SLAC-PUB-3065
LBL-15812
UCB-PTH-83/3
February 1983
(T)

SUPERSYMMETRY AND NON-COMPACT GROUPS IN SUPERGRAVITY*

John Ellis
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Mary K. Gaillard
Physics Department and Lawrence Berkeley Laboratory
University of California, Berkeley, California 94720

Murat Günaydin
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure
Paris, France

Bruno Zumino
Physics Department and Lawrence Berkeley Laboratory
University of California, Berkeley, California 94720

Submitted to Nuclear Physics B

* Work supported in part by the Department of Energy, contracts DE-AC03-76SF00515 and DE-AC03-76SF00098, as well as the National Science Foundation, research grants PHY-82-03424 and PHY-81-18547.

ABSTRACT

We study the interplay of supersymmetry and certain non-compact invariance groups in extended supergravity theories (ESGTs). We use the $N = 4$ ESGT to demonstrate that these symmetries do not commute and exhibit the infinite dimensional superinvariance algebra generated by them in the rigid limit. Using this result, we look for unitary representations of the full algebra. We discuss the implications of our results in the context of attempts to derive a relativistic effective gauge theory of elementary particles interpreted as bound states of the $N = 8$ ESGT.

1. Introduction

At the present time extended supergravity theories (ESGTs) are the most promising candidates for unifying gravitation with the other fundamental particle interactions. Even if attempting such an ultimate unification is probably still premature, it is clear that the ESGTs are sufficiently stimulating for the elucidation of their structure to be worthwhile. One may gain insights which will prove valuable in the construction of the ultimate theory. With this aim in view, we continue in this paper mathematical investigations of the symmetries of ESGTs.

In their important work on the $N = 8$ ESGT, Cremmer and Julia have shown that the extended supergravity theories for $N = 5, 6, 8$ each have an invariance of the equations of motion under a non-compact group.[1] The first non-compact internal symmetry group of this kind was discovered in the $N = 4$ ESGT by Cremmer, Ferrara and Scherk.[2] The largest on-shell invariances of these theories have the form $G_{gl} \times H_{loc}$ where the local invariance group H_{loc} is isomorphic to the maximal compact subgroup H_{gl} of the non-compact global invariance group G_{gl}. Under the action of the non-compact group H_{gl} the vector field strengths are transformed into their duals and together they form a linear representation of G_{gl}. The scalar fields are valued on the coset space G/H. In a manifestly gauge invariant formultion they transform linearly under both G_{gl} and H_{loc}, whereas the spinor fields ($s = 1/2, s = 3/2$) are all inert under G_{gl} and transform like some non-trivial linear representations of H_{loc}.[1,3] In the gauges in which only physical scalar degrees of freedom appear, scalar and spinors transform non-linearly under G_{gl}, which is the only manifest symmetry in this gauge. The potential problem with ghosts due to the non-compactness of G is avoided by the gauging of its maximal compact subgroup. The "gauge fields" associated with the invariance under H_{loc} of these theories are composites of the scalar fields as in the two dimensional CP^N models.[4]

Cremmer and Julia suggested that the composite gauge fields of H_{loc} may become dynamical on the quantum level.[1] Their suggestion was motivated by analogy with the CP^N models in two dimensions,[4,5] whose study in the large N limit shows that the composite gauge fields develop a pole at $p^2 = 0$ in their propagators and become dynamical on the quantum level.[4] Nissimov and Pacheva[6,7] have extended this analysis to the three dimensional $(2+1)$ supersymmetric generalized non-linear σ-models and shown that in the large N limit these theories have a phase in which the composite gauge fields and their superpartners develop poles at $p^2 = 0$ and become propagating,[6,7] with supersymmetry remaining unbroken.

It is well-known that the fundamental fields that enter the largest ESGT ($N = 8$) in four dimensions do not have a rich enough structure to accommodate the basic fields of a realistic gauge theory of strong, weak and electromagnetic interactions.[8] Thus it was thought that some of the fields entering such a theory might have to be made composites of the fundamental fields of $N = 8$ ESGT in order to make contact with elementary particle physics.[9,10] The suggestion of Cremmer and Julia[1] that the composite gauge fields of $SU(8)_{loc}$ in $N = 8$ ESGT may become dynamical on the quantum level was an important step in this direction.

Another step in this direction was taken by Maiani and three of the present authors[11] (EGMZ) who postulated that in addition to massless gauge fields other massless bound states (fermionic and bosonic) may form. The low energy effective theory could then be a grand unified theory based on SU(5) with three generations of quarks and leptons.[12] Since the dynamics of these theories is, as yet, unknown there is still a lot of arbitrariness in such attempts. Subsequent work has discussed various alternatives and possible improvements on the original EGMZ approach.[12–18]

In the four dimensional ESGTs one has the option of introducing additional couplings to turn the elementary vector fields into non-Abelian gauge fields while preserving supersymmetry. For the case of $N = 8$ ESGT this has been done recently by DeWit and Nicolai.[19] This gauging of the vector fields breaks the non-compact global invariance group. For example, the gauged $N = 8$ ESGT has $SO(8)_{loc} \otimes SU(8)_{loc}$ invariance as opposed to the $E_{7(7)} \otimes SU(8)_{loc}$ invariance of the ungauged theory. Note, however, that in the special $SU(8)$ gauge containing only physical degrees of freedom, only $SO(8)$ remains an invariance of the explicitly gauged theory. The relevance of the gauged theory is not yet clear and we will restrict ourselves to the ungauged theory.

EGMZ chose the zero mass shell supergauge multiplet of bound states from which to construct a realistic GUT. In addition to the particles needed for a realistic GUT this supergauge multiplet contains many unwanted helicity states.[11,12,20] These unwanted helicity states cannot be made supermassive in a phenomenologically acceptable way[21] without introducing a large and possibly infinite number of additional supermultiplets of bound states.[13,15] Thus the question of what kind of bound states[1] can be formed in ESGTs is important for attempts at extracting an effective low energy GUT from them.

