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PATTERNS OF SYMMETRY BREAKING IN GAUGED SUPERSYMMETRIC SIGMA MODELS

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ABSTRACT

We consider supersymmetric nonlinear sigma models, in which the chiral superfields serve as coordinates on a compact coset space G/H, and study the effects of gauging various subgroups $S \subseteq G$. The general result is that S spontaneously breaks down much more than expected from experience with nonsupersymmetric models. In particular the rank of S is almost never preserved. Significantly, if S is large enough to break supersymmetry, then no scalars remain in the massless sector. If S is small, supersymmetry is unbroken and the model has an indeterminate vacuum; as a result S breaks either completely or not at all, and the mass spectrum is determined only up to an arbitrary multiplicative constant (possibly zero).

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1. Introduction

A particularly fruitful approach to the study of the low-energy modes of a strongly coupled field theory has been that of the nonlinear sigma model. This is a model in which the scalar fields, which represent the Goldstone bosons of the underlying theory, serve as coordinates on a coset space G/H; here G is an arbitrary compact Lie group characterizing the symmetries of the underlying Lagrangian, and H is the unbroken subgroup, under which the various fields of the theory are assumed to transform linearly. Thus the Goldstone bosons are in one-to-one correspondence with the broken generators of G.

Before proceeding to the supersymmetric case, let us recall some salient features of nonsupersymmetric nonlinear sigma models. In particular, let us focus on what happens in such models when one gauges a subgroup S of the full symmetry group G. (It is straightforward to do so; one need only work out the gauge-covariant derivative appropriate to the manifold question.[1]) Of course, gauging S breaks G *explicitly* to the product of S with those elements of G which commute with S. As for *spontaneous* symmetry breakdown, its pattern[2] depends crucially on the overlap between S and H (which is dynamically determined). If S is wholly contained in H, then S remains unbroken. If not, then the gauge mesons corresponding to the broken generators gain mass at tree level through the Higgs mechanism, and S spontaneously breaks down to its largest subgroup that is contained in H.

Correspondingly, gauging S effectively divides the Goldstone bosons into three classes.[2] Those that correspond to the broken generators of S are eaten. Those that correspond to the generators of G/H which commute with S remain massless. Those that remain — the "pseudo-Goldstone bosons" — gain mass from radiative corrections at the one-loop level; these masses are characteristically of order gf_π, where f_π is the usual Goldstone boson decay constant and g is the gauge coupling constant.

How does the theory determine the alignment of S relative to H? The common wisdom is that S seeks to break down "as little as possible". More concretely, in the important case when G/H is a symmetric space, the gauge group positions itself via G-transformations so as to minimize the sum of the squares of the induced vector boson masses.[3] Suppose for example that S is isomorphic to a subgroup of H, and furthermore that there exists a group element $g \in G$ whose action serves to rotate S completely into H. Then S will not break down.

Supersymmetric sigma models differ from their nonsupersymmetric counterparts in several noteworthy respects. For one thing, the scalars and vectors each have fermionic partners that must be included in the phenomenological Lagrangian; this is naturally accomplished by superfield methods. More strikingly, as Zumino has shown,[4] supersymmetry severely restricts the construction of such models. One must take as one's space a certain type of complex manifold known as Kählerian.[5] An important subclass of Kähler manifolds are the Grassmann spaces $G_{p,q} \equiv U(p+q)/U(p) \times U(q)$. These turn out to be the easiest to analyze; consequently we will confine our attention to $G_{p,q}$ through Section 4.

We will consider the effects of gauging various special unitary subgroups S of the full symmetry group $U(p+q)$ of $G_{p,q}$. (Our analysis generalizes that of Ong[6] and Bagger and Witten[7], who examined the extreme case in which all of $U(p+q)$ is gauged.) We find, in general, that the gauge group breaks down much more severely than one would expect based on experience with nonsupersymmetric models (note that $G_{p,q}$ is a symmetric space). In particular, the rank of the gauge group is almost never preserved. Furthermore, if S is too big to be included in the unbroken subgroup $U(p) \times U(q)$, supersymmetry is itself spontaneously broken — and, surprisingly, *all* scalar particles receive mass *at tree level* or are eaten.

If, on the other hand, S is smaller than $U(p) \times U(q)$, then supersymmetry is unbroken, but even this case contains a surprise: If S is small enough, the model has an indeterminate vacuum. That is, to all orders of perturbation theory, the gauge group does not know whether to break down completely or not at all, and the mass-spectrum is determined only up to an arbitrary multiplicative constant (possibly zero).

This paper is organized as follows. In Section 2 we set up the Lagrangian for $G_{p,q}$, exhibit the nonlinear realization of its symmetries, and show how to gauge them. Sections 3 and 4 discuss the patterns of symmetry breaking that result when the gauge group is, respectively, larger or smaller than the unbroken subgroup of $G_{p,q}$. In Section 5 we consider gauging other Kähler manifolds, namely $SO(2p)/U(p)$ and $Sp(2p)/U(p)$, and find qualitatively the same behavior as for $G_{p,q}$. Possible applications are mentioned in Section 6. Finally, in the Appendix we generalize Zumino's result[4] to the case when the Kähler potential depends on vector as well as chiral superfields, and then utilize this formula to obtain the Lagrangian for our models in component form.

2. Grassmann Models

Both Ong[6] and Bagger and Witten[7] have recently given prescriptions for gauging the supersymmetrized $G_{p,q}$. In Ong's formulation, the chiral superfields live in $p \times (p+q)$ dimensional matrices, taken to be equivalent if they differ by left-multiplication by a nonsingular $p \times p$ matrix. This auxiliary symmetry is implemented, following Aoyama,[8] by the introduction of nondynamical exponentiated $U(p)$ vector superfields which are to be eliminated by the equations of motion. The actual $U(p+q)$ symmetry (right-multiplication), in contrast, is promoted to a *bona fide* gauge symmetry in the usual manner, by the introduction of exponentiated vector superfields endowed with kinetic energy terms. Bagger and Witten showed in general how to gauge supersymmetric models of the form G/H. Their treatment relies on the identification of G with the isometry group of the manifold; the Lagrangian they construct is expressed entirely in terms of geometric invariants.

