SLAC-PUB-3125
May 1983
T/E

HELICITY FORMALISM FOR TRANSITION AMPLITUDES*

G. PASSARINO**

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

Transition amplitudes between states with spin ≤ 1 are considered and directly evaluated in terms of momenta and polarization vectors. A special algorithm is derived to reduce expressions where γ-matrices of different lines are saturated. The application of the method is illustrated for radiative and non-radiative processes, including mass effects.

Submitted to Physical Review D

---

*Work supported in part by the Department of Energy, contract DE-AC03-76SF00515 and in part by the Institute Nazionale di Fisica Nucleare (INFN).

**On leave from Istituto di Fisica Teorica, Universitá di Torino, Italy.

1. Introduction

In recent years the theory of strong, weak and electromagnetic interactions has developed to a point that we are more in a position to make very accurate comparisons between theoretical predictions and experimental results. In this investigation we need to compute higher order Feynman diagrams. The standard procedures, where we square the amplitude for a given process and use a covariant sum over polarizations, has become almost intractable. However alternative techniques have been recently developed for analyzing bremsstrahlung cross sections[1] and transition amplitudes between Dirac spinors.[2] Motivated by these ideas we show that a unified approach can be formulated in which the amplitude for an arbitrary process, radiative or not, is directly computable in terms of the invariants which specify the process and for any set of spin indices. The formalism is discussed in Section 2.

2. Evaluation of Transition Amplitudes

What we need is a convenient procedure which eliminates spinors, spin 1 external wave functions and γ-matrices in terms of momenta and polarization vectors. In order to deal with spinors we first develop a method which makes use of an explicit representation of the γ-matrices. Using the conventions of Ref. 2 we find

$$\begin{gathered} u_\lambda(p_i)\,\bar{u}_\rho(p_j) = -N(p_i)N(p_j)(\not{p}_i + im_i)\Gamma_{\lambda\rho}(\not{p}_j + im_j) \\ N^{-2}(p) = 2p_0(p_0+m), \qquad \Gamma_{\lambda\rho} = \frac{1+\gamma^4}{4}(\delta_{\lambda\rho} + i\gamma^5\vec{\gamma}\cdot\vec{\sigma}_{\rho\lambda}) \end{gathered} \tag{1}$$

where $u(p)$ denotes a Dirac spinor and $\lambda = \pm 1$ gives the spin assignment in the $\vec{p}$ rest frame. Having in mind applications to high energy physics we restrict our analysis to massless particles. v-spinors are then converted into u-spinors by $v_\lambda = -\lambda\gamma^5 u_{-\lambda}$. Next we process by reducing the numerator structure of an arbitrary diagram. Vector and axial couplings of internal particles are replaced with a combination of scalar and pseudoscalar couplings. The basic reduction formula reads

$$\gamma^\mu S_n u_\lambda(p)\,\bar{u}_\rho(q) S_m \gamma^\mu = -\lambda\rho\Big[S_m^R u_{-\rho}(q)\,\bar{u}_{-\lambda}(p) S_n^R - \gamma^5 S_m^R u_{-\rho}(q)\,\bar{u}_{-\lambda}(p) S_n^R \gamma^5 - \bar{u}_{-\lambda}(p) S_n^R S_m^R u_{-\rho}(q) + \gamma^5\,\bar{u}_{-\lambda}(p) S_n^R \gamma^5 S_m^R u_{-\rho}(q)\Big] \tag{2}$$

where S_n stands for an arbitrary string of n γ-matrices and S_n^R is the same string in the reversed order. The formula can be easily proved by using the Chisholm identities.[3] In this way an arbitrary diagram consisting of n fermion lines connected by internal vector bosons is replaced by a collection of diagrams each formed by n disconnected fermion lines. However each application of the Chisholm identities doubles the number of terms. When many internal photons are present we could in principle avoid this problem by using the Kahane algorithm[3] which appears to minimize the number of terms in the final expression. The resulting amplitude is computed by introducing[1]

$$S_{\lambda\rho}(p,q) = -2p_0 q_0\,\bar{u}_\lambda(p) u_\rho(q), \quad P_{\lambda\rho}(p,q) = -2p_0 q_0\,\bar{u}_\lambda(p)\gamma^5 u(q)\,. \tag{3}$$