There are theoretical arguments indicating that in ESGTs the spectrum of bound states may be infinitely rich.[13,15,22] One these arguments is the analogy with two and three dimensional generalized σ-models. In the phase of these theories in which the composite gauge fields become dynamical the bound states form linear representations of the global invariance group.[7,23] This result in two dimensional theories may be related to Coleman's theorem.[24] However, this linear realization of a global symmetry on the bound states, even though it is realized non-linearly on the basic fields of a $(2+1)$ three-dimensional Lagrangian,[7] is perhaps a hint that the same phenomenon may occur in four dimensional theories. If this is the case then in ESGTs with non-compact global invariance groups the bound states must come in infinite towers since all the unitary representation of non-compact groups are infinite dimensional.[13,15,22,25]

Indications for this possibility come from the study of other two-dimensional theories. Makhankov and Pashaev in their study of the non-linear Schrödinger equation with a non-compact $SU(1,1)$ invariance find that the spectrum of soliton solutions is far richer than in the compact case and suggest that this may be understood in the language of unitary realizations of the non-compact invariance group.[26] Studies of σ-models with a non-compact global invariance group[27,28] indicate that gauge bosons are not generated dynamically in 2 or 3 dimensions, for reasons related to the absence of a dynamically generated mass gap. However, no such mass gap is necessary for the gauge fields to become dynamical in 4 dimensions. Another difficulty is that these studies suggest that the non-compact global symmetry is spontaneously broken. We do not regard these arguments as conclusive, and recall the French adage, "Ce n'est que les optimistes qui fassent quelque chose dans ce monde."

There are other suggestions that the physical spectra of ESGTs may contain an infinite number of states. For example, Grisaru and Schnitzer have argued that the scattering amplitudes in ESGTs Reggeize.[29] Furthermore, Green and Schwarz[30] have been able to obtain the $N = 8$ ESGT from a 10 dimensional superstring theory by dimensional reduction in the limit where the radii of the compactified dimensions and the Regge slope parameter approach zero. If the $N = 8$ ESGT and the dimensionally reduced superstring theory coincide in a certain limit this would be compatible with the existence of an infinite set of bound states. In perturbation theory such a coincidence has been established in a limit with zero Regge slope for the superstring theory.[30] However, this theory seems to give a different infinite spectrum from that of Grisaru and Schnitzer[29] when the superstring Regge slope parameter is non-zero. Since the only dimensionful parameter in ESGTs is the Planck mass M_{Pl}, if they Reggeize as suggested by the work of Grisaru and Schnitzer[29] then the slopes of the Regge trajectories would have to be proportional to $1/M_{Pl}^2$. In this case the spectrum of massive states would have to start around the Planck mass.

From the point of view of the unitary realizations of the non-compact symmetry groups of ESGTs the representations which naturally suggest themselves are those that can be constructed in terms of the basic fields in the respective theories. Oscillatorlike unitary representations of these groups have already been constructed using bosonic operators transforming like the vector field [25,22,31] in ESGTs. Remarkably enough the unitary representations that can be constructed over the Hilbert spaces of analytic functions of the scalar fields of the ESGTs are unitarily equivalent to the oscillatorlike unitary representations.[32] Fermionic operators which transform non-linearly, as do the fermionic fields of ESGTs, suggest in a rather straightforward way the construction of induced representations in terms of composite operators constructed from fermions and scalars.

However, a suitable bound state spectrum must form a countable set of normalized states on which the full superinvariance algebra can be realized. The latter realization cannot be achieved trivially in terms of arbitrary representations of the non-compact groups.

Our aim in this article is to study the larger superinvariance algebras generated by supersymmetry and non-compact symmetry generators in ESGTs with special emphasis on their unitary realizations. To minimize algebraic complications we consider the simplest case of $N = 4$ ESGT with $SU(4) \times SU(1,1)_{gl} \otimes U(4)_{loc}$ invariance. After summarizing the salient features of this theory we stress the fact that the action of the non-compact invariance group G in the "special" gauge corresponds to a simultaneous action of G_{gl} and induced H_{loc} transformations. The generators of G do not commute with the supersymmetry generators, as has previously been mentioned in Ref. 13. We give here the algebra generated by G and the supersymmetry generators in the global limit. The rigid limit is the asymptotic limit of large spatial coordinates $\vec{x}$ in which all fields vanish, except for the scalars which tend to some constants and the vierbein which reduces to the Kronecker δ-function. This rigid algebra has a structure similar to that of a Kac-Moody algebra.[33] Just as the Kac-Moody algebras can be though of as extensions of ordinary Lie algebras by functions on a circle, our algebra can be regarded as an extension of an ordinary algebra by functions defined on the open unit disc. The open unit disc enters the picture because it is the domain on which the scalar fields of the theory take their values. We then argue that this infinite dimensional superinvariance algebra may have a unitary realization on the bound states. With this aim in view, we study irreducible unitary representations of $SU(1,1)$ and investigate how they may be used to represent the full superinvariance algebra. Our discussion can be extended to higher ESGTs for $N = 5, 6, 7, 8$: we comment explicitly on the most interesting case of $N = 8$.

2. $N = 4$ Extended Supergravity Theory

There are two different formulations of $N = 4$ ESGT which are referred to as the $SO(4)$ formulation[34,35,1] and the $SU(4)$ formulation.[2] In the $SU(4)$ formulation, of the six $s = 1$ fields entering the theory three are most naturally defined as vectors and the remaining three as axial vectors, while in the $SO(4)$ formulation they are most naturally all taken to be vectors. The non-compact global on-shell invariance group $SU(4) \times SU(1,1)$ of these theories was first discovered in the $SU(4)$ formulation.[2] Below we shall summarize the salient features of the $SO(4)$ formulation following references 1 and 34.

The fundamental fields entering the $SO(4)$ formulation are the vierbein $e^a_\mu(x)$, $s = 3/2$ fields $\psi^i_\mu(x) (i = 1, \ldots 4)$, the vector fields $A^{ij}_\mu(x) = -A^{ji}_\mu$ $(i,j = 1 \ldots 4)$, $s = 1/2$ fields χ^i, a scalar field $A(x)$ and a pseudoscalar field $B(x)$. These two real scalar degrees of freedom correspond to the special gauge discussed above.