We will employ a rather more straightforward method of gauging $G_{p,q}$, one in which the auxiliary degrees of freedom are eliminated at the outset. Consider first the ungauged model. In "stereographic coordinates," its Lagrangian may be written[4]

$$\mathcal{L} = \int d^4\theta f_\pi^2 tr\ log(1_p + \Phi\bar{\Phi}) \ ; \tag{1}$$

here Φ is a $p \times q$ matrix of chiral superfields, $\bar{\Phi}$ is its Hermitian conjugate, and 1_p denotes the $p \times p$ unit matrix. The Goldstone bosons then correspond to the $2pq$ real scalar components of Φ, represented by the complex $p \times q$ matrix A.

We can factor $1_p + \Phi\bar{\Phi}$ as

$$1_p + \Phi\bar{\Phi} = (1_p,\ i\Phi)\begin{pmatrix} 1_p \\ -i\bar{\Phi} \end{pmatrix} \ .$$

Then $\mathcal{L}$ is trivially invariant under the replacement

$$(1, i\Phi) \rightarrow (1, i\Phi)e^{iT} \quad , \quad \begin{pmatrix} 1 \\ -i\bar{\Phi} \end{pmatrix} \rightarrow e^{-iT}\begin{pmatrix} 1 \\ -i\bar{\Phi} \end{pmatrix} \tag{2}$$

where $\exp(iT)$ is an element of $U(p+q)$. We can implement (2) directly as a transformation of Φ and $\bar{\Phi}$ as follows: Consider the effect of an infinitesimal $U(p+q)$ transformation $exp(i\epsilon T)$, with

$$T = \begin{pmatrix} H_1 & X \\ \bar{X} & H_2 \end{pmatrix} \begin{matrix} \}p \\ \}q \end{matrix} \quad .$$

Under this transformation

$$(1, i\Phi) \rightarrow (1, i\Phi)(1 + i\epsilon T) = M(\Phi)(1, i\Phi - \epsilon\Phi H_2 + \epsilon H_1\Phi + i\epsilon X + i\epsilon\Phi\bar{X}\Phi) \qquad (3)$$

where

$$M(\Phi) = 1 + i\epsilon H_1 - \epsilon\Phi\bar{X} \quad .$$

Since M is itself a chiral superfield, it is killed by the Grassmannian integration in (1):

$$\int d^4x\ d^4\theta\ tr\ log M(1 + \Phi\bar{\Phi})\bar{M} = \int d^4x\ d^4\theta\ tr\ log(1 + \Phi\bar{\Phi}) \quad .$$

Therefore the Lagrangian is invariant under the transformation

$$\delta\Phi = i\epsilon(\Phi H_2 - H_1\Phi) + \epsilon X + \epsilon\Phi\bar{X}\Phi \quad , \qquad (4)$$

which provides the desired nonlinear realization of $U(p+q)$. Indeed, it is now manifest that Φ transforms linearly under the $U(p) \times U(q)$ spanned by H_1 and H_2; X and $\bar{X}$ are the $2pq$ broken generators.

It is obvious how to promote a subgroup S of $U(p+q)$ to a local symmetry. As usual, we introduce $V = V_iT_i$, where the V_i are vector superfields and the T_i are $(p+q) \times (p+q)$ Hermitian matrices that generate S. The gauge-invariant Lagrangian is then simply given by

$$\mathcal{L} = \int d^4\theta f_\pi^2\ tr\ log(1, i\Phi)e^{gV}\begin{pmatrix} 1 \\ -i\,\bar{\Phi} \end{pmatrix} + \mathcal{L}_{gauge} \qquad (5)$$

with $\mathcal{L}_{gauge}$ the gauge-superfield kinetic energy. Let us henceforth choose the normalization $tr\ T_iT_j = \delta_{ij}$.

3. Supersymmetry-breaking Cases

We first consider the case where the T_i generate an $SU(p+k)$ subgroup S of $U(p+q)$, with $1 \leq k \leq q \leq p$. This case will be worked out in detail; subsequent cases will be dealt with more succinctly.

There are in general several inequivalent embeddings of $SU(p+k)$ in $U(p+q)$, that is, embeddings which cannot simply be rotated into one another by a $U(p+q)$ matrix.[9] To resolve this ambiguity, let us specify that the fundamental representation of $U(p+q)$ transform under S as a fundamental plus $q-k$ singlets. Then, by a $U(p+q)$ rotation, the T_i can be positioned to lie entirely in the upper left-hand $(p+k) \times (p+k)$ submatrix of V (thereby minimizing the overlap between S and the broken generators). We are thus gauging $2pk$ broken generators. If this is indeed the preferred orientation of the gauge group, these generators will eat $2pk$ of the $2pq$ Goldstone bosons and leave $2p(q-k)$ massive scalars in the spectrum. Our naive expectation is therefore that $SU(p+k) \rightarrow SU(p) \times SU(k) \times U(1)$ (preserving the rank of the gauge group). However, this turns out not to be the case.

