SLAC-PUB-3088
March 1983
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SUPERSPACE GEOMETRY OF FERMIONIC STRINGS *

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ABSTRACT

We calculate the effective action in Polyakov's fermionic string theory in a manifestly supersymmetric formalism, including the effects of string boundaries. Supersymmetry causes divergence cancellations such that no Liouville interaction term is generated by renormalization of the string fields. For open surfaces, supersymmetry is spontaneously broken by the presence of the boundary, leading to a linear divergence proportional to the length of the boundary.

Submitted to Physical Review D.

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* Work supported by the Department of Energy, contract DE-AC03-76SF00515

1. Introduction

A string is the natural one-dimensional extension of the classical point particle. Whereas a particle tries to minimize the length of its world line, the string moves so as to minimize its world surface area. Surface dynamics appear in many contexts, in particular domain wall fluctuations[1] and dynamics of large Wilson loops in non-abelian gauge theories;[2] the low energy excitations of the QCD flux tube connecting two quarks are well described by a string model (the "dual" model).

A particularly interesting class of string theories arises when fermions are glued to the string. Such theories are found in $N = \infty$ lattice gauge theories,[3] the three-dimensional Ising model,[4] and dual models.[5] The structure of the string is enriched by the polarization information carried by the fermions. For instance, the fermionic dual model string may be considered as a one-dimensional chain of fermionic partons[6] with nearest-neighbor interactions.

Because of the these many-fold applications, an understanding of the partition function or quantum mechanics of strings is of considerable importance. However, the quantized string is extremely singular since the zero-point oscillations of the string modes cause the mean displacement to diverge. Early attempts at formulating the quantum theory had many inconsistencies – they were only consistent in unphysical spacetime dimensions (10 for the fermionic string, 26 for the bosonic string), the ground state was a tachyon, Green's functions were only known on-shell, etc. Because of these difficulties, the subject lay dormant for many years. Recently interest has been revived by the work of Polyakov[7] on the path integral quantization of an action originally proposed by Brink, Di Vecchia, and Howe[8] and Deser and Zumino.[9] The basic string variables in this model are a set of coordinate functions $X^{\mu}(z)$ describing the location of the string world sheet in spacetime. Here the index μ runs over the directions in the d-dimensional imbedding spacetime and z is a set of intrinsic coordinates on the world sheet. Coordinate reparametrizations leave the physical configuration of the string unchanged, thus we should make the action invariant under general coordinate transformations of the z's. This may be achieved by coupling the Lagrangian covariantly to a metric g_{ab} on the intrinsic coordinates. The simplest action that we might write down is

$$S = \int d^2z\,\sqrt{g}\;g^{ab}\partial_a X^\mu \partial_b X_\mu \tag{1.1}$$

The equations of motion derived by varying both g_{ab} and X^μ are just those which generate minimal area surfaces.[7,8] Polyakov has evaluated the path integral for this action in the gauge $g_{ab} = e^\phi \delta_{ab}$. The local scale factor ϕ drops out of the action (1.1), allowing an explicit evaluation of the functional determinants arising from gauge fixing and integrating over X^μ. The only dependence on the field ϕ lies in the regularization of the determinants; this is a short distance effect, and so the effective action calculated from these determinants is local:

$$S_{eff} = \frac{26-d}{48\pi}\int d^2z\Big[(\partial\phi)^2 + \mu^2 e^{2\phi}\Big] + \text{ boundary terms } . \tag{1.2}$$

Thus the consistency of the quantized bosonic string in other than 26 dimensions should be restored by the dynamics of the ϕ field. An elegant formalism has been developed for the analysis of the bosonic string with arbitrary world surface topology[10] which exploits the complex structure of two dimensional manifolds and employs heat kernel methods for the determinant calculations.

The most natural way to introduce fermions into the theory is to supersymmetrize it. The action (1.1) becomes two-dimensional supergravity coupled to a set of d scalar supermultiplets. The effective action may be calculated in a manner analogous to the bosonic string using the component fields g_{ab}, X^μ and their partners ψ_a^α and χ_α^μ (α is a two-dimensional spinor index). The present work was initiated with the goal of using superspace methods in order to maintain manifest supersymmetry in the calculation. Indeed we have found that the elegant calculus of Ref. 10 has a natural superspace generalization which clarifies the discussion of cancellations due to the supersymmetry, especially for string surfaces with boundaries.

In Section 2 we review the superspace formulation of two-dimensional supergravity and introduce the generalization of the complex tensor calculus and heat kernel methods of Ref. 10 to the supersymmetric case. In Sections 3 and 4 we analyze the fermionic string path integral using this formalism and find results which differ from a previous calculation in component fields.[11] In particular, the μ^2 term in (1.2) corresponds to a divergent cosmological constant in the original theory (1.1); in the supersymmetric theory, this divergence cancels between bosons and fermions and no such term is generated (this result was first obtained in Ref. 12). The presence of a boundary on the world surface spontaneously breaks the supersymmetry, so that near the boundary this cancellation fails and generates a divergent boundary cosmological term $\lambda \int_{\partial M} dx\, e^{\phi(x)}$. Thus at least in the interior of the parameter space manifold the action is that of a free field. Section 5 contains a discussion of these results. Three appendices are included: Appendix A lists the superspace notations and conventions, Appendix B sketches the determinant calculations used in Section 4, and Appendix C derives the effective action for an alternate choice of boundary conditions on the string fields.

Much of the analysis presented here is a direct corollary of the investigations of Ref. 10. Some details only touched on in the following are carefully discussed there, and the reader may find an understanding of that paper helpful.

2. Two Dimensional Supergravity

The superspace formulation of two-dimensional supergravity is highly redundant. All two dimensional manifolds are conformally flat, and so describable locally by a single superfield;[13] yet the vierbein e_A^M comprises 16 superfields, and the gauge freedoms of local diffeomorphisms and rotations remove only 5 of these. The other redundant components must be removed by constraints on the vierbein. Such constraints can be applied covariantly by fixing some components of the torsion tensor T_{AB}^C, defined by

$$[\nabla_A, \nabla_B\} = T_{AB}^C \nabla_C + R_{AB} M \tag{2.1}$$

where

$$\nabla_A = e_A^M \partial_M + \phi_A M \tag{2.2}$$

is the covariant derivative and R_{AB}, ϕ_A, and M are the curvature, spin connection, and Lorentz generator, respectively.

