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COMMENT ON QUANTIZATION IN THE TEMPORAL GAUGE*

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ABSTRACT

A contradictory result recently reported for canonical quantization in the temporal gauge, and claimed to render the validity of the quantization procedure itself doubtful, is shown to be incorrect and the result of the application of a nonexistent hermiticity property with respect to the unphysical degrees of freedom of the gauge field.

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In a recent Brief Report,[1] the validity of the customary formalism of the quantization of gauge theories in the temporal gauge was questioned on the basis of a contradictory result that appears to be a straightforward consequence of the formalism. Since temporal gauge quantization is usually considered to be a sound and reliable (if calculationally cumbersome) canonical quantization procedure for nonabelian gauge theories such as QCD,[2] a result such as the one above, if based on a valid derivation, would obviously have serious consequences as regards the internal consistency of such theories. Upon scrutiny, however, the equation that constitutes the basis of the claim in Ref. 1 turns out to be false, the faulty step in its derivation being a tacit use of the property of hermiticity for a certain operator beyond its proper domain of validity. The object of this note is to identify the faulty step and to explain in some detail the somewhat nontrivial manner in which it arises.

We start with a brief account of the original derivation and establish the notation at the same time. Following Ref. 1, consider the (pure) gauge theory defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}\, F^a_{\mu\nu}\, F^{a\mu\nu} \quad , \tag{1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu\, A^c_\nu \quad . \tag{2}$$

When canonically quantized in the temporal gauge $A^a_0 = 0$ in the standard way, there results the Hamiltonian system defined by (various operators are henceforth understood to be at equal times)

$$\mathcal{H} = \frac{1}{2}\, \pi^a_i\, \pi^a_i + \frac{1}{4}\, F^a_{ij} F^a_{ij} \quad , \tag{3}$$

$$\left[A^a_i(x), \pi^b_j(y)\right] = i\delta^{ab}\, \delta_{ij}\, \delta^3(\vec{x} - \vec{y}) \quad , \tag{4}$$

subject to the constraint that the physical states of the theory satisfy

$$G^a(x)|\alpha\rangle = 0 \quad , \tag{5}$$

where

$$G^a = \partial_i \pi_i^a + g f^{abc} A_i^b \, \pi_j^c \quad . \tag{6}$$

As usual, Eq. (5) is the implementation of the residual invariance with respect to time-independent gauge transformations and states that only those solutions of the Hamiltonian system (3)-(4) that satisfy (5) belong to the Hilbert space of physical states.

The contradictory result in Ref. 1 is arrived at by taking the matrix element of the commutator equation ($x^o = y^o$)

$$\left[G^a(x), A_i^b(y)\right] = -i\left[\delta^{ab}\partial_i + g f^{acb} \, A_i^c(x)\right] \, \delta^3(\vec{x} - \vec{y}) \quad , \tag{7}$$

between two color-neutral physical states $|\alpha\rangle$ and $|\alpha'\rangle$. It is then stated that the left-hand side of the resulting equation vanishes because of the constraints (5). Since the matrix element of A_i^c on the right-hand side also vanishes on account of the color neutrality of the states, there follows the absurd result $\langle\alpha|\alpha'\rangle \; \partial_i \delta^3(\vec{x} - \vec{y}) = 0$, which implies the vanishing of $\langle\alpha|\alpha'\rangle$ for any pair of color-neutral (physical) states α and α'. If valid, this result would obviously render the entire theory inconsistent.

The statement that

$$\left\langle \alpha \left| \left[G^a(x), A_i^b(y)\right] \right| \alpha' \right\rangle = 0 \quad \text{(false)} \quad , \tag{8}$$

is in general false, however, the reason being the fact that the hermiticity property

$$\left\langle \alpha \left| G^a(x) \; A_i^b(y) \right| \alpha' \right\rangle = \left\langle G^a(x)\alpha \left| A_i^b(y) \right| \alpha' \right\rangle \quad \text{(false)} \quad , \tag{9}$$

tacitly assumed in arriving at Eq. (8), does not hold. This is because the scalar product implicit in the above matrix elements involves only the physical degrees of freedom of a given field configuration A_i^a, and as such cannot support hermiticity with respect to the unphysical ones which in fact occur in Eq. (9) and which are precisely the ones that give rise to the absurd result exhibited above. Moreover, this lack of hermicity relative to the unphysical degrees of freedom has nothing to do with the nonabelian aspect of the theory and fully survives in the abelian limit. We shall therefore take advantage of this circumstance and demonstrate the above assertion in an explicit realization of the theory in the abelian limit (i.e., for the free photon field).