We begin by calculating the (tree-level) effective potential Γ. Let the chiral and vector superfields (in WZ gauge) have component fields (A, ψ^α, F) and (λ^α, v^n, D) respectively; these are of course matrices like Φ and V. It will prove convenient to set

$$J = (1_p, iA),\ B = (J\bar{J})^{-1} = (1_p + A\bar{A})^{-1},\ \text{and}\, C = (1_q + \bar{A}A)^{-1} \quad . \tag{6}$$

Then by Taylor-expanding *exp* and *log* in (5) one obtains:

$$\begin{aligned}\Gamma(A,D) &= -\int d^4\theta f_\pi^2\ tr\ log\left\{J\ exp\left(\frac{1}{2}gD\theta^2\bar{\theta}^2\right)\ \bar{J}\right\} - \frac{1}{2}\ trD^2 \\ &= -\frac{1}{2}gf_\pi^2\ tr(\bar{J}BJD) - \frac{1}{2}\ trD^2 \quad .\end{aligned} \tag{7}$$

(It will be shown in the Appendix that the auxiliary field F is irrelevant to the tree-level calculation of the effective potential.) Eliminating the auxiliary field D yields the expression

$$\Gamma(A) = \left(\frac{g^2 f_\pi^4}{8}\right) \sum_i (tr\,\bar{J}BJT_i)^2 \quad . \tag{8}$$

Finally, if we introduce the $(p+q) \times (p+q)$ dimensional projection matrix

$$Q = \begin{pmatrix} 1_{p+k} & 0 \\ 0 & 0 \end{pmatrix} \quad ,$$

we can perform the indicated sum easily enough and obtain

$$\begin{aligned} \Gamma(A) &= \left(\frac{g^2 f_\pi^4}{8}\right)\left[tr(Q\,\bar{J}\,BJ)^2 - \frac{1}{p+k}\,(trQ\,\bar{J}\,BJ)^2\right] \\ &= \left(\frac{g^2 f_\pi^4}{8}\right)\left[p-q+k+tr\,\tilde{C}^2 - \frac{1}{p+k}(p-q+k+tr\,\tilde{C})^2\right] \end{aligned} \tag{9}$$

where $\tilde{C}$ denotes the bottom right-hand $(q-k) \times (q-k)$ submatrix of C.

The potential is minimized when

$$\tilde{C} = \left(\frac{p-q+k}{p-q+2k}\right) \times 1_{q-k} \ , \tag{10}$$

which defines the space of vacua for the theory. Equation (10) implies that

$$\Gamma(A) \geq \left(\frac{g^2 f_\pi^4}{8}\right)\frac{k(p-q+k)}{(p-q+2k)} > 0 \tag{11}$$

so that supersymmetry is, in fact, spontaneously broken as advertised. This result was obtained previously by Ong[6] and by Bagger and Witten[7] for the case $k = q$. Furthermore, (11) illustrates Ong's simple counting argument proving that supersymmetry must be broken, roughly speaking, whenever the gauge group cannot be contained in the unbroken subgroup (with some exceptions: see (34) and footnote 16).

Let us pick a specific vacuum consistent with (10) and expand around it. The simplest choice is given by

$$A_{vac} = \alpha \begin{array}{c} \begin{array}{cc} k & q-k \end{array} \\ \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right) \end{array} \begin{array}{l} \\ k \\ q-k \\ p-q \end{array} \qquad \text{with} \quad \alpha = \left(\frac{k}{p-q+k}\right)^{1/2} \quad . \tag{12}$$

(We will shortly demonstrate that all minima of $\Gamma(A)$ are gauge-equivalent to (12).) Let us define the shifted scalar fields

$$A' = A - A_{vac} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \\ A_5 & A_6 \end{pmatrix} \quad , \tag{13}$$

where the divisions are the same as in (12), and likewise label the $SU(p+k)$ generators

$$T=\begin{array}{c} \\ \end{array}\begin{array}{ccccc|c} k & q-k & p-q & k & q-k & \\ t_1 & t_5 & t_8 & t_{10} & 0 & k \\ \bar{t}_5 & t_2 & t_6 & t_9 & 0 & q-k \\ \bar{t}_8 & \bar{t}_6 & t_3 & t_7 & 0 & p-q \\ \bar{t}_{10} & \bar{t}_9 & \bar{t}_7 & t_4 & 0 & k \\ 0 & 0 & 0 & 0 & 0 & q-k \end{array} \tag{14}$$

The transformation law (4) then takes the form

$$\begin{aligned} \delta A' &= i\epsilon A_{vac}H_2 - i\epsilon H_1 A_{vac} + \epsilon X + \epsilon A_{vac}\bar{X}\,A_{vac} + O(A') \\ &= \begin{pmatrix} t_{10} & -i\alpha t_5 \\ t_9 & -i\alpha t_2 \\ t_7 & -i\alpha \bar{t}_6 \end{pmatrix} + O(A')\ . \end{aligned} \tag{15}$$

We can identify the broken generators of S as those which appear in the inhomogeneous term of (15). This implies that

$$SU(p+k) \rightarrow SU(k) \times SU(p-q+k) \times U(1) \tag{16}$$

where the unbroken symmetry is generated by t_1, t_3, t_4, t_8 and $\bar{t}_8$. It is interesting to note that the rank of the gauge group has decreased by $q-k$. The gauge symmetry has indeed broken down much more than expected (unless $k=q$, which is the case discussed in Refs. 6 and 7).

As for the Goldstone bosons, (15) implies that they are all eaten except for the $(q-k)^2$ "Hermitian" scalar fields $A_4^H \equiv \sqrt{\frac{1}{2}}(A_4+\bar{A}_4)$; these transform as singlets under the unbroken gauge group. Since the broken generators of $U(p+q)$ to which they correspond do not commute with S, we expect the A_4^H to become massive. In fact, thanks to their coupling to the auxiliary gauge field D, and in contradistinction to the nonsupersymmetric case, they gain mass at tree level.

Our results have ostensibly been based on the particular choice of vacuum given in (12). We will now establish that in fact all vacua are gauge equivalent, and shall do so in a way that easily generalizes to other cases. Let A_0 be some minimum of Γ. We are free to apply a (global) $U(p+k)\times U(q-k)$ transformation to A_0 without changing the value of Γ. (Here we are assuming that the $U(p+k)$ and $U(q-k)$ generators lie in the upper left-hand and lower right-hand subblocks of T, respectively, so that they merely rotate the $SU(p+k)$ gauge fields among themselves, hence leave the form of Γ invariant.) Consider first the infinitesimal change in A_0 due to the broken generators t_{10}, t_9, and t_7 in (14). Choose these generators to be equal and opposite to their counterparts in A_0, which live in the regions designated A_1, A_3 and A_5, respectively, in (13). One then obtains

$$\delta\, tr(\bar{A}_L A_L) = -2\epsilon\, tr\Big[\bar{A}_L A_L + (\bar{A}_L A_L)^2\Big] , \qquad A_L \equiv \begin{pmatrix} A_1 \\ A_3 \\ A_5 \end{pmatrix} \tag{17}$$

as is readily verified using (4). In this way A_L can be quickly driven to zero.

