d = 4 \quad . \end{cases} \tag{6.2}$$

For $d \geq 5$, $c(d)$ is divergent due to the sharp boundaries of the configuration (4.1). When the boundaries are smooth and of order $1/\mu$, $c(d)$ is expected to be of order of $\ell n\, \mu/m$ for $d = 5$ and $(\mu/m)^{d-5}$ for $d \geq 5$ (see (5.6)). This surface energy comes from the "zero-reflection" term.

The non leading L-dependent term comes from the n-"reflection" term, which is exponentially small as $-e^{-2nmL}$ for $mL \gg 1$. That is, the Casimir energy provides an attractive force between boundaries at $x = \pm L/2$, which decays exponentially due to the fluctuations between boundary being massive. The absence of the power series in $1/mL$ can also be explained by the following argument. Imagine $m \to \infty$, such that the fluctuation is completely excluded from $|x_1| < L/2$. Since we have two half-space with no tunneling between them, the Casimir energy of such a configuration is physically expected to be independent from L. (In fact, it is zero when dimensionally regularized.) On the other hand, suppose that (5.1) had a Taylor series in $1/mL$ like

$$E = E_{kinetic} + L_\ell^{d-2} \, L \, m^d \left(\sum_{n=0}^{\infty} \frac{\gamma_n}{(Lm)^n} + \ldots \right) \quad .$$

Then, in general, E has an m-independent term, $\propto L_\ell^{d-2}/L^{d-1}$. When we take $m \to \infty$, this term survives, because the dimensional regularization let us get rid of all non-logarithmic divergences. This contradicts the L-independence of the Casimir energy in that limit.

Ambjørn and Wolfram[6] have calculated the Casimir energy for a scalar field of mass M completely confined in a membrane of thickness L.[17] Our result is consistent with theirs. They showed that the surface energy of the boundary between the regions of mass M and ∞ is,

$$\frac{2}{(4\pi)^{(d+1)/2}} \, \Gamma\left(\frac{1-d}{2}\right) \frac{\pi}{2} M^{d-1},$$

which coefficient coincide with our first ($\pi/2$) term in (4.10). Since our surface energy is of the boundary between the masses m and 0, comparison is possible only for $M \to 0$ and $m \to \infty$. In this limit, their surface energy results in zero, while ours diverges as $\infty^{(d-1)}$. This is due to uncommutativity of the limiting procedures and dimensional regularization: If we have taken our m to be ∞ in the beginning, we would have gotten the zero answer since dimensional regularization automatically subtracts the non-logarithmic divergences.[18]

As is well-known, the one-loop approximation in the $\lambda\phi^4$-theory has limited range of validity.[13] In the physically interesting four-dimensional case, the one-loop effective potential must be taken seriously only for a range of ϕ_c between the origin and its minimum. For the surface energy term in (6.1), it is most natural to expect that the higher orders would bring in extra powers of $\lambda \, \ell n \, \phi$. Therefore, (6.1) should be valid only in that range. In gauge theories, where the minima of the effective potential is real, the type of the calculation done in this paper is complicated due to the mixing of gauge modes and Higgs modes at the boundary. At this moment one could only guess that the result might be similar to (6.1) with λ replaced by g^2, where g is the gauge coupling constant.

It is also interesting to see how the Casimir energy appears for "spherical" Higgs configurations. For example, let us take a spherical $\phi_c(x)$ with radius R in a four-dimensional space time. After the separation of the angular variables, the Casimir energy would be approximately expressed as follows,

$$E_c \sim \int_0^\infty \frac{dk_r}{\pi} \sum_{\ell=0}^{\infty} (2\ell+1) \, \frac{d\delta_\ell(k_r)}{dk_r} \cdot \frac{k_r}{2} \quad .$$

For $\ell > k_r R$, the configuration would be "hidden" by the centrifugal potential. Therefore, E_c receives contributions only from $\ell < k_r R$. For large R, δ have a leading $R \, k_r$ term, which together with a $k_r^2 R^2$-term from ℓ-summation, yield

$$\sim R^3 \int \; dk_r \cdot k_r^2 \cdot k_r \quad .$$

This is the correction to the volume energy. The nonleading R^o term in δ would then yield a R^2 term, the surface energy, which should be $4\pi R^2$ times the surface energy per unit area given by (6.1), $\frac{1}{2} \, c(d) \, m^{d-1}$.

There is at least one other case where we can investigate the Casimir energy rather straightforwardly. That is a case of the scalar field partially confined in a membrane due to a finite mass. It is of particular interest in connection with the dimensional calculation of the completely confined system[6]. The pseudo-reflection amplitude expansion given in this paper is again effective for evaluating the continuous spectrum. It is, however, necessary to employ different methods for the evaluation of the summation over the discrete spectrum. The results including this technology will be published elsewhere.[19]

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11. From (2.5), it is easy to see that the ς-function regularization and the dimensional regularization scheme yield the same answer for E_C. For example, we want to calculate E_C in the 4-dimensional space-time, integrating over the $d-4$ extra dimension yields

$$L_\ell^{d-4} \int \frac{dp_\ell^{d-4}}{(2\pi)^{d-4}} \sqrt{p^2} = A(d) \cdot (p^2)^{(d-3)/2}$$

where the coefficient A satisfies $A(4) = 1$. Therefore the role of ς-regularization parameter s $[\varsigma(s) \equiv \frac{1}{2}\sum_n \omega_n^{-s}]$ is played by $(3-d)/2$. The d-dependence of A causes a finite difference in case when the rest has poles, which, however is irrelevant and can be removed by appropriate renormalization conditions.