For S and P we get

$$S_{\lambda\rho}(p,q) = p\cdot q\;\delta_{\lambda\rho} + i(\vec{p}\times\vec{q})\cdot\vec{\sigma}_{\lambda\rho}\;, \quad P_{\lambda\rho}(p,q) = (p_0\vec{q} - q_0\vec{p})\cdot\vec{\sigma}_{\lambda\rho} \tag{4}$$

[1]Matrix elements of this type have been tabulated for light cone perturbation theory in Ref. 4.

As a bonus for working with an explicit representation of γ-matrices we can avoid arbitrary phases. For simple processes the amplitude can be immediately computed. Consider $\mu^- e^-$ scattering in massless QED

$$\frac{d\sigma}{dt}(\lambda_e \lambda\mu \to \rho_e \rho_\mu) = 2\,\frac{\alpha^2}{s^3 t^2} |\, M(\lambda_e \lambda\mu \to \rho_e \rho_\mu)\,|^2 \tag{5}$$

It follows

$$\begin{aligned} M(\lambda_e \lambda_\rho \to \rho_e \rho_\mu) = -\lambda_e \rho_\mu \Big[& S_{\rho_e, -\rho_\mu} S_{-\lambda_e, \lambda_\mu} - P_{\rho_e, -\rho_\mu} P_{-\lambda_e, \lambda_\mu} \\ & - S_{\rho_e \lambda_\mu} S_{-\lambda_e, -\rho_\mu} + P_{\rho_e \lambda_\mu} P_{-\lambda_e, -\rho_\mu} \Big] \end{aligned} \tag{6}$$

Thus in the c.m.s. of the scattering particles

$$\begin{aligned} M(++ \to ++) &= M(-- \to --) = u^2 \\ M(++ \to -+) &= M(-- \to +-) = M(++ \to +-) \\ &= M(-- \to -+) = -i(tu^3)^{1/2} \\ M(++ \to --) &= M(-- \to ++) = -tu \\ M(+- \to ++) &= M(-+ \to --) = M(+- \to --) \\ &= M(-+ \to ++) = is(tu)^{1/2} \\ M(+- \to -+) &= M(-+ \to +-) = st \\ M(+- \to +-) &= M(-+ \to -+) = -su \end{aligned} \tag{7}$$

In agreement with the well-known result

$$\sum_{spin} |M|^2 = 2s^4 \left(1 + \frac{t}{s} + \frac{1}{2}\,\frac{t^2}{s^2} \right) .$$

Calculations of QED processes with polarized particles, including higher order corrections can be found in Ref. 5. In general we have expressions like

$$\bar{u}_\lambda (p_i) \prod_{\ell=1}^{n} \not{Q}_\ell u_\rho(p_j) \, , \tag{8}$$

where the Q_ℓ are linear combinations of external momenta (even when internal loops are present). This can be reduced to a product of S, P functions since

$$\not{Q}_\ell = 2i \sum_m a_{m\ell} p_{om} \sum_\tau u_\tau(p_m)\, \bar{u}_\tau(p_m) \quad \text{for} \quad Q_\ell = \sum_m a_{m\ell} p_m \, . \tag{9}$$