Our next step is to choose an appropriate $U(p)$ matrix $\exp(iH_1)$ which rotates the remaining entries of A_0 into the subblock labeled A_4 in (13). Finally, through the joint action of two cleverly chosen $U(q-k)$ matrices that live in the bottom right-hand square and in the region labeled t_2 in (14), one can reduce the surviving $(q-k)^2$ elements of A_0 to diagonal form with real, nonnegative entries.

In short, this argument proves that every (constant) $p \times q$ matrix is gauge-equivalent to a matrix of the form

$$\begin{array}{cc} \begin{array}{cc} k & \quad q-k \end{array} & \\ \begin{pmatrix} 0 & 0 \\ 0 & \begin{smallmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_{q-k} \end{smallmatrix} \\ 0 & 0 \end{pmatrix} & \begin{array}{c} k \\ q-k \\ p-q \end{array} \end{array} \qquad (\mu_i \geq 0) \quad . \tag{18}$$

Of these matrices, only (12) satisfies (10), establishing the gauge-equivalence of the space of vacua.

In order to obtain the full mass spectrum and representation content of the theory, it is unfortunately necessary to descend from the superfield formalism and work out the Lagrangian in component form; this is done in the Appendix. It is then a simple matter to plug the shifted scalar fields into (A.6) and collect the quadratic terms. One obtains[10]

$$
\begin{aligned}
\mathcal{L}_{mass} = & \frac{-g^2 f_\pi^2}{4} \frac{\alpha^2}{(1+\alpha^2)^2} \left\{ tr \left(\frac{f_\pi}{1+\alpha^2} A_4^H \right)^2 - \frac{1}{p+k} \left[tr \left(\frac{f_\pi}{1+\alpha^2} A_4^H \right) \right]^2 \right\} \\
& - \frac{g f_\pi}{\sqrt{2}} tr \left\{ f_\pi \psi_1 \lambda_{13} + f_\pi \psi_5 \lambda_{15} + \frac{1}{\sqrt{1+\alpha^2}} \left(\frac{f_\pi \psi_3}{\sqrt{1+\alpha^2}} \ \lambda_{14} \right) \right. \\
& + \frac{i\alpha}{\sqrt{1+\alpha^2}} \left(\frac{f_\pi \psi_2}{\sqrt{1+\alpha^2}} \ \lambda_5 + \frac{f_\pi \psi_6}{\sqrt{1+\alpha^2}} \ \lambda_7 \right) \\
& \left. + \frac{i\alpha}{1+\alpha^2} \left(\frac{f_\pi \psi_4}{1+\alpha^2} \ \lambda_6 \right) + h.c. \right\} \\
& - \frac{g^2 f_\pi^2}{4} tr \left\{ \frac{\alpha^2}{(1+\alpha^2)^2} \ v_2^2 + \frac{\alpha^2}{1+\alpha^2} (\bar{v}_5 \, v_5 + \bar{v}_6 \, v_6) + \bar{v}_7 \, v_7 \right. \\
& \left. + \frac{1}{1+\alpha^2} \ \bar{v}_9 \, v_9 + \bar{v}_{10} \, v_{10} \right\}
\end{aligned}
\tag{19}
$$

where $\psi_1, \ldots, \psi_6$ and $v_1, \ldots, v_{10}$ are labeled analogously to (13) and (14), respectively, and where

$$
\lambda = \left(\begin{array}{ccccc} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & 0 \\ \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & 0 \\ \lambda_9 & \lambda_{10} & \lambda_{11} & \lambda_{12} & 0 \\ \lambda_{13} & \lambda_{14} & \lambda_{15} & \lambda_{16} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} k \\ q-k \\ p-q \\ k \\ q-k \end{array} \quad . \tag{20}
$$

In (19), the matter fields have been grouped with the appropriate normalization constants. The case $k = q$ is precisely Ong's result.[6]

As promised, the gauge generators corresponding to the unbroken $SU(k) \times SU(p-q+k) \times U(1)$ are the only remaining massless vector fields. Likewise the matter spinors have combined with their counterparts in the vector multiplet via the supersymmetric Higgs mechanism to yield $2pq$ massive Dirac spinors, leaving $(p+k)^2 - pq - 1$ massless gauginos.

The representation content is most easily read off if one interchanges the second and third (blocks of) rows and likewise columns in both (14) and (20). For example, the $k(p-q+k)$ (massless) gauginos λ_4 and λ_{12} together transform like a $(p-q+k,\ \bar{k})$ under the unbroken $SU(p-q+k) \times SU(k)$.

Let us briefly discuss other cases. Gauging other subgroups in addition to $SU(p+k)$ leaves the above results virtually intact. Consider the gauge group $SU(p+k) \times \tilde{G}$ with $\tilde{G}$ a subgroup of $SU(q-k)$. Then $\tilde{G}$ leaves the space of vacua (10) unchanged. The generators of $\tilde{G}$ can be thought of as living in the bottom right $(q-k) \times (q-k)$ subblock of (14); denote the corresponding gauge particles v and λ and the Yang-Mills coupling constant $\tilde{g}$. Then (19) holds verbatim if one substitutes $v_2 - \frac{\tilde{g}}{g} v$ for v_2 and $\lambda_6 - \frac{\tilde{g}}{g}\lambda$ for λ_6. If, in addition, one gauges a (traceless) $U(1)$ proportional to $(k-q)1_{p+k} \oplus (p+k)1_{q-k}$, the only change is in (12); one must now take

$$\alpha = \left[\frac{kg^2 + pg_1^2}{(p-q+k)g^2 + qg_1^2}\right]^{1/2} \tag{21}$$

with g_1 the $U(1)$ coupling constant. Our general result is thus

$$SU(p+k) \times \tilde{G} \times U(1) \rightarrow SU(k) \times SU(p-q+k) \times U(1) \times \tilde{G} \times U(1) \tag{22}$$

when $\tilde{G} \subseteq SU(q-k)$.