12. Although $\Delta V(0) = 0$ when dimensionally integrated, we leave it unintegrated for clarity.

13. S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

14. Actually, the pole at $d = 5$ in (5.3) is of "fake" IR origin. While the original integral in (5.3) is IR-convergent for $d = 5$, it is not so when dimensionaly calculated. This is because the average value 1/2 of $\sin^2$ yields zero when integrated. Nevertheless, this property of the integrand is irrelevant for this discussion; any extra decay factor of k/μ kills the fake IR divergence and leads to (5.4).

15. We can see this property explicitly by summing the L-dependent terms $(e_c)_{nth}$ of (4.7) as follows

$$\sum_{n=1}^{\infty} (e_c)_{nth}\Big|_{L=0} = \frac{1}{(4\pi)^{d-1}\Gamma(\frac{d-1}{2})} \int_0^\infty dx\, x^{d-2}\, \ell n\left[1 - \left(\frac{x - \sqrt{x^2+m^2}}{m}\right)^4\right] .$$

The above integral can be evaluated analytically and yield

$$\sum_{n=1}^{\infty} (e_c)_{nth}\Big|_{L=0} = -c(d)\, m^{d-1} = -(e_c)_{oth} .$$

16. See Milton's papers in Refs. 5 and 6.

17. The $d = 2$ case was done in Ref. 4 and P. Hayes, Ann. Phys. 121, 32 (1979).

18. In Ref.19, we calculate the surface energy of a boundary between masses m_1 and m_2, and show (if $m_1 \geq m_2$) that this energy takes the form:

$$\frac{2}{(4\pi)^{(d+1)/2}}\,\Gamma\left(\frac{1-d}{2}\right)\left[\frac{\pi}{2}(m_1^{d-1}+m_2^{d-1})-m_1^{d-1}F\left(\frac{m_2^2}{m_1^2}\right)\right].$$

We find $F(0)=B(\frac{1}{2},\frac{d}{2})$, and so this agrees with the result of this paper in the limit $m_2 \to 0$. In the limit $m_1 \to \infty$, the first and third terms diverge. However, these terms have no piece proportional to m_1^0. Therefore they are set to zero by an analytic regulator. This gives the result of Ambjørn and Wolfram.

19. H. Aoyama, SLAC preprint PUB-3253

FIGURE CAPTIONS

1. A typical shape of $m^2(x)$ of (3.2). The amplitudes of the incoming and outgoing waves of (3.3) are written.
2. The circle C is traced by the value $A_p^2 - B_p^{*2} e^{-2ip'L}$ as we change L. It is easy to see that the angle oscillates around a constant value due to (3.10).
3. The complex plane of p and the contour C for the integration (4.7). The integral picks up the cut on the imaginary axis when closed in the upper half plane.

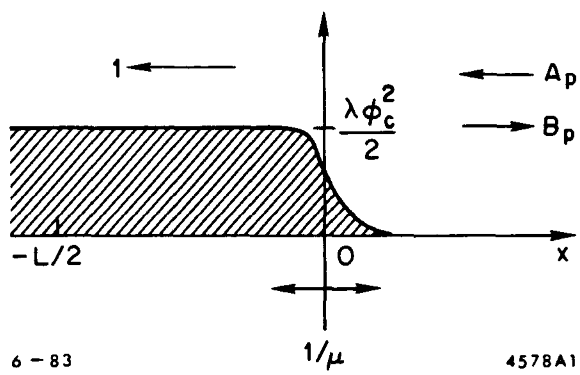

Fig. 1

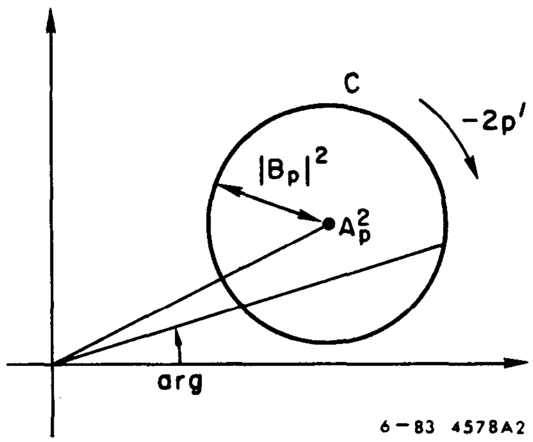

Fig. 2

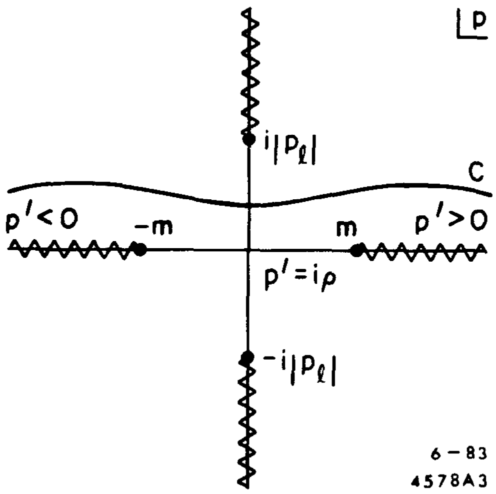

Fig. 3