Following Ref. 13, we choose the constraints

$$T^c_{\alpha\beta} = 2i\gamma^c_{\alpha\beta} \tag{2.3a}$$

$$T^\gamma_{\alpha\beta} = 0 \tag{2.3b}$$

$$T^c_{ab} = 0\,. \tag{2.3c}$$

The first of these ensures that the supersymmetry algebra $\{Q_\alpha, Q_\beta\} = 2i\,\gamma^a_{\alpha\beta}P_a$ is maintained, and enables one to express e^M_a in terms of e^M_α. The next two constraints allow one to determine the spin connection ϕ_A in terms of the vierbein just as (2.3c) allows one to solve for the spin connection in ordinary general relativity. One may verify that the vierbein of the conformal form

$$e_\alpha = e^\psi D_\alpha\,; \qquad e_a = e^{2\psi}\Big[\partial_a + i\,\gamma_a^{\bar{\alpha}\beta}(D_\alpha\psi)\,D_\beta\Big] \tag{2.4}$$

indeed satisfies (2.3a-c) and the spin connection is determined to be

$$\phi_\alpha = -2\gamma^{5\beta}_\alpha\, D_\beta e^\psi\,; \qquad \phi_a = -\epsilon^b_a \partial_b e^{2\psi} \tag{2.5}$$

The most general vierbein satisfying (2.3a-c) is locally gauge equivalent to (2.4) because e^M_a is determined from e^M_α, and (2.3b) provides two constraints on e^M_α. This leaves 6 superfield degrees of freedom: 2 vector and 2 spinor fields generating diffeomorphisms, 1 field generating local rotations, and the conformal factor ψ.

Having chosen a gauge in which the vierbein takes the form (2.4), we may classify the set of tensors on the manifold which carry tangent space indices. This is most conveniently done in complex coordinates

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}(x_1 + ix_2) \qquad \bar{x} = \frac{1}{\sqrt{2}}(x_1 - ix_2) \\ \theta &= \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \qquad \bar{\theta} = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2)\,. \end{aligned} \tag{2.6}$$

Tensor indices may be freely raised and lowered with the tangent space metric

$$\delta_{ab} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \delta_{\alpha\beta} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2.7}$$

Thus an arbitrary tensor with raised and lowered x, $\bar{x}$, θ, and $\bar{\theta}$ indices may be turned into one with only x's and θ's by raising and lowering indices – a raised $\bar{x}$ is equivalent to a lowered x index, etc. The γ matrices are Clebsch-Gordon coefficients between vectors and spinors, so we may use them to replace an x index by two θ indices (only $\gamma^{x}_{\theta\theta} = -\gamma^{\bar{x}}_{\bar{\theta}\bar{\theta}} = \sqrt{2}$ is nonzero in these coordinates).

Tensors are classified by their transformation law under the tangent space group $0(2) = U(1)$. A spinor χ^{α} transforms as

$$\chi^{\theta} \to e^{-i\frac{\beta}{2}}\chi^{\theta} \; ; \qquad \chi_{\theta} \to e^{i\frac{\beta}{2}}\chi_{\theta} \tag{2.8}$$

under a rotation by angle β. Thus a general tensor with n_{+} raised θ indices and n_{-} lowered θ indices transforms like a spin $(n_{+} - n_{-})/2$ object. The rotational transformation properties of a given tensor depend only on the different $n_{+} - n_{-}$, so all tensors lie in one of the spaces T^{n} defined by

$$T^{n} = \{T | T \to e^{i\frac{n}{2}\beta}\, T \text{ under a rotation by angle } \beta\} \, . \tag{2.9}$$

The covariant derivatives (2.2) on T^{n} take a particularly simple form in the basis (2.4)

$$\begin{aligned} \nabla^{n}_{\theta} &= e^{\psi} D_{\theta} + n(D_{\theta} e^{\psi}) = e^{-(n-1)\Psi} D_{\theta}\, e^{n\Psi} \\ \nabla^{n}_{\bar{\theta}} &= e^{\psi} D_{\bar{\theta}} - n(D_{\bar{\theta}} e^{\psi}) = e^{(n+1)\psi} D_{\bar{\theta}} e^{-n\Psi} \end{aligned} \tag{2.10}$$

Defining an inner product on T^{n} by

$$< T, S >= \int_{M} d^{2}z\, e^{-1} T^{*} S \qquad T, S \in T^{n} \tag{2.11}$$

we see that formally $\nabla_{\theta} = -(\nabla_{\bar{\theta}})^{+}$, neglecting boundary contributions (these will be discussed in Section 4). Note that ∇^{n}_{θ} maps T^{n} into T^{n+1} whereas $\nabla^{n}_{\bar{\theta}}$

maps $\mathcal{T}^n$ into $\mathcal{T}^{n-1}$. We may define two distinct Laplace operators $\Delta_n^{(\pm)} : \mathcal{T}^n \to \mathcal{T}^n$ by

$$\begin{aligned} \Delta_n^{(+)} &= \nabla_{\bar\theta}^{(n+1)} \nabla_\theta^{(n)} \\ \Delta_n^{(-)} &= \nabla_\theta^{(n-1)} \nabla_{\bar\theta}^{(n)} \end{aligned} \tag{2.12}$$

In Section 3, we will find that the effective action for the fermionic Polyakov string is determined by the determinants of these Laplacians. Although we cannot calculate $\det \Delta$ for a general vierbein, we may calculate $\delta(\det \Delta)/\delta\psi$ and hence determine the dependence of this quantity on the conformal factor.

One conventional definition of the determinant of a Laplacian Δ is

$$\ell n \,\mathrm{sdet}' \Delta = -\int_\epsilon^\infty \frac{dt}{t} \mathrm{str}' \left[e^{-t\Delta} \right] \tag{2.13}$$

where the prime indicates that the operation is to be carried out over the space orthogonal to the zero modes of Δ (for a definition of sdet and str, see Appendix A). The lower limit ϵ of the integral over t regulates the divergences of the determinant by cutting off the large eigenvalues of the Laplacian. However, the heat kernel $e^{-t\Delta_n^{(\pm)}}$ does not have a diffusive behavior and does not tend to zero as $t \to \infty$ since the spectrum of Δ_n is not bounded below; thus $e^{-t\Delta_n^{(\pm)}}$ has spurious poles and divergences in its behavior even in flat space. To remedy this situation, we calculate instead

$$\ell n \,\mathrm{sdet}' \Delta_n^{(\pm)} = \frac{1}{2} \ell n \,\mathrm{sdet}' \left(\Delta_n^{(\pm)} \right)^2 = -\frac{1}{2} \int_\epsilon^\infty \frac{dt}{t} \mathrm{str}' \, e^{-t\Delta_n^{(\pm)2}} \tag{2.14}$$

which is perfectly well behaved since $(\Delta_n^{(\pm)})^2 = -\frac{1}{2}\Box$ in flat space. If we perform a variation of $\Delta_n^{(+)}$ with respect to the conformal factor ψ, we find

$$\delta \Delta_n^{(+)} = (n+2)\delta\psi \Delta_n^{(+)} + n \Delta_n^{(+)} \delta\psi - 2n \nabla_{\bar\theta}^{(n+1)} \delta\psi \nabla_\theta^{(n)} \, . \tag{2.15}$$

Proceeding along the lines of Ref. 10, one may prove that

$$\delta \ell n \,\mathrm{sdet}' \Delta_n^{(+)} = (-)^n \left[2(n+1) \,\mathrm{str}' \left(\delta\psi e^{-\epsilon \Delta_n^{(+)2}} \right) + 2n \,\mathrm{str}' \left(\delta\psi e^{-\epsilon \Delta_{n+1}^{(-)2}} \right) \right] \tag{2.16}$$

where the different signs arise from the fact that we are taking the strace and not the trace. Equation (2.16) is tractable because it only involves the local structure of the manifold; a diffusing particle cannot travel far in an infinitesimal time. We defer calculation of the determinants until we have properly treated the boundary conditions on the spaces $\mathcal{T}^n$.