Finally, consider gauging $SU(p-k) \times SU(q+k) \times U(1)$. (To avoid any ambiguity, we assume that the defining representation of $U(p-q)$ transforms under this $SU(p-k) \times SU(q+k)$ as a $(p-k,1)+(1,q+k)$.) We will focus on $p > q+k$, since $p = q+k$ does not break supersymmetry, while $p < q+k$ is a special case of (22).

One can rotate the generators of $SU(p-k)$ and $SU(q+k)$ to lie in the upper left-hand and lower right-hand subblocks of (14), respectively; then the $U(1)$ is proportional to $(q+k)1_{p-k} \oplus (k-p)1_{q+k}$. The effective potential is found to be

$$\begin{aligned}\Gamma = &\left(\frac{g_1^2 f_\pi^4}{8}\right) [(p-k)(q+k)(p+q)]^{-1} \left[p(k-p) + (p+q) tr\,\tilde{B}\right]^2 \\ &+ \left(\frac{g_2^2 f_\pi^4}{8}\right)\left[tr\,\tilde{B}^2 - \frac{1}{p-k}(tr\,\tilde{B})^2\right] \\ &+ \left(\frac{g_3^2 f_\pi^4}{8}\right)\left[q+k-p+tr\,\tilde{B}^2 - \frac{1}{q+k}(q+k-p+tr\,\tilde{B})^2\right] .\end{aligned} \tag{23}$$

Here g_1, g_2, g_3 are the coupling constants for $U(1)$, $SU(p-k)$ and $SU(q+k)$, respectively, and $\tilde{B}$ is the upper left-hand $(p-k) \times (p-k)$ submatrix of $B \equiv (1_p + A\bar{A})^{-1}$.

Now, the null-space of $\bar{A}$ has dim $\geq p-q$ so that B has at least $p-q$ eigenvalues equal to unity. There is a well-known theorem[11] which states that the eigenvalues λ_i of a Hermitian matrix M must interlace with the eigenvalues λ'_i of the matrix M' obtained from M by deleting (say) the last row and column: $\lambda_1 \leq \lambda'_1 \leq \lambda_2 \leq \lambda'_2 \leq \ldots$. Therefore $\tilde{B}$ must have (at least) $p-q-k$ eigenvalues equal to unity; the remaining eigenvalues $\lambda_1, \ldots, \lambda_q$ are constrained only to lie between zero and one. Γ is minimized when

$$\lambda_1 = \ldots = \lambda_q = \lambda_{opt} = \frac{(q+k)\,[g_1^2 q + g_2^2(p-q-k)]}{g_1^2 q(p+q) + g_2^2(q+k)(p-q-k) + g_3^2 k(p-k)} \tag{24}$$

at which point $\Gamma > 0$, so that supersymmetry is spontaneously broken, as it must be.

By the same arguments as before, one can show that the vacua are all gauge-equivalent; a convenient choice is

$$A_{vac} = \beta \begin{pmatrix} 1_q \\ 0 \end{pmatrix} \quad , \quad \beta = \left(\lambda_{opt}^{-1} - 1\right)^{1/2} \quad . \tag{25}$$

Expanding around A_{vac} yields the result

$$SU(p-k) \times SU(q+k) \times U(1) \to SU(k) \times SU(p-k-q) \times SU(q) \times U(1) \times U(1) \quad . \tag{26}$$

Note that the rank of the gauge group has once again decreased.

4. Supersymmetry-preserving Cases

We next consider gauging $S = SU(n)$ with $n, q \leq p$, assuming as before that the defining representation of $U(p+q)$ transforms under S as a fundamental plus $p+q-n$ singlets. Then S can be rotated to lie in the upper left-hand $n \times n$ submatrix of V; in that case the gauge group is contained entirely in the unbroken subgroup of $G_{p,q}$. It is obvious from (7) that Γ will be minimized when $\tilde{B}$, the upper left-hand $n \times n$ submatrix of B, is proportional to the identity. Then $\Gamma = 0$, and supersymmetry is unbroken.

There are actually two cases to consider here. We focus first on the case $q < n \leq p$. By the interlacing theorem, it follows that $\tilde{B}$ must have at least $n-q$ eigenvalues equal to unity. Therefore $\tilde{B}$ must in fact be equal to the identity in order for Γ to vanish. $A = 0$ is the simplest vacuum one can think of, and indeed all other vacua are gauge-equivalent to it. We conclude that the gauge symmetry remains unbroken, as naively expected, and that all particles stay massless.

More interesting is the case when $n \leq q \leq p$; then interlacing no longer poses any restriction. Any vacuum is gauge-equivalent to one of the form

$$A_{vac} = \gamma \begin{pmatrix} 0 & 1_n \\ 0 & 0 \end{pmatrix} \tag{27}$$

where γ is an arbitrary real, non-negative constant. Let

$$A' \equiv A - A_{vac} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \tag{28}$$

with fermionic partners $\psi_1, \ldots, \psi_4$. Then the massive fields are the $n^2 - 1$ "traceless Hermitian" scalars in A_2, the $n^2 - 1$ gauge mesons, and the $2(n^2 - 1)$ Dirac spinors composed of the gauginos mixed with the corresponding "traceless" ψ_2 fields; all of these have mass $\sqrt{\frac{1}{2}}\, g f_\pi\, \frac{\gamma}{1+\gamma^2}$. (Note the symmetry $\gamma \leftrightarrow \gamma^{-1}$.)

The model is thus *indeterminate*: The gauge symmetry either breaks down completely or, if $\gamma = 0$, not at all, and the masses can assume any value up to $\frac{g f_\pi}{\sqrt{8}}$. By the magic of supersymmetry, this peculiar situation must to persist to any order in perturbation theory (a result reminiscent of supersymmetric SU(5)[12]). One should probably not take this too seriously; presumably, in any realistic application, the vacuum degeneracy would be lifted by perturbations.