The Lagrangian $\mathcal{L}$ reads as[34]

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{4\kappa^2} eR - \frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\,\bar{\psi}^i_\nu\,\gamma_5\gamma_\mu D_\nu \psi^i_\rho - \frac{1}{8} e F^{ij}_{\mu\nu} G^{\mu\nu}_{ij} \\
&+ \frac{i}{2} e\;\bar{\chi}^i\,\gamma^\mu\,\hat{D}_\mu\,\chi^i + \frac{e}{2a^2}(\partial_\mu A \partial_\nu A + \partial_\mu B \partial_\nu B)\,g^{\mu\nu} \\
&- \frac{\kappa}{16} e\,\sqrt{a}\,(\hat{F} + \hat{H})^{ij}_{\mu\nu} P^{\mu\nu}_{ij} + \frac{i\kappa^2}{4a}\epsilon^{\mu\nu\rho\lambda}\,\bar{\psi}^i_\mu\,\gamma_\rho \psi^i_\lambda A \partial_\nu B \\
&- \frac{3\kappa^2 e}{4a}\bar{\chi}^i\,\gamma_5\gamma^\mu\chi^i A\,\hat{D}_\mu\,B - \frac{\kappa}{2\sqrt{2}}\,\frac{e}{a}\,\bar{\psi}^i_\mu\,\partial_\nu(A + iB\gamma_5)\gamma^\nu\gamma^\mu\chi^i
\end{aligned} \tag{2.1}$$

where

$$
\begin{aligned}
a \equiv & 1-\kappa^2(A^2+B^2) \\
P^{ij}_{\mu} \equiv & \bar{\psi}^i_\mu \psi^j_\nu - \bar{\psi}^j_\mu \psi^i_\nu - \frac{i}{e}\epsilon^{\rho\sigma}_{\mu\nu}\bar{\psi}^i_\rho \gamma_5 \psi^j_\sigma \\
Q^{ij}_{\mu\nu} \equiv & \frac{i}{2}\epsilon^{ijk\ell}\,\bar{\psi}^k_\lambda \sigma_{\mu\nu}\gamma^\lambda \chi^\ell \\
H^{ij}_{\mu\nu} \equiv & g_1 F^{ij}_{\mu\nu} - g_2 F^{*ij}_{\mu\nu} - g_3 \tilde{F}^{*ij}_{\mu\nu} - g_4 \tilde{F}^{ij}_{\mu\nu} \\
G^{ij}_{\mu\nu} \equiv & H^{ij}_{\mu\nu} + \frac{\kappa\sqrt{a}}{2}\Big[(1+g_1)P - g_2 P^* - g_3\tilde{P}^* - g_4\tilde{P}\Big]^{ij}_{\mu\nu} \\
& + \kappa\sqrt{2a}\Big[(1+g_1)Q - g_2 Q^* - g_3\tilde{Q}^* - g_4\tilde{Q}\Big]^{ij}_{\mu\nu}
\end{aligned}
\tag{2.2}
$$

and the symbol $*$ denotes dual with respect to internal indices

$$
F^*_{ij} \equiv \frac{1}{2}\epsilon_{ijk\ell}F^{k\ell} \quad etc.
$$

whereas $\sim$ denotes dual with respect to space-time indices

$$
\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}
$$

The G_i are functions of the scalar fields

$$
\begin{aligned}
g_1 - i g_4 \equiv & \frac{1+\bar{z}^2}{1-\bar{z}^2} \\
g_3 - i g_2 \equiv & \frac{2i\,\bar{z}}{1-\bar{z}^2}
\end{aligned}
\tag{2.3}
$$

where $z \equiv \kappa(A+iB)$. $\hat{D}$ denotes a supercovariant derivative operation:

$$
\begin{aligned}
\hat{D}_\mu A \equiv & \partial_\mu A - \sqrt{\frac{1}{2}}\kappa a\,\bar{\psi}^i_\mu \chi^i \\
\hat{D}_\mu B \equiv & \partial_\mu B - i\sqrt{\frac{1}{2}}\kappa a\,\bar{\psi}^i_\mu \gamma_5 \chi^i
\end{aligned}
\tag{2.4}
$$

and

$$\hat{F}^{ij}_{\mu\nu} \equiv F^{ij}_{\mu\nu} - \frac{i\kappa}{\sqrt{2a}}\Big[\epsilon^{ijk\ell}\,\bar{\psi}^k_{[\mu}\gamma_{\nu]}\chi^\ell - \kappa\,\bar{\psi}^{[i}_{[\mu}\gamma_{\nu]}(A+iB\gamma_5)\chi^{j]}\Big]$$
$$- \frac{\kappa}{\sqrt{a}}\Big[\bar{\psi}^i_{[\mu}\psi^i_{\nu]} + \kappa\epsilon^{ijk\ell}\,\bar{\psi}^k_\mu(A+iB\gamma_5)\psi^\ell_\nu\Big] \qquad (2.5)$$

The purely bosonic part of the above Lagrangian can be written as[1]:

$$\mathcal{L}_{bosonic} = -\frac{1}{4\kappa^2}eR + \frac{e}{2\kappa^2}\,\frac{\partial_\mu z g^{\mu\nu}\partial_\nu\bar{z}}{(1-z\bar{z})^2}$$
$$-\frac{e}{8}F^{ij}_{\mu\nu}M^{\mu\nu\rho\lambda}_{ijk\ell}F^{k\ell}_{\rho\lambda} \qquad (2.6)$$

where

$$M^{\mu\nu\rho\lambda}_{ijk\ell} \equiv \frac{1}{4}(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk})\Big(g_1 g^{\mu\nu,\rho\lambda} - g_4\epsilon^{\mu\nu\rho\lambda}\Big)$$
$$-\frac{1}{4}\epsilon_{ijk\ell}\Big(g_2 g^{\mu\nu,\rho\lambda} + g_3\epsilon^{\mu\nu\rho\lambda}\Big) \qquad (2.7)$$

with $g^{\mu\nu,\rho\lambda} \equiv g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho}$. The largest on-shell invariance of the full Lagrangian is $SU(4)\times SU(1,1)_{gl}\otimes U(4)_{loc}$. In the special gauge the scalar fields are valued on the coset space $SU(4)\times SU(1,1)/U(4)\approx SU(1,1)/U(1)$ and transform non-linearly under the global group. When restricted to the special gauge, the Lagrangian is invariant only under an $SU(1,1)\times SU(4)$ group for which the diagonal $U(1)$ in $SU(1,1)_{gl}$ and $U(4)_{loc}$ acts on the dimensionless complex field z as

$$SU(1,1):\quad z=\kappa(A+iB) \quad\rightarrow\quad z' = \frac{(\alpha z+\beta)}{(\bar{\beta}z+\bar{\alpha})} \qquad (2.8)$$

with $|\alpha|^2-|\beta|^2=1$.