Finally we must take into account a possible multi-photon radiation. Once the reduction formula is applied the formalism of Ref. 1, namely the use of circularly polarized photon states, becomes particularly simple since each fermion line with its emitted photons can be analyzed independently from the rest of the diagram. Even when strong cancellations between different diagrams are not expected we use the fact that the polarization vector ϵ_μ^λ of Ref. 1 is explicitly given in terms of the external momenta and the previous formulas suffice in evaluating the amplitude. We also need a explicit representation for massive vector boson wave functions. A convenient way is to write

$$\epsilon_i^\lambda(k) = \delta_i^\lambda + \frac{1}{M(M+k_0)} k^\lambda k_i \;, \epsilon_0^\lambda(k) = \frac{1}{M} k^\lambda$$

$$\lambda, i = 1,2,3 \qquad k^2 = -M^2$$

From Ref. 2 we learn that a specific reference to spinor components can be avoided[2] if we allow for arbitrary phases. Here we derive the formalism for the massless limit with a new version of the reduction formula. Let u_λ and v_λ be eigenstates of $\frac{1}{2}(1+\lambda\gamma^5)$ and $\frac{1}{2}(1-\lambda\gamma^5)$ respectively. The only property we need is

$$u_\lambda(p)\, \bar{u}_\lambda(p) = -i \frac{\not{p}}{2p_0}\, \pi_{+\lambda} \;, \quad v_\lambda(p)\, \bar{v}_\lambda(p) = -i \frac{\not{p}}{2p_0}\, \pi_{-\lambda} \tag{10}$$

---

[2]See also Ref. 6.

with $\pi_\lambda = \frac{1}{2}(1+\lambda\gamma^5)$. Hence we may use $v_\lambda = -\lambda\gamma^5 u_{-\lambda}$. Next we derive

$$u_\lambda(p)\,\bar{u}_{-\lambda}(q) = \frac{e^{-i\psi_-}}{2\sqrt{2}}(p_0 q_0 p\cdot q)^{-1/2}\,\not{p}\,\not{q}\,\pi_{-\lambda} \tag{11}$$

where $e^{-i\psi_-}$ is an unspecified phase. When the helicity is the same $\bar{u}_\lambda\, u_\lambda = 0$ and a different procedure must be used

$$u_\lambda(p)\,\bar{u}_\lambda(q) = \frac{e^{-i\psi_+}}{2\sqrt{2}}(p_0 q_0)^{-1/2}(2p\cdot n\, q\cdot n - p\cdot q)^{-1/2}\,\not{p}\,\not{n}\,\not{q}\,\pi_\lambda \tag{12}$$

where n_μ is an arbitrary vector normalized to $n^2 = 1$. If R is the operator which reverses the order of a string of γ-matrices, $RS = S^R$, we get

$$R\; u_\lambda(p)\,\bar{u}_\rho(q) = u_{-\rho}(q)\,\bar{u}_{-\lambda}(p)\;.$$

Application of the Chisholm identities gives now[3]

$$\begin{aligned}
\gamma^\mu S_n u_\lambda(p)\,\bar{u}_\lambda(q) S_m \gamma^\mu &= -2S_m^R u_{-\lambda}(q)\,\bar{u}_{-\lambda}(p) S_n^R \\
&\qquad (n+m\ even) \\
\gamma^\mu S_n u_\lambda(p)\,\bar{u}_\lambda(q) S_m \gamma^\mu &= \bar{u}_{-\lambda}(p) S_n^R S_m^R u_{-\lambda}(q) - \gamma^5\,\bar{u}_{-\lambda}(p) S_n^R \gamma^5 S_m^R u_{-\lambda}(q) \\
&\qquad (n+m\ odd) \\
\gamma^\mu S_n u_\lambda(p)\,\bar{u}_{-\lambda}(q) S_m \gamma^\mu &= \bar{u}_{-\lambda}(p) S_n^R S_m^R u_\lambda(q) - \gamma^5\,\bar{u}_{-\lambda}(p) S_n^R \gamma^5 S_m^R u_\lambda(q) \\
&\qquad (n+m\ even) \\
\gamma^\mu S_n u_\lambda(p)\,\bar{u}_{-\lambda}(q) S_m \gamma^\mu &= -2S_m^R u_\lambda(q)\,\bar{u}_{-\lambda}(p) S_n^R \\
&\qquad (n+m\ odd)\;.
\end{aligned} \tag{13}$$