5. Generalization to Other Kähler Manifolds

It is important to know whether the phenomena discussed in the last two sections are peculiar to $G_{p,q}$ or characteristic of supersymmetric sigma models in general. Fortunately, the symmetric Kähler manifolds of the form G/H, where G is a compact, connected simple Lie group, have been completely classified.[13] They are: (i) $SU(p+q)/SU(p) \times SU(q) \times U(1)$, (ii) $Sp(2p)/U(p)$, (iii) $SO(2p)/U(p)$, (iv) $SO(q+2)/SO(q) \times U(1)$, (v) $E_6/SO(10) \times U(1)$, and (vi) $E_7/E_6 \times U(1)$. Case (i) is equivalent to $U(p+q)/U(p) \times U(q)$ considered previously. In this section we shall focus on cases

(ii) and (iii). For the sake of brevity, let us denote both $Sp(2q)$ and $SO(2q)$ by $S(2q)$; then they can be treated simultaneously.

It is a pleasant surprise that the appropriate Lagrangians for (ii) and (iii) are still given[13] by (1), (5) and (A.6). Now, however, Φ is a $p \times p$ matrix of chiral superfields subject to the constraint $\Phi^T = \pm\Phi$, the upper sign referring henceforth to (ii) and the lower to (iii). Thus $C = B^T$. The global $S(2p)$ symmetries are generated by [14]

$$T = \begin{pmatrix} H & X \\ \bar{X} & -H^T \end{pmatrix} \quad ; \tag{29}$$

here H and X are $p \times p$ matrices satisfying $H = \bar{H}$ and $X^T = \pm X$. It is obvious from (4) that the (anti)symmetry of Φ is preserved by the action of T, and that H indeed generates the unbroken $U(p)$ subgroup.

We will first consider the consequences of gauging an $S(2k)$ subgroup of $S(2p)$. Let us assume for convenience that the generators of $S(2k)$ can be rotated into the form

$$T = \left(\begin{array}{cccc} T_2 & 0 & T_1 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{T}_1 & 0 & -T_2^T & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} k \\ p-k \\ k \\ p-k \end{array} \tag{30}$$

where of course $T_1^T = \pm T_1$ and $T_2 = \bar{T}_2$ as above. The naive expectation is then $S(2k) \rightarrow U(k)$. Needless to say, this underestimates the breaking of the gauge group, as we shall presently show.

As before, we seek to minimize the effective potential $\Gamma(A)$. It is convenient to reduce A to "canonical form" at the outset, reversing our usual procedure. Let A_0 be some minimum of Γ:

$$A_0 = \begin{pmatrix} A_1 & A_2 \\ \pm A_2^T & A_3 \end{pmatrix} \begin{array}{l} k \\ p-k \end{array} \qquad , \quad A_1^T = \pm A_1 \quad , \quad A_3^T = \pm A_3 \quad .$$

We are free to apply a global $S(2k) \times S(2p-2k)$ transformation to A_0 without changing the value of Γ; here $S(2k)$ is generated by (30), and $S(2p-2k)$ by

$$\tilde{T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & T_4 & 0 & T_3 \\ 0 & 0 & 0 & 0 \\ 0 & \bar{T}_3 & 0 & -T_4^T \end{pmatrix} , T_3^T = \pm T_3 \quad , \quad \bar{T}_4 = T_4 \tag{31}$$

which manifestly commutes with (30).

We begin by choosing $T_1 = -A_1$, $T_2 = T_3 = T_4 = 0$. Then A_1 can be quickly driven to zero, precisely as in (17). Our next step is to choose $T_3 = -A_3(1+\bar{A}_3 A_3) + (\bar{A}_2 A_2)^T A_3(\bar{A}_2 A_2)$, $T_1 = \mp A_2 \bar{T}_3 A_2^T$, $T_2 = T_4 = 0$. With these choices

$$\delta A_1 = 0, \quad \delta A_2 = \epsilon A_2 \bar{T}_3 A_3, \quad \delta tr(\bar{A}_3 A_3) = -2\epsilon\, tr(\bar{T}_3 T_3)$$

so that A_3 can likewise be made to vanish.[15] Finally, with a judicious choice of T_2 and T_4, one can put the $k \times (p-k)$ matrix A_2 in the convenient form

$$A_2 = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{pmatrix}$$

where all the λ_i are real and nonnegative. This exhausts our gauge freedom.

With this choice of A_1, A_2 and A_3, it is easy to calculate the effective potential. One obtains:

$$\Gamma = \frac{g^2 f_\pi^4}{16}\left[\max\{2k-p,0\} + \sum_i \left(\frac{1-\lambda_i^2}{1+\lambda_i^2}\right)^2\right], \tag{32}$$

which is minimized when $\lambda_i = 1$. Once again there are two cases to consider. Suppose first that $k \leq p/2$. Then supersymmetry remains unbroken, but the gauge symmetry breaks down completely. (Note that there is no indeterminacy here, unlike the case to be discussed at the end of this section.) On the other hand, if $k > p/2$, supersymmetry breaks and one finds:

$$S(2k) \rightarrow U(2k-p)\ . \tag{33}$$

In addition to the $(2k-p)^2$ gauge mesons corresponding to the unbroken symmetries, the massless sector consists of $(k - \frac{1}{2}p)(p \pm 1)$ Dirac gauginos transforming under the unbroken unitary group as $p-k$ covariant vectors plus a contravariant two-index (anti)symmetric tensor. There are no massless scalars.