The vector field strengths $F^{ij}_{\mu\nu}$ and their duals defined by $\tilde{G}^{\mu\nu}_{ij} = (4\kappa^2/e)$ $(\delta\mathcal{L}/\delta F^{ij}_{\mu\nu})$ are transformed into one another under $SU(1,1)$ and together form a 2 dimensional spinor representation. As a consequence $SU(1,1)$ interchanges the Bianchi identities $\partial_\mu(e\tilde{F}^{\mu\nu ij}) = 0$ for the vector fields with their equation of motion $\partial_\mu(e\tilde{G}^{\mu\nu}_{ij}) = 0$. The vierbein e^a_μ is a singlet of $SU(1,1)$.

As for the spinor fields in the theory, they undergo induced local axial $U(1)_A$ rotations under the action of $SU(1,1)$. To determine the $U(1)_A$ rotations consider first the part of the Lagrangian containing only $s = 0$ and $s = 1/2$ fields with all the other fields set to zero:

$$\begin{aligned} \mathcal{L}_{\frac{1}{2}=0} &= \frac{i}{2}\bar{\chi}^i\gamma^\mu D'_\mu\chi^i + \frac{\partial_\mu z\partial^\mu \bar{z}}{2(1-z\,\bar{z})^2} \\ D'_\mu &= \partial_\mu - \frac{3}{2}\gamma_5 A_\mu \quad \textit{where} \\ A_\mu &\equiv \frac{1}{2}\,\frac{\bar{z}\,\partial_\mu z - z\partial_\mu\bar{z}}{(1-\bar{z}\,z)} \end{aligned} \tag{2.9}$$

The second term in the above Lagrangian is invariant under the $SU(1,1)$ transformation (2.8). In the first term the composite gauge field A_μ undergoes a gauge transformation under (2.8).

$$\begin{aligned} SU(1,1): \quad A_\mu \to A'_\mu &= A_\mu + i\ \Lambda_\mu \\ \Lambda_\mu &\equiv exp\Big(\frac{1}{2}\omega(\alpha,\beta,z,\bar{z})\Big)\partial_\mu exp\Big(-\frac{1}{2}\omega(\alpha,\beta,z,\bar{z})\Big) \end{aligned} \tag{2.10}$$

where

$$exp\left(-i\omega(\alpha,\beta,z,\bar{z})\right) = \frac{\beta\,\bar{z}+\alpha}{\bar{\beta}\,z+\bar{\alpha}} \tag{2.11}$$

Thus to make the Lagrangian invariant $s = 1/2$ fields χ^i must undergo the $U(1)_A$ rotations:

$$SU(1,1): \quad \chi^i \to exp\Big(i\frac{3}{4}\omega(\alpha,\beta,z\,\bar{z})\gamma_5\Big)\chi^i \tag{2.12}$$

Similarly the part of the Lagrangian involving only $s = 3/2$ and $s = 0$ fields

$$\mathcal{L}_{\frac{3}{2},0} = \frac{1}{2}\epsilon^{\mu\nu\rho\lambda}\,\bar{\psi}^i_\mu\,\gamma_\mu\gamma_5 D''_\rho \psi^i_\lambda + \frac{1}{2}\,\frac{\partial_\mu z \partial^\mu\,\bar{z}}{(1 - z\,\bar{z})^2} \tag{2.13}$$

will be invariant under $SU(1,1)$ if the gravitino fields ψ^i_μ undergo an induced $U(1)_A$ rotation

$$SU(1,1): \quad \psi^i_\mu \to \psi^{i\prime}_\mu = exp\Big(\frac{i}{4}\omega(\alpha,\beta,z,\bar{z})\gamma_5\Big)\,\psi^i_\mu \tag{2.14}$$

For the full theory, including vector fields, the $SU(1,1)$ is an invariance only of the equation of motion, however.

One important feature of this theory, which it shares with the higher $N > 4$ ESGTs, is that the scalar fields are constrained to have values on a certain bounded homogeneous domain. For the $N = 4$ theory the z fields satisfy the constraint $(1 - z\,\bar{z}) > 0$, i.e. they take values on the open unit disc Δ in the complex z-plane. The group $SU(1,1)$ under which the field z undergoes a non-linear transformation maps the domain Δ into itself. In the higher supersymmetry theories also the non-compact symmetry group acts as the automorphism group of the domain on which the scalar fields take their values. This complements the theorem[36] connecting $N = 1$ supersymmetry with Kähler manifolds and the connection[37] between local $N = 2$ supersymmetry and quaternionic manifolds.

3. Supersymmetry and the Non-compact Invariance Group

One intriguing aspect of ESGTs for $4 \leq N \leq 8$ is the fact that they have some non-compact invariance groups whose generators do not in general commute with the supersymmetry generators.[13,18,22] In this section we will point out the source of this non-commutativity and construct the infinite dimensional superalgebra generated by supersymmetry and the non-compact group generators in the case of the $N = 4$ ESGT.