An alternative approach can e found by means of the identity[3]

$$tr(\gamma^\mu S)tr(\gamma^\mu S') = 2tr(S+S^R)S'\;.$$

As an example we consider

$$\bar{u}_4\,\gamma^\mu u_1\,\bar{u}_3\,\gamma^\mu u_2 = tr(\gamma^\mu u_1\,\bar{u}_4)tr(\gamma^\mu u_2\,\bar{u}_3) \quad u_i = u_{\lambda_i}(p_i)$$

$$= 2\,\bar{u}_\rho\,(p_3)u_\lambda(p_1)\,\bar{u}_\lambda\,(p_4)u_\rho(p_2) + 2\,\bar{u}_\rho\,(p_3)u_{-\lambda}(p_4)\,\bar{u}_{-\lambda}\,(p_1)u_\rho(p_2)$$

$$\lambda = \lambda_1, \lambda_2 \quad \rho = \lambda_2, \lambda_3 \tag{14}$$

The first term contributes only for $\rho = -\lambda$, the second only for $\rho = \lambda$. Helicity conservation prevents in general from a proliferation of diagrams in the repeated use of the reduction formula. Scalar and pseudoscalar bilinear forms can now be derived.

$$\bar{u}_\lambda\,(p)u_\rho(q) = e^{i\psi_s}\Big(\frac{p\cdot q}{2p_0q_0}\Big)^{1/2}\Lambda^-(\lambda,\rho)\;,$$

$$\bar{u}_\lambda\,(p)\gamma^5 u_\rho(q) = -\rho\, e^{i\psi_s}\Big(\frac{p\cdot q}{2p_0q_0}\Big)^{1/2}\Lambda^-(\lambda,\rho) = e^{i\psi_p}\Big(\frac{p\cdot q}{2p_0q_0}\Big)^{1/2}\Lambda^-(\lambda,\rho) \tag{15}$$

where $\Lambda^\pm(\lambda,\rho) = \frac{1}{2}(1 \pm \lambda\rho)$. Also using Eq. (12) with $n = Q$ we get

$$\bar{u}_\lambda\,(p)\;\not{Q} u_\rho(q) = e^{i\psi_v}\Big(\frac{2p\cdot Q\,q\cdot Q - Q^2 p\cdot q}{2p_0q_0}\Big)^{1/2}\Lambda^+(\lambda,\rho)\;,\; etc. \tag{16}$$

The arbitrariness of the phases becomes relevant whenever different diagrams interfere. In the following example we use $e^{i\psi_p} = -\rho e^{i\psi_s}$ to derive

$$g_s^2\,\bar{u}_1\,u_2\,\bar{u}_3\,u_4 + g_p^2\,\bar{u}_1\,\gamma^5 u_2\,\bar{u}_3\,\gamma^5 u_4 = e^{2i\psi_s}\,\frac{1}{2}\Big(\frac{p_1\cdot p_2\,p_3\cdot p_4}{E_1E_2E_3E_4}\Big)^{1/2}(g_s^2 + \lambda\rho g_p^2)$$

$$\lambda_1 = \lambda_2 = \lambda,\;\; \lambda_3 = \lambda_4 = \rho$$

As an application we consider the radiative Coulomb potential scattering of an electron. The amplitude is

$$M_\gamma = -\frac{Ze^2}{|\vec{q}\,|^2}\,tr\,\Big[\Big(\frac{p\cdot\epsilon^\sigma}{p\cdot k} - \frac{p'\cdot\epsilon^\sigma}{p'\cdot k}\Big)\,\not{n} - \frac{1}{2}\,\frac{\not{n}\,\not{k}\,\not{\epsilon}^\sigma}{p\cdot k} + \frac{1}{2}\,\frac{\not{k}\,\not{\epsilon}^\sigma\,\not{n}}{p'\cdot k}\Big]u_\lambda(p)\,\bar{u}_\lambda\,(p')\;,$$