3. The Fermionic String

The extension of the action (1.1) to the fermionic string was first considered in Refs. 8 and 9. In this section we review the description of the string in superspace,[13] deriving the equations of motion and boundary conditions for the string fields as well as the Faddeev-Popov determinant for the conformal gauge (2.4). This will set the stage for the computation in Section 4 of the effective action coming from the integration over the gauge group and string fields.

The action for the string may be written

$$\begin{aligned} S = \frac{1}{2} A \int_M d^2z\, e^{-1} e^\alpha X^\mu \cdot e_\alpha X^\mu + B\Big[\frac{1}{2}\int_M d^2z\, e^{-1} R + \int_{\partial M} ds\, k\Big] \\ + C\int_M d^2z\, e^{-1} + D\int_{\partial M} ds + E\int_{\partial M} ds\, k \end{aligned} \tag{3.1}$$

where $\mu = 1, \cdots, d$ (the dimension of the imbedding space of the string), M is the parameter space manifold, X^μ and the unconstrained components of e_A^M are dynamical variables, and

$$e = \operatorname{sdet} e_A^M \tag{3.2a}$$

$$R = -\frac{1}{2}\gamma^{5\,\alpha\beta}\nabla_\alpha\nabla_\beta = -\frac{1}{2}\gamma^{5\,\alpha\beta}\nabla_\alpha\phi_\beta \tag{3.2b}$$

$$\int_{\partial M} ds\, k = \int_{\partial M} dx \int d^2\theta\, e^{-1}\Big[-\frac{1}{4}\, n_a\gamma^{5\,\alpha\beta} e_\alpha^m n_m \nabla_\beta t^a\Big] \tag{3.2c}$$

Here n^a and t^a denote the unit tangent and normal vectors to the boundary. Possible terms involving the torsion T_{AB}^C may be expressed in terms of R by using the constraints (2.3) and the Bianchi identities.[13] Thus the action (3.1) includes all the terms which are allowed under the requirements of general covariance and renormalizability. The B term in (3.1) is the superspace version of the Gauss-Bonnet invariant

$$\frac{1}{2}\int_M d^2 z e^{-1} R + \int_{\partial M} ds\, k = 2\pi\chi(M) \tag{3.3}$$

where the Euler characteristic $\chi(M)$ depends only on the topology of the manifold:

$$\chi(M) = 2 - 2\cdot(\#\text{ handles of M}) - (\#\text{ boundaries of M}) \tag{3.4}$$

Since (3.3) is metric-independent, it will have no influence on the dynamics of the string. We will find that supersymmetry prevents renormalization of the C term for either Neumann or Dirichlet boundary conditions on X^μ. The D term is the length of the boundary in superspace, hence it is an integral over one bosonic and two fermionic coordinates. In the gauge (2.4) this term will be independent of ψ since the factors of e^ψ cancel in the vierbein determinant in the measure. An important consequence of this fact is that any boundary cosmological term which does depend on ψ must break supersymmetry.

Variation of the action yields

$$\begin{aligned} \delta S = A\int_M d^2 z\, e^{-1}\Big[&\delta X^\mu \Delta_0 X^\mu - \Big(\delta H_a^{\ a} - \delta H_\beta^{\ \beta}\Big)\frac{1}{2} e^\alpha X^\mu \cdot e_\alpha X^\mu \\ &+ \delta H^{\alpha\beta} e_\beta X^\mu \cdot e_\alpha X^\mu + \delta H^{\alpha b} e_\alpha X^\mu \cdot e_b X^\mu\Big] \\ &+ C\int_M d^2 z\Big(\delta H_\alpha^{\ \alpha} - \delta H_a^{\ a}\Big) + \text{boundary terms} \end{aligned} \tag{3.5}$$

where

$$\Delta_0 = -e e^\alpha e^{-1} e_\alpha \tag{3.6a}$$

$$\delta H_A^{\ B} = (\delta e_A^{\ M}) e_M^{\ B} \tag{3.6b}$$

We concentrate on the A term which will result in the "classical" string equations of motion. It is straightforward to show that

$$\begin{aligned}\delta T_{AB}^{C} = {} & H_A^D T_{DB}^C - T_{AB}^D H_D^C + (-)^{ab} H_B^D T_{AD}^C \\ & + \nabla_A H_B^C - (-)^{ab} \nabla_B H_A^C + (H_A^D \phi_D - \delta\phi_A) M_B^C - (-)^{ab} (H_B^D \phi_D - \delta\phi_B) M_A^C\end{aligned} \tag{3.7}$$

which, combined with $\delta T_{\alpha\beta}^{c} = 0$, yeilds

$$H_a^b = (\gamma^b \gamma_a)_\beta^\alpha H_\alpha^\beta - \frac{i}{2} \gamma_a^{\alpha\beta} \nabla_\alpha H_\beta^b \tag{3.8}$$

Substituting this result into (3.5) yields the equations of motion

$$\begin{aligned}\Delta_0 X^\mu &= 0 \\ (\gamma^b \gamma_a)^{\alpha\beta} \nabla_\beta X^\mu \cdot \nabla_b X^\mu &= 0\end{aligned} \tag{3.9}$$

Equations (3.9), when written in component fields, reproduce the usual fermi string equations.[5]. In contrast to the bosonic case, an algebraic solution for the vierbein in terms of the matter fields is not possible due to the necessity of using tangent space indices.

Now we turn to a discussion of boundary terms. The boundary term in (3.5) arising from the matter field variation is

$$\int_{\partial M} dx\, d^2\theta\ \delta X^\mu \theta \cdot \not{n} \cdot D X^\mu = 0 \tag{3.10}$$

where n^a is a unit vector normal to the boundary. Let

$$X(x,\theta) = A(x) + i\theta \cdot \chi(x) + i\theta^2 F(x)\ . \tag{3.11}$$

Then there are two choices of boundary conditions for which (3.10) holds:

$$\left.\begin{aligned} A &= 0 \\ \chi_\theta &= \pm \left[\frac{n_z}{n_{\bar{z}}}\right]^{1/2} \chi_{\bar{\theta}} \end{aligned}\right\} \text{ on } \partial M \tag{3.12}$$

and

$$\left.\begin{aligned} n \cdot \partial A &= 0 \\ \chi_\theta &= \pm\left[\tfrac{n_z}{n_{\bar z}}\right]^{1/2} \chi_{\bar\theta} \end{aligned}\right\} \text{ on } \partial M \tag{3.13}$$

where $n_z, n_{\bar z}$ are the components of $\vec{n}$ in the basis (2.6). These conditions are sufficient to ensure that (3.10) vanishes provided the variation δX also obeys (3.12) and (3.13). The choice of boundary conditions depends on the physical problem we wish to model. The fluctuations of surfaces with a fixed boundary provides a toy model of the Wilson loop. When we vary the action, the boundary of the string is held fixed; thus (3.12) is the appropriate boundary condition. In the case of the dual model, the end of the string is free; hence the string variables satisfy the condition (3.13). The Neumann boundary conditions (3.13) are just those of Ref. 11, where they are derived by performing supersymmetry transformations on $n \cdot \partial A = 0$. We will complete the discussion of the boundary conditions on X^μ below when we discuss the problem for fields belonging to any of the spaces $\mathcal{T}^n$.