We choose the generators of the $SU(1,1)$ group such that they satisfy the commutation relations

$$
\begin{aligned}
[L_-, L_+] &= 2iL_0 \\
[L_0, L_+] &= iL_+ \\
[L_0, L_-] &= -iL_-
\end{aligned}
\tag{3.1}
$$

where L_0 corresponds to the generator of the $U(1)$ subgroup. A general element of the $SU(1,1)$ group can be represented in the form

$$U(g) = exp(\omega^0 L_0 + \omega L_+ + \omega^* L_-)$$

In a unitary representation the generators must satisfy

$$L_-^\dagger = -L_+ \qquad\qquad L_0^\dagger = -L_0$$

For the 2-dimensional representation of $SU(1,1)$ we shall choose

$$L_0 = \frac{i}{2}\sigma_3 \qquad\qquad L_- = i\sigma_- \qquad\qquad L_+ = -i\sigma_+$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, and denote

$$
\begin{aligned}
g &= exp\left(\frac{i}{2}\omega_0\sigma_3 + i\omega^*\sigma_- - i\omega\sigma_+\right) \\
&= \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \ , \qquad\qquad |\alpha|^2 - |\beta|^2 = 1
\end{aligned}
\tag{3.2}
$$

Now if we consider the scalar field $z(x)$ as an operator we must demand

$$U(g^{-1}) z(x) U(g) = \frac{\alpha z(x) + \beta}{\bar{\beta}\, z(x) + \bar{\alpha}} \tag{3.3}$$

This implies that

$$
\begin{aligned}
&[L_0, z] = -iz \\
&[L_-, z] = iz^2 \\
&[L_+, z] = i
\end{aligned}
\tag{3.4}
$$

Similarly for the conjugate operator $\bar{z}(x)$ we find

$$
\begin{aligned}
&[L_0, \bar{z}] = i\,\bar{z} \\
&[L_-, \bar{z}] = -i \\
&[L_+, \bar{z}] = -i\,\bar{z}^2
\end{aligned}
\tag{3.5}
$$

Similarly from the transformation properties of the $s = 1/2$ fields χ^i and $s = 3/2$ fields ψ^i_μ:

$$U(g^{-1})\chi^i(x)U(g) = exp\left(i\frac{3}{4}\omega(\alpha, \beta, z, \bar{z})\gamma_5\right)\chi^i(x) \tag{3.6}$$

and

$$U(g^{-1})\psi^i_\mu(x)U(g) = exp\left(\frac{i}{4}\omega(\alpha, \beta, z, \bar{z})\gamma_5\right)\psi^i_\mu(x) \tag{3.7}$$

we obtain

$$
\begin{aligned}
&\left[L_0, \chi^i(x)\right] = \frac{3i}{4}\gamma_5\chi^i(x) \\
&\left[L_-, \chi^i(x)\right] = \frac{-3i}{4}z(x)\gamma_5\chi^i(x) \\
&\left[L_+, \chi^i(x)\right] = \frac{-3i}{4}\bar{z}(x)\gamma_5\chi^i(x)
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
&\left[L_0, \psi^i_\mu(x)\right] = \frac{i}{4}\gamma_5\psi^i_\mu(x) \\
&\left[L_-, \psi^i_\mu(x)\right] = -\frac{i}{4}z(x)\gamma_5\psi^i_\mu(x) \\
&\left[L_+, \psi^i_\mu(x)\right] = -\frac{i}{4}\bar{z}(x)\gamma_5\psi^i_\mu(x)
\end{aligned}
\tag{3.9}
$$

To calculate the commutators of supersymmetry generators Q^i with the non-compact symmetry group generators it is simplest to use the action of Q^i on the vierbein $e^a_\mu(x)$ or on the scalar fields $z(x)$, $\bar{z}(x)$:

$$\begin{aligned}
\Big[Q^i, e^a_\mu(x)\Big] &= -i\kappa\gamma^a\psi^i_\mu(x) \\
\Big[Q^i_L, z(x)\Big] &= \sqrt{2}\,\kappa a(x)\chi^i_L(x) \\
\Big[Q^i_R, z(x)\Big] &= 0 \\
\Big[Q^i_L, \bar{z}(x)\Big] &= 0 \\
\Big[Q^i_R, \bar{z}(x)\Big] &= \sqrt{2}\,\kappa a(x)\chi^i_R(x)
\end{aligned} \tag{3.10}$$

where $a(x) = 1 - z(x)\,\bar{z}(x)$. Using the Jacobi identities one finds

$$\begin{aligned}
\Big[L_0, Q^i\Big] &= -\frac{i}{4}\gamma_5 Q^i \\
\Big[L_+, Q^i\Big] &= \frac{i}{4}\bar{z}\,\gamma_5 Q^i \\
\Big[L_-, Q^i\Big] &= \frac{i}{4}z\gamma_5 Q^i
\end{aligned} \tag{3.11}$$

Comparing these commutation relations with those of the ψ^i_μ fields (see Eq. (3.9)) we see that under a global $SU(1,1)$ transformation

$$U(g^{-1})Q^iU(g) = exp\Big(-\frac{i}{4}\omega(\alpha,\beta,z,\bar{z})\gamma_5\Big)Q^i \tag{3.12}$$

The important feature of this $SU(1,1)$ is that it does not commute with the supersymmetric transformations. By multiple commutation of the generators L_+, L_-, L_0 with the supersymmetry generators Q^i one generates scalar field dependent supersymmetry generators of the form $z^n\,\bar{z}^m\,Q^i$ or $z^n\,\bar{z}^m\,\gamma_5 Q^i$. As suggested in reference 13 it is simplest to study the algebra generated by L_+, L_-, L_0

and Q^i in the rigid limit, i.e. by going to spatial infinity where all fields vanish asymptotically except for the scalar fields and the vierbein e_a^μ which simply becomes the Kronecker δ-function. In this limit the constant scalar fields which we define as the commuting operators Z and $\bar{Z}$ commute with Q^i. Interpreting the operators $z^n(x)\ \bar{z}^m(x)Q^i$ as "generators" of generalized supersymmetry transformations may look puzzling since the generators of a symmetry must become integrated charges independent of space-time. What we are implicitly assuming is that the corresponding integrated charges, which will in general be integrals over the basic fields and their canonical momenta, act on the basic fields of the theory in the same way as the $Z^n\,\bar{Z}^m\,Q^i$. In general, however, these generalized fermionic charges will not be representable in the form of products of the Q^i with scalar field operators. We assume that the algebra of these charges remains valid, and look for other representations. Defining

$$Q_i^{n,m} \equiv Z^n\,\bar{Z}^m\,Q_i \qquad , \qquad P_\mu^{n,m} \equiv Z^n\,\bar{Z}^m\,P_\mu \tag{3.20}$$