$$n^2 = 1\;. \tag{17}$$

Using (1) $\not\epsilon^{\sigma} = \frac{1}{\sqrt{2N}}(\not k\ \not p'\ \not p\,\pi_{-\sigma} - \not p'\ \not p\ \not k\,\pi_{\sigma})$ we find

$$M_{\gamma} = -\frac{Ze^2}{|\vec{q}|^2}\frac{e^{i\psi}}{N}(EE')^{-1/2}(2p\cdot n\,p'\cdot n - p\cdot p')^{-1/2}\Big[M_{+}^{\sigma}\Lambda^{+}(\sigma,\lambda) + M_{-}^{\sigma}\Lambda^{-}(\sigma,\lambda)\Big] \tag{18}$$

$$\begin{aligned} M_{+}^{\sigma} &= p\cdot p'(p\cdot p' - 2p\cdot n\,p'\cdot n + p\cdot k - p\cdot n\,k\cdot n) \\ &\quad - p\cdot n\,p'\cdot n\,p\cdot k + (p\cdot n)^2 p'\cdot k + 2p'\cdot n\,\epsilon(n,p,p',k)\sigma \\ M_{-}^{\sigma} &= p\cdot p'(p\cdot p' - 2p\cdot n\,p'\cdot n + p'\cdot k - p'\cdot n\,k\cdot n) \\ &\quad - p\cdot n\,p'\cdot n\,p'\cdot k + (p'\cdot n)^2 p\cdot k - 2p\cdot n\,\epsilon(n,p,p',k)\sigma \end{aligned} \tag{19}$$

where $\Lambda^{\pm}(\sigma,\lambda) = \frac{1}{2}(1 \pm \lambda\sigma), \epsilon(n,p,p',k) = \epsilon^{\mu\nu\alpha\beta} n_{\mu} p_{\nu} p'_{\alpha} k_{\beta}$. Since $\Lambda^{\pm}$ are projectors $M_{\pm}^{\sigma}$ do not interfere.

Even in a situation where masses are not negligible there are many advantages in computing directly the amplitude. The relevant formalism has been developed in Ref. 2. The reduction formula is now more complicated and we derived it only for a simple example. Let

$$M = \bar{u}_4\,\gamma^{\mu} u_1\,\bar{u}_3\,\gamma^{\mu} u_2$$

where $u_i = u(p_i, n_i, \lambda_i)$ denotes a Dirac spinor with polarization $\lambda_i = \pm 1$ along n_i with $n_i^2 = 1, n_i\cdot p_i = 0$. Thus

$$\begin{aligned} M &= tr\,(\gamma^{\mu} u_1\,\bar{u}_4)\ tr\,(\gamma^{\mu} u_2\,\bar{u}_3) = tr\,(\gamma^{\mu} M_{14}^{odd})tr\,(\gamma^{\mu} M_{23}^{odd}) \\ M_{ij} &= M_{ij}^{odd} + M_{ij}^{even} = N_{ij}(\not p_i + im_i)\Gamma_{ij}(\not p_j + im_j) \\ \Gamma_{ij} &= 1 + i\gamma^5(\lambda_i\ \not n_i + \lambda_j\ \not n_j) + \lambda_i\lambda_j\ \not n_i\ \not n_j \end{aligned} \tag{20}$$

N_{ij} is a normalization factor and even (odd) denotes the part of M_{ij} with an even (odd) number of γ-matrices. We can easily prove that

$$\begin{aligned} M_{ij}^{odd} + M_{ij}^{even} &= u(p_i, n_i, \lambda_i)\,\bar{u}\,(p_j, n_j, \lambda_j) \\ M_{ij}^{odd} - M_{ij}^{even} &= -u(-p_i, -n_i, \lambda_i)\,\bar{u}\,(-p_j, -n_j, \lambda_j) \\ M_{ij}^{R} &= u(p_j, -n_j, \lambda_j)\,\bar{u}\,(p_i, -n_i, \lambda_i) \end{aligned} \tag{21}$$