One might wish to gauge the generators of (31) in addition to those of (30). In that case

$$\Gamma = \frac{f_\pi^4}{16}\left[g^2\max\{2k-p,0\} + \tilde{g}^2\max\{p-2k,0\} + (g^2+\tilde{g}^2)\sum_i \left(\frac{1-\lambda_i^2}{1+\lambda_i^2}\right)^2\right] \tag{34}$$

where $\tilde{g}$ is the $S(2p-2k)$ coupling constant. Let us assume without loss of generality that $k \geq p/2$. Then we find

$$S(2k) \times S(2p-2k) \rightarrow U(2k-p) \times S(2p-2k) \tag{35}$$

so that essentially the symmetry does not break down any further than in (33) (cf. (16) and (22)). Note that supersymmetry is broken unless $k = p/2$.[16]

Finally, let us consider gauging a subgroup of $SU(p)$ that can be rotated to lie entirely in the unbroken subgroup generated by H and $-H^T$ in (29). (This is *not* possible for the case $k \leq p/2$ discussed above.) One then finds the same indeterminacy described in Section 4. All in all, it thus appears that the behavior of these models is qualitatively very similar to that of $G_{p,q}$.

6. Discussion

We have seen that the spontaneous breaking of the gauge symmetry in supersymmetric sigma models is generally much more severe than in their nonsupersymmetric counterparts. (The only exceptions occur when $S = G$, in which case $S \rightarrow H$ as naively expected.) This phenomenon has potentially interesting applications. One might for example envision that at some large scale the world is supersymmetric, with a global symmetry group G broken spontaneously to H, and a grand unified gauge group $S \subseteq G$. One might then quite naturally obtain $S \rightarrow SU(5)$ or more directly $S \rightarrow SU(3) \times SU(2) \times U(1)$, even when rank $S > 4$. On the other hand, one might imagine that the symmetry breaking takes place at low energies (1 TeV) and find $SU(2) \times U(1) \rightarrow U(1)$.

In either case, however, it is no small task to arrange that the representation content and number of "families" of the remaining massless matter and gauge fermions correspond to those of the quarks and leptons (and that the ABJ anomalies vanish).[17] For example, consider the case when $G/H = SO(14)/U(7)$ and $S = SO(12)$. Then one finds that $S \rightarrow SU(5) \times U(1)$ with the massless fermions transforming quite encouragingly like a $\bar{5} + 10$ under the unbroken $SU(5)$; unfortunately, there is room for only one family. Still, it is especially heartening to note that one need not worry about massless scalars remaining in the spectrum; in general, thanks to the gauging, they all either gain mass at tree-level or are eaten by broken generators.

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APPENDIX

Derivation of the Lagrangian in Component Form

We begin by presenting a straightforward generalization of Zumino's result.[4] Let the Kähler potential be a function, not only of chiral superfields Φ^i and $\bar{\Phi}^i$, but also of vector superfields V^ℓ. That is,

$$\mathcal{L} = \int d^4\theta \; K(\Phi^i, \bar{\Phi}^i, V^\ell) + \mathcal{L}_{gauge} \; . \tag{A.1}$$

Then as an exercise in the chain rule, we find:

$$\begin{aligned}
\mathcal{L} = & \frac{\partial^2 K}{\partial\Phi^i \partial\bar{\Phi}^j} \left[F^i F^{*j} + \frac{i}{2}\, \partial_m \psi^i \sigma^m \bar{\psi}^j - \frac{i}{2}\, \psi^i \sigma^m \partial_m \bar{\psi}^j - \partial_m A^{*j} \partial^m A^i \right] \\
& - \frac{1}{2}\, \frac{\partial^3 K}{\partial\Phi^i \partial\bar{\Phi}^j \partial\bar{\Phi}^k} \left[\bar{\psi}^j \bar{\psi}^k F^i + i \psi^i \sigma^m \bar{\psi}^j \partial_m A^{*k} \right] \\
& - \frac{1}{2}\, \frac{\partial^3 K}{\partial\bar{\Phi}^i \partial\Phi^j \partial\Phi^k} \left[\psi^j \psi^k F^{*i} - i \psi^k \sigma^m \bar{\psi}^i \partial_m A^j \right] \\
& + \frac{1}{4}\, \frac{\partial^4 K}{\partial\Phi^\ell \partial\bar{\Phi}^k \partial\bar{\Phi}^j \partial\Phi^i} \, (\bar{\psi}^j \bar{\psi}^k)(\psi^i \psi^\ell) \\
& - \frac{1}{2}\, \frac{\partial^3 K}{\partial V^k \partial\Phi^i \partial\bar{\Phi}^j} \, \psi^i \sigma^m \bar{\psi}^j v^k_m \\
& + \frac{i}{2}\, \frac{\partial^2 K}{\partial\Phi^i \partial V^j} \left[\sqrt{2}\, \psi^i \lambda^j + v^j_n \partial^n A^i \right] - \frac{i}{2}\, \frac{\partial^2 K}{\partial\bar{\Phi}^i \partial V^j} \left[\sqrt{2}\, \bar{\psi}^i \bar{\lambda}^j + v^j_n \partial^n A^{*i} \right] \\
& - \frac{1}{4}\, \frac{\partial^2 K}{\partial V^i \partial V^j} v^i \cdot v^j + \frac{1}{2}\, \frac{\partial K}{\partial V^i} D^i + \mathcal{L}_{gauge} \; .
\end{aligned} \tag{A.2}$$

In (A.2), all derivatives of the Kähler potential are to be evaluated at $\theta = \bar{\theta} = 0$; they are thus functions only of the scalar fields A^i and A^{*i}. These fields can be thought of as coordinates on a complex manifold[5] whose metric is given by

$$g_{ij^*}(A^k, A^{*k}) = \left. \frac{\partial^2 K}{\partial\Phi^i \partial\bar{\Phi}^j} \right|_{\theta=\bar{\theta}=0} \; ; \tag{A.3}$$

such a metric is said to be *Kählerian.*

It is now clear that the auxiliary fields F^i and F^{*i}can be ignored in the tree-level computation of the effective potential Γ, as previously asserted, since at the minimum of Γ they assume the value zero. The first four lines of (A.2), which do not contain gauge fields, constitute Zumino's result.