we find

$$\begin{aligned}
\left\{Q_i^{n,m}, Q_j^{p,q}\right\} &= 2\delta_{ij}\gamma^\mu P_\mu^{m+p,m+q} \\
[Q_i^{n,m}, P_\mu^{p,q}] &= 0 \\
[L_0, Q_i^{n,m}] &= i\left(m-n-\frac{1}{4}\gamma_5\right)Q_i^{n,m} \\
[L_+, Q_i^{n,m}] &= inQ_i^{n-1,m} - i\left(m-\frac{1}{4}\gamma_5\right)Q^{n,m+1} \\
[L_-, Q_i^{n,m}] &= i\left(n+\frac{1}{4}\gamma_5\right)Q^{n+1,m} - imQ^{n,m-1}
\end{aligned} \tag{3.21}$$

$$
\begin{aligned}
&[L_0, P_\mu^{n,m}] = i(m-n)P_\mu^{n,m} \\
&[L_+, P_\mu^{n,m}] = inP_\mu^{n-1,m} - imP_\mu^{n,m+1} \\
&[L_-, P_\mu^{n,m}] = inP_\mu^{n+1,m} - imP_\mu^{n,m-1}
\end{aligned}
\tag{3.22}
$$

The above algebra has the structure of a semi-direct product of the $SU(1,1)$ algebra with the algebra of $P_\mu^{n,m}$ and $Q^{n,m}$. The subalgebra generated by the generalized momenta $P_\mu^{n,m}$ and generalized supersymmetry generators $Q^{n,m}$ looks very similar to a Kac-Moody extension of the algebra of ordinary P_μ and Q. The Kac-Moody extension of a Lie algebra L, whose elements satisfy the commutation relations

$$[M_i, M_j] = f_{ijk} M_k$$

where f_{ijk} are the structure constants, has the form

$$\left[M_i^m, M_j^n\right] = f_{ijk} M_k^{m+n}$$

modulo some possible Schwinger (or anomaly) terms. One simple realization of this algebra is given by the direct product of a representation of the Lie algebra L by functions $e^{in\theta}$ ($n =$ integer) on the circle:

$$M_i^n = M_i \otimes e^{in\theta} \qquad\qquad -\infty < n < \infty$$

The representation of $P_\mu^{n,m}$ and $Q_i^{n,m}$ on the fundamental fields of the $N = 4$ ESGT corresponds to the direct product of the algebra of P_μ and Q_i by polynomials $Z^n \bar{Z}^m$ (n, m integers ≥ 0) defined inside the unit circle. When one goes to the boundary $Z \to e^{i\theta}$ then $Z^n \bar{Z}^m \to e^{i(n-m)\theta}$ and the algebra resembles more closely the form of a Kac-Moody algebra.

The unitary basic representations of the Kac-Moody algebras (or the so-called highest weight representations) are constructed in terms of the vertex operators of the dual resonance model.[38] They involve an anomalous Schwinger term in the commutation relations, but in our case possible c-number terms which could arise are restricted by non-trivial Lorentz invariance properties,[39] with the only possibilities being Lorentz scalars or pseudoscalars in $\left\{Q_i^{n,m}\ ,\ Q_j^{p,q}\right\}$. Furthermore, when we check the Jacobi identities using (3.21, 3.22) we find that even these Lorentz-allowed Schwinger terms must in fact vanish as a consequence of the fractional $U(1)$ charge associated with the spinorial charges.

In Section 1 we have given the arguments as to why we expect the bound states to form unitary representations of the non-compact invariance group. The supersymmetry transformations extend the Lie algebra of the non-compact group $SU(1,1)$ to the infinite dimensional algebra given by Eqs. (3.21-3.22) in the case of $N = 4$ ESGT. Thus we expect the bound states (bosonic and fermionic) to form a unitary representation of this infinite dimensional algebra. The operators $P_\mu^{m,n}$ and $Q^{m,n}$ can be considered as the Fourier coefficients of generalized momentum and supersymmetry generators that are defined on the open unit disc. This is analogous to the Fourier expansion of generalized momenta and position operators in the string theories.[40]

4. Unitary Representations of $SU(1,1)$

As was pointed out above, one important feature of $N = 4$ to 8 ESGTs is the fact that the scalar fields in these theories are constrained to take values in what is called a bounded homogeneous domain. In the case of the $N = 4$ theory this is the open unit disc Δ in the complex plane:

$$\Delta \equiv \{z(x) | (1 - z\,\bar{z}) > 0\} \tag{4.1}$$

If the bound states of these theories form linear representations of the non-compact invariance group then the relevant unitary representations must be those that can be constructed from the elementary fields appearing in the theories. With this aim a class of unitary representations of the non-compact groups of ESGTs ($n = 4$ to 8) has been constructed in terms of boson operator transforming like the vector fields.[25,22,31] In this section we shall study the construction of unitary representations of $SU(1,1)$ using the scalar fields $z(x)$. This construction of unitary representations of $SU(1,1)$ on functions defined over the open unit disc was first given by Bargmann[41] and corresponds to some of the oldest known unitary representations of a non-compact group. Our eventual aim will be to attempt to realize the full algebra on these functions.

As pointed out in Ref. 39, we can represent the full superalgebra if we can represent the operators L_+, L_-, L_0, Z, and $\bar{Z}$ where $Z \equiv \lim_{x\to\infty} Z(x)$ is constrained as in (4.1). One would expect to be able to do this by constructing functions of Z and $\bar{Z}$ which are representations of the $SU(1,1)$ algebra. More generally, we can construct induced representations of $SU(1,1)$ by forming functions of Z and $\bar{Z}$ which multiply a state transforming under $SU(1,1)$ similarly to the fermions of ESGTs, namely

$$\begin{aligned} L_+|\eta> &= -i\eta\,\bar{Z}\,|\eta> \\ L_-|\eta> &= -i\eta Z|\eta> \\ L_0|\eta> &= i\eta|\eta> \end{aligned} \tag{4.2}$$