We now must discuss the measure and gauge-fixing determinant for the vierbein. Since e_a^M is determined from e_α^M we look for the most general metric on the space of e_α^M

$$||\delta e_\alpha||^2 = \int d^2z\, e^{-1}\Big[\delta_{CD}\delta^{\alpha\beta} + a\delta_C^\beta\delta_D^\alpha + b\delta_C^\alpha\delta_D^\beta\Big]\, H_\alpha^C H_\beta^D \tag{3.14}$$

In the first term, $\delta_{\gamma\delta}\delta^{\alpha\beta} = \delta_\gamma^\alpha\delta_\delta^\beta - \delta_\gamma^\beta\delta_\delta^\alpha$ implies that only the $\delta_{dc}\delta^{\alpha\beta}$ term is independent. The constraints (2.3) imply, through their variation (3.7), that $\gamma_\alpha^{a\beta} H_\beta^\alpha$ is related to H_α^b, $\gamma_\alpha^{5\beta} H_\beta^\alpha$, and H_α^α. Thus the only independent terms are

$$||\delta e_\alpha||^2 = \int d^2z\, e^{-1}\Big[H^{ab}H_{ab} + a(\gamma_\alpha^{5\,\beta} H_\beta^\alpha)^2 + b(H_\alpha^\alpha)^2\Big] \tag{3.15}$$

We fix the gauge by specifying

$$e_\alpha = e^{\psi}\,\hat{e}_\alpha \tag{3.16}$$

with $\hat{e}_\alpha$ a suitable background vierbein satisfying the constraints (2.3). Next let us decompose (3.15) into gauge transformations (diffeomorphisms), local tangent space rotations, and conformal transformations. The variation induced by a diffeomorphism $\delta z^M = \Lambda^M$, tangent space rotation W, and conformal transformation ψ is

$$\delta e_A^M \partial_M = [\Lambda^N \partial_N, e_A^M \partial_M] + W M_A^B e_B^M \partial_M + \psi e_A^M \partial_M \tag{3.17}$$

from which one may show

$$H_A^B = \Lambda^C T_{CA}^B - \nabla_A \Lambda^B + (\Lambda^C \phi_C + W) M_A^B + \psi \delta_A^B \tag{3.18}$$

Substituting (3.18) into (3.15), the Jacobian for the change of variables from e_α^M to ψ, W, and Λ^A may be computed to be

$$\left| \frac{\partial(e_\alpha^M)}{\partial(\psi, W, \Lambda^A)} \right| = \left| \begin{array}{c|c} \begin{matrix} \sqrt{b} & 0 \\ 0 & \sqrt{a} \end{matrix} & \text{stuff} \\ \hline 0 & Q_{AB} \end{array} \right| = (\ \text{sdet}\ Q^+ Q\ )^{1/2} \tag{3.19}$$

where a and b are the constants in (3.14) and Q_{AB} is determined from the first term of (3.15):

$$\begin{aligned} \int d^2z\, e^{-1} H^{\alpha b} H_{\alpha b} &= \int d^2z\, e^{-1} (\Lambda^C T_C^{\alpha b} - \nabla^\alpha \Lambda^b)(\Lambda^D T_{D\alpha b} - \nabla_\alpha \Lambda_b) \\ &= \int d^2z\, e^{-1} \Lambda^C (T_C^{\alpha b} + \delta_C^b \nabla^\alpha)(T_{D\alpha b} - \delta_{Db} \nabla_\alpha) \Lambda^D \end{aligned} \tag{3.20}$$

so that

$$(Q^+ Q)_{CD} = (T_C^{\alpha b} + \delta_C^b \nabla^\alpha)(T_{D\alpha b} - \delta_{Db} \nabla_\alpha) \tag{3.21}$$

This expression may be simplified by use of the formula

$$\text{sdet} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \text{sdet}\,(A - BD^{-1}C)\,\text{sdet}\ D \tag{3.22}$$

where the block form denotes any partition of the bose and fermi variables. Choosing $D = (Q^+ Q)_{\gamma\delta}$ we find

$$\text{sdet}\, Q^+ Q = \text{const.} \times \text{sdet}\, [\nabla^\alpha (\gamma_d \gamma_c)_\alpha^\beta \nabla_\beta] \tag{3.23}$$

which, when evaluated in the complex basis (2.10), becomes

$$\operatorname{sdet} Q^+Q = \text{const.} \times (\operatorname{sdet} \Delta_2^{(+)})(\operatorname{sdet} \Delta_{-2}^{(-)}) \ . \tag{3.24}$$

We can now write a complete expression for the string effective action obtained in the conformal gauge (3.16) after integrating over the string fields X^μ, diffeomorphisms Λ^A, and local rotations W:

$$\begin{aligned} S_{eff}[\psi] = & \frac{1}{2}\, ln\left(\operatorname{sdet} \Delta_2^{(+)} \cdot \operatorname{sdet} \Delta_{-2}^{(-)}\right) - \frac{d}{2}\, ln \operatorname{sdet} \Delta_0 \\ & + B \cdot \chi(M) + C \int\limits_M d^2z\, \hat{e}^{-1}\, e^{-2\psi} \\ & + E \int\limits_{\partial M} ds d^2\theta k \end{aligned} \tag{3.25}$$

The contribution of the string and ghost determinants in (3.25) is

$$\begin{aligned} S_{eff}^{matter\ +\ ghost} = & \frac{10-d}{4\pi} \int\limits_M d^2z\, \hat{e}^{-1} \left[\hat{e}^\alpha \psi \cdot \hat{e}_\alpha \psi + \hat{R}\,\psi\right] \\ & + \text{boundary} \quad \text{terms} \end{aligned} \tag{3.26}$$

in agreement with Polyakov's result.[7,12] We will obtain this result, and its boundary corrections, in the following section.