Also

$$\begin{aligned} M &= 2tr\,(M_{14}^{odd} + M_{14}^{odd,R})M_{23}^{odd} \\ &= \frac{1}{2}\Big[\bar{u}_3\,u_1\,\bar{u}_4\,u_2 - \bar{u}_3\,u_1'''\,\bar{u}_4'''\,u_2 + \bar{u}_3\,u_4''\,\bar{u}_1''\,u_2 - \bar{u}_3\,u_4'\,\bar{u}_1'\,u_2 \\ &\quad - \bar{u}_3'''\,u_1\,\bar{u}_4\,u_2''' + \bar{u}_3'''\,u_1'''\,\bar{u}_4'''\,u_2''' - \bar{u}_3'''\,u_4''\,\bar{u}_1''\,u_2''' + \bar{u}_3'''\,u_4'\,\bar{u}_1'\,u_2'''\Big] \end{aligned} \tag{22}$$

where $u_i' = u(-p_i, n_i, \lambda_i)$, $u_i'' = u(p_i, -n_i, \lambda_i)$, $u_i''' = u(-p_i, -n_i, \lambda_i)$. Using the formalism we are able to derive the expression for $u_\lambda(p)\,\bar{u}_\lambda\,(q)$ previously given in the massless case. It turns out that n_μ can be chosen to satisfy $n \cdot p = n \cdot q = 0$. Indeed we start by computing $u(p_1, n_1, \lambda_1)\,\bar{u}\,(p_2, n_2, \lambda_2)$ with

$$\begin{aligned} n_{i\mu} &= \frac{\cos\psi}{m\beta_i}(\vec{p}_i\,, i\beta_i^2 E_i) + \sin\psi n_\mu \\ n_\mu &= (\vec{n}\,, 0), \vec{n}\cdot\vec{p}_i = 0, \vec{n}^2 = 1, \\ 0 \le \psi &\le \pi/2, p_i^2 = -m^2, \beta_i^2 = 1 - \frac{m^2}{E_i^2} \end{aligned} \tag{23}$$

for $\psi = 0$ this corresponds to longitudinal polarization. As usual

$$\begin{aligned} u_1\,\bar{u}_2 &= \frac{1}{2}(E_1 E_2)^{-1/2}\Big(n_+^{-1/2}\Lambda^+ + n_-^{-1/2}\Lambda^-\Big) \\ &\quad\times(-i\,\not{p}_1 + m)\,\frac{1}{2}\,(1 + i\lambda_1\gamma^5\,\not{n}_1)\,\frac{1}{2}\,(1 + i\lambda_2\gamma^5\,\not{n}_2)(-i\,\not{p}_2 + m) \end{aligned} \tag{24}$$

When $\lambda_1 = \lambda_2 = \lambda$ we find $n_+ = -2e^{+i\phi}\sin^2\psi\; p_1\cdot p_2 + O(m)$. Thus

$$u(p_1,n_1,\lambda)\,\bar{u}\,(p_2,n_2,\lambda) = e^{i\Phi}\frac{1}{4\sqrt{2}}(E_1E_2\,p_1\cdot p_2)^{-1/2} \times \not{p}_1[\sin\psi + i(\cos\psi + \lambda\gamma^5)\;\not{n}]\;\not{p}_2 + O(m)\,. \tag{25}$$

In the limit $\psi = 0, m = 0$

$$u_\lambda(p_1)\,\bar{u}_\lambda\,(p_2) = ie^{i\Phi}\frac{1}{2\sqrt{2}}(E_1E_2\,p_1\cdot p_2)^{-1/2}\;\not{p}_1\;\not{n}\;\not{p}_2\,\pi_\lambda\,. \tag{26}$$