We wish to apply this formula to (5). Let B, C and J be defined as in (6). It is convenient to define the $q \times (p+q)$ dimensional matrices

$$I = (0,\ 1_q) \ , \quad L = (i\bar{A},\ 1_q) \tag{A.4}$$

as well. Note the following useful identities (δ_i denotes the ith canonical basis vector):

$$\begin{aligned} &\frac{\partial B}{\partial A_{ij}} = -B\delta_i\delta_j^T \bar{A} B \ , \quad \frac{\partial C}{\partial A_{ij}} = -C\bar{A}\delta_i\delta_j^T C \ , \quad BA = AC \ , \\ &\bar{A}BA = 1 - C \ , \quad \bar{A}BJ = i(I - CL) \ , \quad \bar{J}BJ + \bar{L}CL = 1 \ . \end{aligned} \tag{A.5}$$

Then, after a good deal of algebra, one obtains[10]

$$\begin{aligned} \mathcal{L} =& f_\pi^2 \ tr\{\bar{F}BFC + \bar{\psi}B\psi C\bar{A}\psi C\bar{\psi}AC + (\bar{\psi}BFC\bar{\psi}AC + h.c.) \\ & - \partial_n\bar{A}B\partial^n AC + \frac{i}{2}\ (\partial_n\psi\sigma^n C\bar{\psi}B - h.c.) \\ & + \frac{1}{2}\ \bar{\psi}B\psi C\bar{\psi}B\psi C + \frac{i}{2}\ (B\psi\sigma^n C[\partial_n\bar{A}BA\bar{\psi} + \bar{\psi}BA\partial_n\bar{A}] - h.c.) \\ & + \frac{1}{2}\ g\psi\sigma^n C\bar{\psi}BJv_n\bar{J}B - \frac{1}{2}\ g\psi\sigma^n CLv_n\bar{L}C\bar{\psi}B - \frac{g}{\sqrt{2}}(\psi CL\lambda\bar{J}B + h.c.) \\ & - \frac{1}{2}\ g(v_n\bar{J}B\partial^n ACL + h.c.) - \frac{1}{4}\ g^2 v_n\bar{J}BJv^n\bar{L}CL + \frac{1}{2}\ g\bar{J}BJD\} \\ & + tr\{\frac{1}{2}\ D^2 - \frac{1}{4}(v_{mn})^2 - i\lambda\sigma^n\mathcal{D}_n\bar{\lambda}\} \ , \end{aligned} \tag{A.6}$$

where v_{mn} and $\mathcal{D}_n$ denote the usual gauge field strength and gauge-covariant derivative. The first line vanishes upon elimination of F and $\bar{F}$.

Recall that (A.6) is the appropriate Lagrangian, not only for $G_{p,q}$, but also for $Sp(2\overline{p})/U(p)$ and $SO(2p)/U(p)$. In these cases $C = B^T$, and the matter and gauge fields obey the various symmetry constraints discussed in Section 5.

Footnotes and References

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2. S. Weinberg, Phys. Rev. D 13, 974 (1967).
3. M. E. Peskin, Nucl. Phys. B175, 197 (1980); J. Preskill, Nucl. Phys. B177, 21 (1981). A symmetric space is one in which the commutator of any two broken generators is a conserved generator. (In addition, since H is a subgroup of G, the commutator of a conserved with a broken/conserved generator must perforce be a broken/conserved generator.)
4. B. Zumino, Phys. Lett. 87B, 203 (1979).
5. For a simple introduction to Kähler geometry, see L. Alvarez-Gaumé and D. Z. Freedman, in *Unification of the Fundamental Particle Interactions*, Erice 1980, eds. S. Ferrarra, J. Ellis and P. van Nieuwenhuizen (Plenum Press, New York, 1980).
6. C.-L. Ong, Phys. Rev. D 27, 911 (1983).
7. J. Bagger and E. Witten, Phys. Lett. 118B, 103 (1982).
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9. As an illustration, the defining representation of SU(3) contains two inequivalent SU(2) subgroups: a "spin-$\frac{1}{2}$" representation generated (for example) by the Gell-Mann matrices $\{\lambda_1, \lambda_2, \lambda_3\}$, and a "spin-1" representation generated (for example) by $\{\lambda_2, \lambda_5, \lambda_7\}$.
10. The conventions are those of J. Wess and J. Bagger, *Supersymmetry and Supergravity*, to be published by the Princeton University Press. In particular $\eta_{mn} = (-1,1,1,1)$, $\psi M\lambda \equiv \psi^\alpha M\lambda_\alpha$, $\bar\psi M\bar\lambda \equiv \bar\psi_{\dot\alpha} M\bar\lambda^{\dot\alpha}$, and $\psi\sigma^n M\bar\lambda \equiv \psi^\alpha\sigma^n_{\alpha\dot\beta} M\bar\lambda^{\dot\beta}$ for any matrix M. We work in WZ gauge throughout.
11. See, for example, G. Strang, *Linear Algebra and its Applications* (Academic Press, New York, 1976), exercise 6.4.15.
12. S. Dimopoulos and H. Georgi, Nucl. Phys. B193, 150 (1981).

13. E. Calabi and E. Vesentini, Ann. Math. 71, 472 (1960); the noncompact Kähler manifolds are likewise classified here. See also Ref. 6.

14. This representation of $SO(2p)$ is related to the more familiar antisymmetric one by the unitary transformation $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$.

15. There are troublesome points in the space of A_2 and A_3 for which $T_3 = 0$ even when $A_3 \neq 0$, but they can always be avoided.

16. The case $k = p/2$ provides an interesting exception to what one would generally expect from the counting argument given in Ref. 6; namely, that the spontaneous breakdown of the gauge symmetry must be accompanied by the breakdown of supersymmetry whenever the gauge group is too large to be included in the unbroken subgroup. This exception is possible because of the numerical coincidence $2\left[\frac{1}{2}\cdot 2p(2p \pm 1) - p^2\right] = 4 \cdot \frac{1}{2}p(p \pm 1)$. (This footnote is courtesy of C.-L. Ong.)

17. C.-L. Ong, Phys. Rev. D27, 3044 (1983).