4. The String Effective Action

We will now evaluate the determinants of all the Laplacians (2.12) using (2.16). We will thus find the ψ-dependence of the determinants in (3.25). The first step in this program is to find the appropriate boundary conditions on the tensor spaces $\mathcal{T}^n$ on which the Laplacians act. The analysis is similar to that of Ref. 10. Define the tensor space S^n as the direct sum $\mathcal{T}^n \oplus \mathcal{T}^{-n}$ along with the corresponding differential operators $D^n = \nabla_\theta^n \oplus \nabla_{\bar\theta}^{-n}$ and $L_n^{(\pm)} = \Delta_n^{(\pm)} \oplus \Delta_{-n}^{(\mp)}$. First note the identity

$$\left\langle \Phi_1, D^n \Phi_2 \right\rangle - \left\langle D^{n^+} \Phi_1, \Phi_2 \right\rangle = 2\, Im \int\limits_{\partial M} d\,\bar{x}\, d^2\theta e^{-\psi} \Phi_1^* i \sqrt{2}\, \theta n_x \Phi_2 \qquad . \tag{4.1}$$

$$\text{for } \ \Phi_1 \epsilon\, S^{n+1}, \Phi_2 \epsilon\, S^n$$

We would like the right hand side to vanish, for then the operators D^n would have well-defined adjoints. This will define the proper boundary conditions for $S^{(n+1)}$ in terms of those for S^n. If we denote the components of $\phi^n \epsilon \mathcal{T}^n$ and ψ by

$$\begin{aligned} \phi^n &= A^n + i\theta\chi^n + i\,\bar\theta\,\bar\chi^n + i\,\bar\theta\,\theta F^n \\ \psi &= \phi + i\theta\eta + i\,\bar\theta\,\bar\eta + i\,\bar\theta\,\theta\rho \end{aligned} \tag{4.2}$$

then the integral on the RHS of (4.1) is

$$0 = Im \int_{\partial M} d\bar{x}\ \sqrt{2}\, i\Big[-A^{-(n+1)}\bar\chi^n + \bar\chi^{-(n+1)}A^n + \bar\eta\, A^{-(n+1)}A^n\Big]e^{-\phi} \tag{4.3}$$

For the present discussion we choose coordinates in which ∂M is the x_1 axis, so that $n_x = -n_{\bar x}$. The boundary condition (3.12) $\chi^0 = \pm\,\bar\chi^0$ implies $A^1 = \pm A^{-1}$ and $\eta = \pm\,\bar\eta$ from the first and last terms of (4.3). Using the second term we find $\bar\chi^2 = \pm\chi^{-2}$.

Additional conditions may be found from the requirement $\Delta_n\phi^n\epsilon\mathcal{T}^n$; for instance, we find that F^n and A^n obey the same boundary conditions because Δ_n maps the F^n part of the space $\mathcal{T}^n$ into the A^n part of that space. We also need $\nabla^n_\theta\phi^n\epsilon\mathcal{T}^{n+1}$ and $\nabla^n_{\bar\theta}\phi^n\epsilon\mathcal{T}^{n-1}$; for instance, the constraint $\bar\chi^2 = \pm\chi^{-2}$ implies $\partial_x A^1 = \partial_{\bar x}A^{-1}$.

Some boundary conditions are dictated by physical considerations. Diffeomorphisms of the coordinates x and $\bar x$, which belong to the space S^2, must map the boundary into itself. This means that the normal component of the vector field specifying an infinitesimal diffeomorphism must vanish on the boundary, resulting in the boundary condition $A^2 = A^{-2}$ for our choice of coordinates. Also, conditions (3.12) or (3.13) are required by the variational problem for X^μ. Starting from physical conditions such as these we can generate boundary conditions for all the spaces S^n through the requirements given above. We thus find, for $n \neq 0$

$$\begin{aligned} (boson)^n &= (boson)^{-n} & (fermion)^n &= \pm\overline{(fermion)}^{-n} \\ \partial_x(boson)^n &= \partial_{\bar x}(boson)^{-n} & \partial_{\bar x}(fermion)^n &= \mp\partial_x\overline{(fermion)}^{-n} \end{aligned} \tag{4.4}$$

where $(boson)^n$ and $(fermion)^n$ denote the bosonic and fermionic parts of ϕ^n, respectively. These are just the mixed boundary conditions of Ref. 10.

For $\mathcal{T}^0$ these boundary conditions are replaced by (3.12) or (3.13). By requiring invariance under supersymmetry transformations ϵ which preserve ∂M, namely $\epsilon = \pm\bar{\epsilon}$, we may find a complete set of boundary conditions for the space $\mathcal{T}^0$. The transformations of the component fields are

$$\begin{aligned} \delta A^0 &= i\,(\epsilon\chi^0 + \bar{\epsilon}\,\bar{\chi}^0) \\ \delta\chi^0 &= -\left(\sqrt{2}\,\epsilon\partial_x A^0 + \bar{\epsilon}\,F^0\right) \\ \delta\,\bar{\chi}^0 &= \sqrt{2}\,\bar{\epsilon}\,\partial_{\bar{x}}A^0 + \epsilon F^0 \\ \delta F^0 &= i\,\sqrt{2}\,(\bar{\epsilon}\,\partial_{\bar{x}}\chi^0 - \epsilon\partial_x\,\bar{\chi}^0) \end{aligned} \tag{4.5}$$

from which we find the additional conditions[11]

$$\begin{aligned} &\partial_x\,\bar{\chi}^0 = \pm\,\partial_{\bar{x}}\chi^0 \\ &F^0 = 0 \qquad\qquad \text{if } n\cdot\partial A^0 = 0 \\ \text{or}\qquad & \\ &n\cdot\partial F^0 = 0 \qquad\qquad \text{if } A^0 = 0 \end{aligned} \tag{4.6}$$

This last condition violates the requirement $\Delta_0\phi^0 \in \mathcal{T}^0$; that is, the component equations of the eigenvalue problem $\Delta_0\phi^0 = \lambda\phi^0$ are

$$\begin{aligned} i\partial_x\,\bar{\chi}^0 &= \lambda\chi^0 \\ -i\partial_{\bar{x}}\chi^0 &= \lambda\,\bar{\chi}^0 \\ \Box A^0 &= \lambda F^0 \\ F^0 &= \lambda A^0 \end{aligned} \tag{4.7}$$

which has no solutions unless A^0 and F^0 obey identical boundary conditions. The correct boundary conditions on $\mathcal{T}^0$ are thus either

$$\begin{aligned} A^0 &= F^0 = 0 \\ \chi &= \pm\,\bar{\chi} \\ \partial_x\,\bar{\chi} &= \mp\,\partial_{\bar{x}}\chi \end{aligned} \tag{4.8}$$

or

$$n \cdot \partial A^0 = n \cdot \partial F^0 = 0$$

$$\chi = \pm \bar{\chi} \tag{4.9}$$

$$\partial_x \bar{\chi} = \mp \partial_{\bar{x}} \chi$$

These boundary conditions spontaneously break supersymmetry and will give rise to nonsupersymmetric boundary terms in the effective action. Although (4.8-4.9) seem to be the most natural choice, it is possible to choose supersymmetric boundary conditions (3.12-3.13) and (4.6). In this case Δ_0 would not have eigenvectors because $\Delta_0 \phi^0 \not\epsilon \mathcal{T}^0$; nevertheless, $(\Delta_0)^2 \phi^0 \epsilon \mathcal{T}^0$ and therefore the heat kernel $e^{-t(\Delta_0)^2}$ appearing in (2.16) is well defined. The derivation of the effective action given below may thus also be carried through for supersymmetric boundary conditions; we sketch this analysis in Appendix C.