As a final application we consider the bremsstrahlung amplitude for $e^+e^- \to F^+F^-\gamma$ where the F mass effects are explicitly taken into account. Moreover we only include final states radiation simulating in this way a QCD 3-jet cross section for heavy quarks. The amplitude is

$$\begin{aligned} M &= \bar{u}\,(p_3,n_3,\lambda_3)[\gamma^\mu\Delta(p_4+k)\;\not{\epsilon}^\sigma + \not{\epsilon}^\sigma\Delta(-p_3-k)\gamma^\mu]v(p_4,n_4,\lambda_4) \\ &\quad \bar{v}\,(p_1,\lambda_1)\gamma^\mu u(p_2,\lambda_2)\;\;\Delta^{-1}(p) = i\;\not{p} + m \end{aligned} \tag{27}$$

where factors due to couplings and an overall s^{-1} from the photon propagator have been omitted. Using (1)

$$\begin{aligned} \epsilon^\sigma_\mu &= \frac{1}{\sqrt{2}\,N}(p_4\cdot k\,p_{3\mu} - p_3\cdot k\,p_{4\mu} + \sigma\epsilon_{\mu\nu\alpha\beta}p_4^\nu p_3^\alpha k^\beta) \\ \not{\epsilon}^\sigma &= \frac{1}{\sqrt{2}\,N}(\not{p}_3\;\not{p}_4\;\not{k}\,\pi_\sigma - \not{k}\;\not{p}_3\;\not{p}_4\,\pi_{-\sigma})\;,\;\pi_\sigma = \frac{1}{2}(1+\sigma\gamma^5) \end{aligned} \tag{28}$$

we find

$$\begin{aligned} M &= \frac{i}{\sqrt{2}\,N}\bar{u}_3\,\gamma^\mu\Big[p_3\cdot p_4 + m^2\frac{p_3\cdot k}{p_4\cdot k} + im\frac{p_3\cdot k}{p_4\cdot k}\;\not{k}\,\pi_{-\sigma} + \not{k}\;\not{p}_3\,\pi_\sigma\Big]v_4\,\bar{v}_1\,\gamma^\mu u_2 \\ &\quad + \frac{i}{\sqrt{2}\,N}\bar{u}_3\Big[p_3\cdot p_4 + m^2\frac{p_4\cdot k}{p_3\cdot k} - im\frac{p_4\cdot k}{p_3\cdot k}\,\pi_{-\sigma}\;\not{k} + \pi_\sigma\;\not{p}_4\;\not{k}\Big]\gamma^\mu v_4\,\bar{v}_1\,\gamma^\mu u_2 \end{aligned} \tag{29}$$

Next we use

$$v_4\,\bar{v}_1 = \frac{1}{4}N_{14}(\not{p}_4 - im)(1 - i\lambda_1\lambda_4\,\not{n}_4)\,\not{p}_1\,\pi_{-\lambda_1}$$

$$u_2\,\bar{u}_3 = \frac{1}{4}N_{23}\,\pi_{-\lambda_2}\,\not{p}_2(1 + i\lambda_2\lambda_3\,\not{n}_3)(\not{p}_3 + im)$$

$$\begin{aligned} N_{if} &= (mp_i\cdot n_f + p_i\cdot p_f)^{-1/2}\Lambda^-(\lambda_i,\lambda_f) + (-mp_i\cdot n_f + p_i\cdot p_f)^{-1/2}\Lambda^+(\lambda_i,\lambda_f) \\ &= \sum_{k=\pm} N_{ifk}\Lambda^k(\lambda_i,\lambda_f) \end{aligned} \tag{30}$$

Thus $M = (i/16\sqrt{2}N)N_{14}N_{23}trT$. After applying Chisholm identities and rearranging the terms in the trace we find

$$\begin{aligned} T &= \Lambda^-(\lambda_1,\lambda_2)t \\ t &= \sum_{ijk=\pm} tr\,D_{ijk}\Lambda^i(\sigma,\lambda_1)\Lambda^j(\lambda_1,\lambda_4)\Lambda^k(\lambda_2,\lambda_3) \end{aligned} \tag{31}$$