In the configuration representation of the Hamiltonian system (3)-(4) in the abelian limit, with

$$\pi_i(x) = -i\ \frac{\delta}{\delta A_i(x)}\ , \tag{10}$$

the states of the system are realized as ("square-integrable") wave functionals of the field configuration, e.g., $\alpha[A] = \langle A|\alpha\rangle$, subject to the constraint (5) which (together with suitable boundary conditions that serve to exclude the longitudinal electric fields caused by charges at spatial infinity) simply asserts that $\alpha[A]$ is independent of $\partial_i A_i$ and it is a functional of F_{ij} only. Moreover, the inner product is given by

$$\langle\alpha|\alpha'\rangle = \int \mathcal{D}[F]\ \alpha^*[F]\ \alpha'[F]\ , \tag{11}$$

where $\mathcal{D}[F]$ denotes functional integration with respect to F_{ij}. Note that there is no integration with respect to $\partial_i A_i$ in Eq. (11).

It is now trivial to see why Eq. (9) fails. In the configuration representation, where $\partial_i \pi_i = i\nabla^2 \delta/\delta(\partial_i A_i)$, it reads

$$\begin{aligned} &\int \mathcal{D}[F]\ \alpha^*[F]\ (i)\nabla_x^2\ \frac{\delta}{\delta[\partial_i A_i(x)]}\ A_j(y)\ \alpha'[F] \\ &= \int \mathcal{D}[F] \left\{ i\nabla_x^2\ \frac{\delta}{\delta[\partial_i A_i(x)]}\ \alpha[F] \right\}^*\ A_j(y)\ \alpha'[F] \quad \text{(false)}\ . \end{aligned} \tag{9'}$$

While the right-hand side vanishes, the left-hand side is given by $\langle\alpha|\alpha'\rangle(i\partial/\partial\vec{y}_j)\,\delta^3(\vec{x}-\vec{y})$, which is recognized as the anomalous term causing the contradiction in Ref. 1; in general, the G^a are not hermitian operators in the unconstrained version of the Hamiltonian system (4)-(5).[3] It is now clear that the transposition $\langle\alpha|G^a|\beta\rangle = \langle G^a\alpha|\beta\rangle$, where $|\beta\rangle = A_i^b|\alpha'\rangle$, can only be true if $\langle\alpha|\alpha'\rangle = 0$.

This last conclusion effectively turns the contradictory result of Ref. 1 on its head and renders it innocuous.

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REFERENCES

1. Y. Kakudo et al., Phys. Rev. D 27, 1954 (1983).

2. J. D. Bjorken, in Proceedings of the SLAC Summer Institute, 1979, edited by A. Mosher, Stanford Linear Accelerator Center Report No. 224 (unpublished), p. 219.

3. Even if one insists on (incorrectly) enlarging the inner product in Eq. (11) to include integration over $\partial_i A_i$, Eq. (9) will fail, simply because $A_j(y)\ \alpha'[F]$ is manifestly square-nonintegrable in the variable $\partial_i A_i$ and therefore a fortiori not in the domain of hermiticity of the operator $-i\delta/\delta(\partial_i A_i)$. A simple analog is the failure of $\int d\varsigma\ v^*(\varsigma)(-i\partial/\partial\xi)\xi u(\varsigma) = \int d\varsigma\ [-i\partial/\partial\xi\ v(\varsigma)]^* \xi u(\varsigma)$ (false), where ς is the counterpart of F_{ij} and ξ that of $\partial_i A_i$. This failure occurs with or without an integration with respect to ξ.