Then if we construct a state

$$f(Z,\bar{Z})|\eta> \tag{4.3}$$

the operators $L_\pm$ and L_0 will be represented by

$$\begin{aligned}
-iL_+ &= \frac{\partial}{\partial Z} - \bar{Z}^2\frac{\partial}{\partial \bar{Z}} - \eta\bar{Z} \\
&= \frac{1}{2}e^{-i\phi}\left[(1-\rho^2)\frac{\partial}{\partial\rho} - i\frac{(1+\rho^2)}{\rho}\frac{\partial}{\partial\phi} - 2\rho\eta\right] \\
-iL_- &= -\frac{\partial}{\partial \bar{Z}} + Z^2\frac{\partial}{\partial Z} - \eta Z \\
&= \frac{-1}{2}e^{i\phi}\left[(1-\rho^2)\frac{\partial}{\partial\rho} + i\frac{(1+\rho^2)}{\rho}\frac{\partial}{\partial\phi} + 2\rho\eta\right] \\
-iL_0 &= \frac{\partial}{\partial \bar{Z}} - \frac{\partial}{\partial Z} + \eta = \frac{i\partial}{\partial\phi} + \eta
\end{aligned} \tag{4.4}$$

as follows from Eqs. (3.4) and (3.5), where we have made a change of variables $Z = e^{i\phi}\rho$. We look for a state $|m,\nu,\eta>$ which satisfies the eigenvalue equations

$$-iL_0|m,\nu,\eta> = (m+\eta)|m,\nu,\eta> \tag{4.5}$$

$$K|m,\nu,\eta> = -\nu(\nu-1)|m,\nu,\eta> \tag{4.6}$$

where

$$K \equiv -\frac{L_+L_- + L_-L_+}{2} + L_0^2 \tag{4.7}$$

is the $SU(1,1)$ Casimir invariant which takes real values for unitary representations. We write states in the form

$$\begin{aligned}
|m,\nu,\eta> &= exp(-im\phi)\,\rho^{|m|}(1-\rho^2)^\nu u_{m,\nu,\eta}(\rho)|\eta> \\
&\equiv exp(-im\phi)\,f_{m,\nu,\eta}(\rho)|\eta>
\end{aligned} \tag{4.8}$$

Considering the case $m > 0$, we find that $u_{m,\nu,\eta}$ must be a solution of the hypergeometric equation

$$\rho(1-\rho)\frac{d^2u}{d\rho^2} + [\gamma - (\alpha+\beta+1)\rho]\frac{du}{d\rho} - \alpha\beta\rho = 0 \tag{4.9}$$

with

$$\gamma = 1+m \qquad \alpha = \nu+\eta+m \qquad \beta = \nu-\eta \tag{4.10}$$

The solutions are the hypergeometric series $u_{m,\nu,\eta} = F(\alpha,\beta;\gamma;\rho^2)$ which converge for $0 \le \rho^2 < 1$. Since the operator

$$\begin{aligned} L_-^\dagger L_- &= -L_+L_- = -\frac{1}{2}\{L_+,L_-\} - \frac{1}{2}[L_+,L_-] \\ &= K + L_0 - L_0^2 \end{aligned} \tag{4.11}$$

has positive eigenvalues:

$$\begin{aligned} \ell(\ell-1) - \nu(\nu-1) = (\ell-\nu)(\ell+\nu-1), \\ \ell = m+\eta \end{aligned} \tag{4.12}$$

the eigenvalues of the Casimir K are restricted[41] to three classes:*

1. The principal series: $\nu = \frac{1}{2} + i\lambda$; λ, ℓ arbitrary.
2. The supplementary series: $O < \nu < \frac{1}{2}, \ell = n + \eta_0$, with n integer and $O < \eta_0 < \nu$ or $1-\nu < \eta_0 < 1$.
3. The discrete or bounded series $\nu > 0, \ell \ge \nu$ or $\ell \le -\nu$.

Matrix elements which satisfy the hermiticity requirement

$$< m+1,\nu,\eta|L_+|m,\nu,\eta > = - < m,\nu,\eta|L_-|m+1,\nu,\eta > \tag{4.13}$$

---

* There are further restrictions[41] if one wishes to represent the group and not just the algebra.

are defined by integration over the invariant measure

$$\frac{d\phi\rho d\rho}{(1-\rho^2)^2} \tag{4.14}$$

For $\nu \leq \frac{1}{2}$, as is the case for the principal and supplementary series, this integral is divergent for $\rho \to 1$. In addition the hypergeometric series $F(\alpha, \beta; \gamma; \rho^2)$ is divergent at $\rho = 1$ unless one of the following conditions is fulfilled

(a)

$$Re(\alpha + \beta - \gamma) < 0 \ , \tag{4.15}$$

which cannot be achieved in our case for which

$$\alpha + \beta - \gamma = 2\nu > 0 \ , \tag{4.16}$$

(b) Either α or β is a non-positive integer:

$$\alpha \ \text{ or } \ \beta = -n \leq 0 \ , \tag{4.17}$$

in which case $F(\alpha, \beta; \gamma; \rho^2)$ is a polynomial of finite degree. From Eq. (4.10) we see that (4.17) is achieved by:

$$\nu = \eta - n \ . \tag{4.18}$$

(c) Either $\alpha = \beta$ or $\beta = \gamma$, in which case

$$F(\alpha, \beta; \alpha; \rho^2) = (1 - \rho^2)^\beta$$

This is achieved by either

$$\nu + \eta = 1 \tag{4.19}$$

or

$$\nu = \eta + 1 + m \quad . \tag{4.20}$$

The case (4.20) is not useful since it does not allow the required spectrum of L_0 eigenvalues for fixed ν and η. The same is true of (4.18) for $n \neq 0$. Equation (4.18) for $n = 0$ and Eq. (4.19) give, respectively

$$\ell = \nu + m \tag{4.21a}$$

$$\ell = 1 - \nu + m \tag{4.21b}$$

Equations (4.18, 4.21a) necessarily correspond to the discrete series since ℓ differs from ν by a positive integer. For the case (4.19, 4.21b) we obtain

$$f_{m,\nu,\eta}(\rho) = \rho^m (1-\rho^2)^{\nu+\beta} = (1-\rho^2)^{1-\nu} \tag{4.22}$$

which again is not square integrable over the measure (4.14) for $\nu > 0$.