If we let $G(z_1, z_2; t)$ be the bulk heat kernel, then in the coordinates in which ∂M is the x_1 axis the full heat kernel on the spaces S^n, $n \neq 0$, is

$$G_B(z_1, z_2; t) = G(z_1, z_2; t) \pm G(z_1, z_2'; t) \quad \begin{matrix} +\text{Neumann} \\ -\text{Dirichlet} \end{matrix} \tag{4.10}$$

$$\text{with } z = (x, \theta) \rightarrow z' = (\bar{x}\, \theta)$$

One may verify by expanding in θ_1 and θ_2 that this expression satisfies (4.4). Since we are only interested in the short-time behavior of the diffusion operator, a diffusing particle only feels the local structure of the manifold and hence (3.27) is sufficient to calculate (2.16). The calculation is easily done by treating the operators $\Delta_n^{(\pm)^2}$ locally as perturbations about the flat space Laplacian $\partial^2/\partial x^2$ with bulk heat kernel

$$G_0(z_1, z_2; t) = \frac{1}{4\pi t} e^{-(x_1 - x_2)^2/4t} \delta(\theta_1 - \theta_2) \tag{4.11}$$

The δ-function in θ-space greatly simplifies the calculation compared to the bosonic case (in particular $G_0(z_1 = z_2) \equiv 0$ so the leading divergence cancels) and one finds

$$\operatorname{str} f\, e^{-tL_n^{(+)^2}} = (-)^n \Big(\frac{n+1}{\pi}\Big) \Big[\frac{1}{2} \int_M d^2 z\, e^{-1} R\, f + \int_{\partial M} ds\, d^2\theta\, k\, f\Big]. \tag{4.12}$$

For some details of the calculation, see Appendix B. Inserting this result in (2.16) gives us

$$\delta\,\ell n\,\mathrm{sdet}'\,L_n^{(+)} = (-)^n\Big\{\Big(\frac{2(2n+1)}{\pi}\Big)\Big[\frac{1}{2}\int\limits_M d^2z\,e^{-1}R\,\delta\psi + \int\limits_{\partial M} ds\,d^2\theta\,k\,\delta\psi\Big]$$

$$-\,2(n+1)\,\mathrm{str}\Big(\delta\psi\ \mathrm{Ker}\,(L_n^{(+)})\Big) - 2n\,\mathrm{str}\Big(\delta\psi\ \mathrm{Ker}\,(L_{n+1}^{(-)})\Big)\Big\} \tag{4.13}$$

If we write out the Laplacians $\Delta_n^{(\pm)}$ in components we see that, in a background field where $\psi(x,\theta) = \phi(x)$ (i.e., only the bosonic conformal factor is nonvanishing), sdet$L_n^{(+)}$ is the ratio of the determinants of the Laplacians of Ref. 10 for $n/2$ and $(n-1)/2$. That the results agree is another check on the consistency of our approach.

The kernel terms in (4.13) are important when the topology of M is nontrivial. In this case there are deformations of the vierbein which preserve the gauge choice (3.16) but cannot be expressed as a diffeomorphism; i.e., Ker $(\Delta_3^{(-)}) \neq 0$. An extensive discussion of these "Teichmüller" deformations is presented in Ref. 10 for the bosonic string. We will not consider this problem here, but a similar analysis should be possible with the formalism of this paper.

The determinant of $\Delta_0 = \frac{1}{2}\Big[\Delta_0^{(+)} + \Delta_0^{(-)}\Big]$ must be treated somewhat differently due to the boundary conditions. The flat space heat kernel is

$$G_B^0(z,z';t) = G_0(z,z';t) \pm \frac{1}{4\pi t}e^{-(x-\bar{x}')^2/4t}[(\theta-\theta')^2 + 2\theta_+\theta'_-] \tag{4.14}$$

where $\theta_\pm = \bar\theta \pm \theta$. The $\theta_+\theta'_-$ part of the image term guarantees that the fermion components χ_+ and χ_- obey opposite boundary conditions. However, it also explicitly breaks supersymmetry: the bosonic superfield components obey the same boundary conditions as discussed above. Using (4.14), we find

$$\begin{aligned}\delta\ell n\,\mathrm{sdet}\,\Delta_0 = \frac{1}{\pi}\Big[\frac{1}{2}\int\limits_M d^2z\,e^{-1}R\,\delta\psi + \int\limits_{\partial M} ds\,d^2\theta\,k\,\delta\psi\Big]\\ + \frac{\beta}{2\sqrt{\pi\epsilon}}\int\limits_{\partial M} ds\,\delta\psi\Big|_{\theta=0} + \frac{\beta}{2\pi}\int\limits_{\partial M} ds\,n^a\partial_a\delta\psi\Big|_{\theta=0}\end{aligned} \tag{4.15}$$

where $\beta = 1(-1)$ for Neumann (Dirichlet) boundary conditions. Note that the area divergence cancels but not the perimeter divergence because of the non-supersymmetry of the bosonic components on the boundary.

Combining the scalar determinant arising from the path integral over the X^{μ}'s with the gauge-fixing determinant (3.24) we find

$$\begin{aligned}\delta S_{eff}^{matter\ +\ ghost} = &\left(\frac{10-d}{2\pi}\right)\left[\frac{1}{2}\int_M d^2z\, e^{-1} R\,\delta\psi + \int_{\partial M} ds\, d^2\theta\, k\,\delta\psi\right] \\ &+ \frac{\beta d}{4\sqrt{\pi\epsilon}}\int_{\partial M} ds\, d^2\theta\,\delta(\theta)\,\delta\psi + \frac{\beta d}{4\pi}\int_{\partial M} ds\, d^2\theta\,\theta\cdot\not{n}\cdot\nabla\delta\psi \\ &+ (\text{Ker terms})\ .\end{aligned} \tag{4.16}$$

Equation (4.16) may be easily integrated in the gauge (3.16) to give

$$\begin{aligned}S_{eff}^{matter\ +\ ghost} = &\left(\frac{10-d}{2\pi}\right)\left[\int_M d^2z\ \hat{e}^{-1}\left[\frac{1}{2}(\hat{e}^{\alpha}\,\psi)(\hat{e}_{\alpha}\,\psi) + \frac{1}{2}\hat{R}\,\psi\right]\right. \\ &\left. + \int_{\partial M} d\,\hat{s}\ d^2\theta\ \hat{k}\ \psi\right] \\ &\frac{\beta d}{4\sqrt{\pi\epsilon}}\int_{\partial M} d\,\hat{s}\ d^2\theta\,\delta(\theta)e^{-\psi} \\ &+ \frac{\beta d}{4\pi}\int_{\partial M} d\,\hat{s}\ d^2\theta\,\theta\cdot\not{n}\cdot\hat{e}\,\psi \\ &+ (\text{Ker terms}) + (\text{indep of } \psi)\ .\end{aligned} \tag{4.17}$$

Hats refer to quantities calculated with the background matrix $\hat{e}_{\alpha}$.