The Λ are projectors and the different terms in the sum never interfere in the cross section. However the matrices D for arbitrary polarizations contain up to a maximum of 8 γ-matrices and therefore their trace is cumbersome but clearly not impossible (compare with the standard procedure)

$$\begin{aligned} D_{ijk} &= A_i\,\pi(i\lambda_1)B_{ij}C_k + A_{-i}\,\pi(-i\lambda_1)B_{-ij}C_k \\ &+ A'_iC'_jB'_{ik}\,\pi(i\lambda_1) + A'_{-i}C'_jB'_{-ik}\,\pi(-i\lambda_1) \quad i,j,k = +,- \end{aligned} \tag{32}$$

$$A_+ = A + \not{k}\,\not{p}_3\,, A_- = A + im\frac{p_3\cdot k}{p_4\cdot k}\not{k}\,, A = p_3\cdot p_4 + m^2\frac{p_3\cdot k}{p_4\cdot k}$$

$$A'_+ = A' + \not{p}_4\,\not{k}\,, A'_- = A' - im\frac{p_4\cdot k}{p_3\cdot k}\not{k}\,, A' = p_3\cdot p_4 + m^2\frac{p_4\cdot k}{p_3\cdot k}$$

$$B_{+\pm} = \not{p}_4\,\not{p}_1 \mp m\,\not{n}_4\,\not{p}_1\,, B_{-\pm} = -im\,\not{p}_1 \mp i\,\not{p}_4\,\not{n}_4\,\not{p}_1\,, \tag{33}$$

$$C_\pm = p_2\cdot p_3 \mp mp_2\cdot n_3 - 2im\,\not{p}_2 \mp 2i\,\not{p}_3\,\not{n}_3\,\not{p}_2$$

$$C'_\pm = p_1\cdot p_4 \mp mp_1\cdot n_4 + 2im\,\not{p}_1 \pm 2i\,\not{p}_1\,\not{n}_4\,\not{p}_4\,,$$

$$B'_{+\pm} = im\,\not{p}_2 \pm i\,\not{p}_2\,\not{n}_3\,\not{p}_3\,, B'_{-\pm} = \not{p}_2\,\not{p}_3 \mp m\,\not{p}_2\,\not{n}_3$$

Finally

$$M = \frac{i}{16\sqrt{2}\,N}\Lambda^{-}(\lambda_1,\lambda_2)\sum_{ijk=\pm} N_{14j}N_{23k}\,tr\,D_{ijk}\Lambda^{i}(\sigma,\lambda_1)\Lambda^{j}(\lambda_1,\lambda_4)\Lambda^{k}(\lambda_2,\lambda_3)\ . \tag{34}$$

The method developed here can be of great advantage in computing the amplitude for hard subprocesses in perturbative QCD where each quark is assumed to be collinear with the hadron. In this approximation we only need the kinematics of a four body process to analyze $\pi-\pi$ or $\pi-p$ scattering.

I wish to express my gratitude to Professor S. Drell for the hospitality at the Stanford Linear Accelerator Center. It is a pleasure to thank Professor S. J. Brodsky for reading the manuscript.

REFERENCES

1. P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, Nucl. Phys. B206, 53 (1982).
2. M. Caffo and E. Remiddi, Helv. Phys. Acta 55, 339 (1982); see also G. R. Farrar and F. Neri, RU-83-20.
3. J.S.R. Chisholm, Nuovo Cimento 30, 426 (1963), Comput. Phys. Comm. 4, 205 (1972); J. Kahane, J. Math. Phys. 9, 1732 (1968).
4. G. P. Lepage, S. J. Brodsky, Phys. Rev. D22, 2157 (1980).
5. L. L. Deraad and Y. J. Ng, Phys. Rev. D10, 3440 (1974); D11, 1586 (1975).
6. J. D. Bjorken and M. C. Chen, Phys. Rev. 154, 1135 (1967); L. Michel and A. S Wightman, Phys. Rev. 98, 1190 (1955).