Several remarks are in order. First, the result (4.17) differs from Polyakov[7] by a factor of 4 in the first term only because of a difference in the definition of ψ. Note that the leading divergence cancels even when a boundary is present, in contrast to the result of Ref. 11. Thus the C term in Eq. (3.1) is *not* renormalized. This is to be expected for reasonable boundary conditions on the fields because the propagator G_B is obtained by adding image sources outside of the domain M; these cannot contribute to the short-time diffusion behavior in the bulk of the domain. Neither the calculation just presented nor the calculation with supersymmetric boundary conditions presented in Appendix C agrees with the result of Ref. 11. In both cases, however, the A^0 and χ^0 determinants agree with the results explicitly calculated there; only the contribution of F^0 terms differ. For supersymmetric boundary conditions, our method gives

$$\mathrm{sdet}\,\Delta_0 = \left[\frac{(\det \not{\partial})^2}{(\det\Box)|_D(\det\Box)|_N}\right]^{1/2} \tag{4.18}$$

where the subscripts denote the boundary conditions. We may thus interpret the F^0 component as representing the ratio

$$\mathrm{sdet}\,\Delta_0|_{F\ compt.} = \left[\frac{(\det\Box)|_D}{(\det\Box)|_N}\right]^{\pm 1/2} . \tag{4.19}$$

Hence the properly regularized F^0 determinant gives a contribution which cancels the boundary divergence of the A^0 and χ^0 determinants as shown in Appendix C. For nonsupersymmetric boundary conditions, we can understand the difference between the two calculations because in Ref. 11 it was assumed that a remnant of supersymmetry exists when there is a boundary. The effective action was obtained from the purely bosonic sector by supersymmetrization, but we see that this cannot work since supersymmetry is completely broken for the boundary conditions (4.8-4.9).

The divergences of the determinants come from the small time behavior of $e^{-t\Delta^2}$. Substituting (4.12) into (2.14) we obtain a logarithmic divergence

$$\ell n\ \mathrm{sdet}\,L_n^{(+)} = (-)^n\left\{\frac{(n+1)}{2\pi}\left[\frac{1}{2}\int_M d^2z\,R + \int_{\partial M} ds\,d^2\theta\,k\right] - \dim\mathrm{Ker}\ L_n^{(+)}\right\}\ell n\epsilon$$
$$+\ \text{finite as } \epsilon \to 0\ . \tag{4.20}$$

Similarly, for the scalar determinant we have

$$\ell n \ \mathrm{sdet}\,\Delta_0 = \left\{\frac{1}{4\pi}\left[\frac{1}{2}\int_M d^2z\,R + \int_{\partial M} ds\,d^2\theta\,k\right] - \dim\mathrm{Ker}\ \Delta_0\right\}\ell n\epsilon$$
$$+\frac{\beta}{4\sqrt{\pi\epsilon}}\int_{\partial M} ds\,d^2\theta\,e^{\psi} + \text{finite as }\ \epsilon\to 0\ . \tag{4.21}$$

Combining (4.17), (4.20), (4.21) and (3.25) we arrive at our final result for the effective action

$$S_{eff} = \left[B + \left(\frac{6-d}{4}\right)\ell n\epsilon\right]\chi(M) + +C\int_M d^2z\ \hat{e}^{-1}\,e^{-2\psi}$$
$$+\frac{\beta d}{4\sqrt{\pi\epsilon}}\int_{\partial M} d\,\hat{s}\ d^2\theta\,\delta(\theta)e^{-\psi} + \left(E + \frac{\beta d}{4\pi}\right)\int_{\partial M} ds\,d^2\theta\,k$$
$$+\left(\frac{10-d}{2\pi}\right)\left[\int_M d^2z\ \hat{e}^{-1}\left[\frac{1}{2}\hat{e}^{\alpha}\,\psi\,\hat{e}_{\alpha}\,\psi + \hat{R}\,\psi\right] + \int_{\partial M} d\,\hat{s}\ d^2\theta\ \hat{k}\ \psi\right] \tag{4.22}$$
$$+\,(\text{Ker terms}) + (\text{indep of }\ \psi)$$

where B, C, and E are the bare parameters of Eq. (3.1).

This is our major result. By varying ψ, we obtain the equations of motion of the supersymmetric Liouville theory[14]

$$\hat{\Delta}_0\,\psi = \mu^2 e^{-2\psi} + \hat{R} \tag{4.23}$$

together with the boundary conditions

$$\begin{aligned} n^a\partial_a\phi &= \lambda\,e^{\phi} + \hat{k} \\ \eta &= \pm\,\bar{\eta} \end{aligned} \tag{4.24}$$

(recall $\psi = \phi + i\theta\eta + i\,\bar{\theta}\,\bar{\eta} + \bar{\theta}\,\theta\rho$) for the dual model case, and

$$\begin{aligned} \phi &= 0 \\ \eta &= \pm\,\bar{\eta} \end{aligned} \tag{4.25}$$

for the Wilson loop case.

5. Discussion

We have succeeded in generalizing the differential operator formalism of Ref. 10 to superspace, and have calculated the effective action of Polyakov's formulation of the fermionic string in this manifestly supersymmetric formalism. The calculations are simpler and more transparent than previous calculations in component fields. In particular, the nature of divergence cancellations due to supersymmetry has been clarified. The cosmological constant is not renormalized, which raises the possibility that it may be consistent to set $C = 0$ in Eq. (4.22). This would result in an effective action which is a free field obeying either Dirichlet boundary conditions for the Wilson loop case or "Liouville" boundary conditions (4.24) for the dual model case (due to the perimeter divergence in (4.22)).

The reader should keep in mind, however, that a nonzero value of C is certainly permissible in this theory. One might, in fact, argue that eliminating the cosmological constant (i.e., the Liouville interaction term) is not a natural choice, since any nonzero value of C in Eq. (4.22) may be transformed to any other value by changing ψ by an additive constant. However, classically ψ tends to move to the minimum of its exponential potential, which corresponds to decreasing C. The value of the effective cosmological constant in the quantum theory is a dynamical question which still is not completely understood. Even so, it seems likely that there is no problem in taking C to be small or zero. The assumption that $C = 0$ should simplify the probem of quantizing the effective action. Further, whatever value we choose for the cosmological constant C, it need not be equal to the boundary cosmological constant, as is claimed in Refs. 11 and 14. Indeed, since the boundary term is divergent while C is not, it can be argued that its coefficient ought to be much larger than C.

Several problems still remain even if the action for ψ is a free field. First, the boundary conditions for the dual model case are nonlinear; the low-frequency modes will not be simple harmonic oscillators of the ψ field. The short distance fluctuations of ψ must be cut off at some small proper distance scale depends on ψ itself, which greatly complicates the regularization procedure. Friedan[15] has argued that the full theory should be invariant under rescalings of the background vierbein $\hat{e}_\alpha$, since this is just an arbitrary reference choice in the class of metrics $\{e^\psi \hat{e}_\alpha\}$. Hence the full theory should have no trace anomaly; this restricts the quantization of ψ, since the trace anomaly of the ψ -field stress tensor must cancel that of the string and ghost fields. Friedan[15] showed that straightforward canonical quantization of ψ does not have this property.

The free field effective action for the fermionic string theory should provide a simplified laboratory for the investigation of these remaining difficulties without the additional complication of quantizing a theory with an exponential interaction. The formalism presented here makes the analysis of the supersymmetric string no more difficult, and perhaps simpler, than that of the bosonic string.

APPENDIX A. Notations and Conventions

We denote tangent space (coordinate) indices by letters from the beginning (middle) of the alphabet; latin (greek) letters represent vectors (spinors), and capitals denote a general index. Define an orthonormal frame field $e_A = e_A^M \partial_M$, define the structure constants

$$[e_A, e_B\} = C_{AB}^C e_C \tag{A.1}$$

where [, } denotes the graded commutator. The action of the Lorentz generators is

$$MV^A = V^B M_B^A \quad \text{where} \quad M_A^B = \begin{cases} \epsilon_a^b \text{ on vectors} \\ \frac{1}{2}\gamma_\alpha^{5\beta} \text{ on spinors} \end{cases} \tag{A.2}$$

The covariant derivative $\nabla_A = e_A + \phi_A M$ defines the torsion and curvature through

$$[\nabla_A, \nabla_B\} = T_{AB}^C \nabla_C + R_{AB} M \tag{A.3}$$

from which

$$T_{AB}^C = C_{AB}^C - \phi_{[A} M_{B)}^C \tag{A.4}$$

The flat space vierbein is

$$\begin{aligned} \hat{e}_\alpha &= D_\alpha = \partial_\alpha + i(\theta \not{\partial})_\alpha \\ \hat{e}_a &= \partial_a \end{aligned} \tag{A.5}$$

For a general matrix M_A^B the superdeterminant and supertrace are defined by

$$\text{sdet } M = \det(M_a^b - M_a^\gamma M_\gamma^{-1\delta} M_\delta^b) \det{}^{-1}(M_\alpha^\beta) \tag{A.6}$$

$$\text{str } M = M_a^a - M_\alpha^\alpha \tag{A.7}$$

The δ-function in θ-space is $\delta(\theta_1 - \theta_2) = (\theta_1 - \theta_2)^2$ so that $\int d^2\theta\, \delta(\theta) = 1$.

APPENDIX B. Sketch of the Determinant Calculations

We wish to calculate

$$\mathrm{str}\,(fe^{-t\Delta^2}) = \int\limits_M d^2z\, e^{-1}(z)\, f(z)\, \langle z|e^{-t\Delta^2}|z\rangle \qquad (B.1)$$

for $t \to 0$. We write

$$\Delta^2 = -\partial_x\partial_{\bar{x}} + V \qquad (B.2)$$

where V contains the information about the geometry of the manifold as it deviates locally from flat space. The Green's function appearing in (B.1) may then be expanded in perturbation series

$$G = G_0 + G_0VG_0 + O(\sqrt{t}) \qquad (B.3)$$

where

$$G = e^{-t\Delta^2}$$

$$\text{and} \quad G_0 = e^{-t(-\partial_x\partial_{\bar{x}})} = \frac{1}{4\pi t} e^{-(x-x')^2/4t}(\theta - \theta')^2 \qquad (B.4)$$

(for manifolds without boundary)

Because of the factor $(\theta - \theta')^2$, $G_0(z, z; t) = 0$; hence only the second term contributes:

$$\begin{aligned} G_0VG_0(z,z,t) = \int d^2z' \int\limits_0^t dt' \frac{(\theta-\theta')^2}{4\pi(t-t')} e^{-(x-x')^2/4(t-t')} \\ \times V(z') \frac{(\theta-\theta')^2}{4\pi t'} e^{-(x-x')^2/4t'} \end{aligned} \qquad (B.5)$$

Since $(\theta - \theta')^3 = 0$, only the $\partial_{\theta'}\partial_{\bar{\theta}'}$ part of V contributes. This term is

$$e^{4\psi}[2(n+1)(D_{\bar{\theta}}D_\theta\psi) + 4(D_{\bar{\theta}}\psi)(D_\theta\psi)]\partial_{\bar{\theta}}\partial_\theta \qquad (B.6)$$

for the Laplacians (2.12). In the neighborhood of z we may choose coordinates such that $D_\theta\psi = D_{\bar\theta}\psi = 0$. Then we have that

$$G_0 V G_0(z, z; t) = \left(D_{\bar\theta} D_\theta \psi(z)\right) \cdot \left(\frac{n+1}{2\pi}\right) \tag{B.7}$$

Essentially the calculation boils down to counting factors of $D_{\bar\theta} D_\theta \psi$. The calculation for surfaces with boundary differs only in that we must use the propagators (4.10) or (4.14) which obey the appropriate boundary conditions and in that $n \cdot \partial\phi$ cannot be gauged away.

APPENDIX C.

Effective Action for Supersymmetric Boundary Conditions

The heat kernel formalism may also be used to calculate the scalar determinant when the supersymmetric boundary conditions (3.13) and (4.6) are chosen. The zeroth order heat kernel is

$$G_B^0(z,z';t) = G_0(z,z';t) \pm \frac{1}{4\pi t}\, e^{\frac{-(z-z')^2}{4t}}\, [(\theta_+ + \theta'_+)(\theta_- - \theta'_-)/2] \qquad (C.1)$$

where $\theta_\pm = \bar\theta \pm \theta$. Note that $G_B^0(z = z') \equiv 0$ even at the boundary due to the residual supersymmetry under boundary-preserving supersymmetry transformations, and hence neither area nor perimeter divergences are generated. The variational equation for the determinant becomes

$$\delta\ \ell n\ \operatorname{sdet}\Delta_0 = \frac{1}{\pi}\left[\frac{1}{2}\int_M d^2z e^{-1}R\delta\psi + \int_{\partial M} ds d^2\theta k\delta\psi\right] . \qquad (C.2)$$

Thus we find, instead of (4.17):

$$\begin{aligned} S_{eff}^{matter+ghost} = \left(\frac{10-d}{2\pi}\right)\Bigg[\int_M d^2z\ \hat e^{-1}\left[\frac{1}{2}(\hat e^\alpha\,\psi)(\hat e_\alpha\,\psi) + \frac{1}{2}\hat R\,\psi\right] \\ + \int_{\partial M} d\,\hat s\ d^2\theta\ \hat k\,\psi\Bigg] . \end{aligned} \qquad (C.3)$$

ACKNOWLEDGEMENTS

I sincerely thank O. Alvarez, D. Friedan, S. J. Gates, and especially M. Peskin for their patient and generous assistance.

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