It is nevertheless possible[42] to define state normalization for the principal series, by forming "wave packets" of solutions in terms of the continuous parameters λ. The orthonormalization condition is expressed in terms of a Dirac δ-function:

$$< \ell, \frac{1}{2} + i\lambda | \ell', \frac{1}{2} - i\lambda >= \delta_{\ell\ell'} \delta(\lambda - \lambda') \tag{4.23}$$

Such a continuous spectrum of states is not appropriate for the problem at hand, namely a countable spectrum of states which might be identified with the bound states of $N = 4$ supergravity. It was shown,[39] however, by algebraic construction without explicit reference to the functional form (4.8), that the principal series can be used to represent the full supersymmetry and $SU(1,1)$ algebra with a countable set of states of finite normalization. It was further pointed out that an $SU(1,1)$ singlet cannot be introduced as a discrete state if one wishes to represent the full superalgebra, but an $SU(1,1)$ singlet can be obtained as the limit $\nu \to 0$ of the supplementary series. It may be that if the full algebra can be represented at all on this series, it can be done so only in terms of a continuous spectrum. Since such solutions are uninteresting for the discussion of bound states we shall not pursue them further.

Let us now examine the discrete series which have a finite normalization for $\nu > \frac{1}{2}$ when expressed in terms of functions of Z. If for example we take $n = 0$ in (4.18) we get $\beta = 0$, $F(\alpha, 0; \gamma; \rho) = 1$, and

$$f_{m,\nu,\eta} = f_{m,\nu} = (1-\rho^2)^{\nu}\rho^m \quad ; \quad m \geq 0 \ . \tag{4.24}$$

These are the functions which are conventionally used to represent the discrete series in the literature.* It is clear that we cannot represent the operators Z and $\bar{Z}$ by simple multiplication on these functions. Functions corresponding to the same value of m but different values of ν are not orthogonal because they are eigenfunctions of different differential operators $K(\nu)$ corresponding to the choice of $\eta = \nu$ in Eqs. (4.5). The orthonormality conditions

$$\begin{gathered} < \ell', \nu' | \ell, \nu > = \delta_{\ell\ell'}\delta_{\nu\nu'} \\ |\ell, \nu > \equiv |m = \ell - \nu, \nu, \eta = \nu > = exp(-im\phi)\, \rho^m (1-\rho^2)^{\nu} |\nu > \end{gathered} \tag{4.25}$$

* Sometimes the eigenfunctions of K are taken[41,43] to be the monomials ρ^m, with the factor $(1-\rho)^{\nu}$ being absorbed into the definition of the invariant measure (4.14). This requires a corresponding redefinition of the differential operators (4.5).

are achieved by imposing

$$< \nu' | \nu >= \delta_{\nu\nu'} \tag{4.26}$$

while integration over ϕ gives $\delta_{m\bar{m}'}$. Thus while we may write

$$\bar{Z}\,|\ell, \nu >= |\ell + 1, \nu > \tag{4.27}$$

we also have

$$\begin{aligned} Z|\ell, \nu > &= exp(-i(m-1)\phi)\,\rho^{m+1}(1-\rho^2)^{\nu})|\nu > \\ &= |\ell - 1, \nu > + exp(-i(m-1)\phi)\, f_{m-1,\nu+1}|\nu > \quad , \end{aligned} \tag{4.28}$$

and we cannot identify the last term in (4.28) with the state $|\ell, \nu+1 >$. Solutions of (4.9) with common η and m but different values of ν, are eigenfunctions of the same differential operator $K(\eta)$ but with different eigenvalues $-\nu(\nu+1)$, and thus necessarily orthogonal, and one might be able to represent the operators Z and $\bar{Z}$ by simple multiplication on such functions. However there do not seem to be any solutions other than (4.24) which are both square integrable and possess the required spectrum of L_0 eigenvalues.

We are therefore led to look for representations of the full algebra by methods of algebraic construction which do not directly exploit the functional forms of eigenfunctions of the differential operators (4.4). As mentioned above one such solution has been found[39] by making the ansatz[44] that the actions of the operators L_+ and L_- can be expressed as functions of the actions of the operators L_0, Z and $\bar{Z}$ which are at most linear in L_0. Explicitly, we look for solutions $| >$ with

$$\begin{aligned} -iL_+| > &= \bar{Z}\,[A(Z\,\bar{Z})L_0 + B(Z\,\bar{Z})]\,| > \\ -iL_-| > &= [A^*(Z\,\bar{Z})L_0 + B^*(Z\,\bar{Z})]\,Z| > \end{aligned} \tag{4.29}$$

The solution to the algebra defined in (3.1), (3.4) and (3.5), together with

$$[Z,\bar{Z}]=0 \tag{4.30}$$

gives

$$A|>=-|> \qquad\qquad B|>=\left(-\frac{1}{2}-i\lambda\right)|> \tag{4.31}$$

with

$$Z\bar{Z}|>=|> \quad . \tag{4.32}$$

Using these results to construct the Casimir operator (4.7) gives

$$K|>=\left(\frac{1}{2}+i\lambda\right)\left(+\frac{1}{2}-i\lambda\right)|>=\left(\frac{1}{4}+\lambda^2\right)|> \tag{4.33}$$

which means that the ansatz (4.29) restricts the Hilbert space to that corresponding to the principal series of $SU(1,1)$ representations, as well as restricting the operators Z and $\bar{Z}$ to the unit circle, Eq. (4.32). We may now represent the algebra by states $|\ell,\lambda>$, where we choose some countable set of parameters λ and impose

$$\begin{aligned} <\ell',\lambda'|\ell,\lambda> &= \delta_{\ell\ell'}\delta_{\lambda\lambda'} \\ -iL_0|\ell,\lambda> &= \ell|\ell,\lambda> \quad , \\ \bar{Z}|\ell,\lambda> &= |\ell+1,\lambda> \quad , \\ Z|\ell,\lambda> &= |\ell-1,\lambda> \end{aligned} \tag{4.34}$$

Using

$$\begin{aligned} -iL_+|> &= -\bar{Z}\left(L_0+\frac{1}{2}+i\lambda\right)|> \\ -iL_-|> &= -Z\left(L_0-\frac{1}{2}-i\lambda\right)|> \end{aligned} \tag{4.35}$$

as follows from (4.29-4.32) we then obtain

$$
\begin{aligned}
L_+|\ell,\lambda> &= -\left(\ell+\frac{1}{2}+i\lambda\right)|\ell+1,\lambda> \\
L_-|\ell,\lambda> &= -\left(\ell-\frac{1}{2}-i\lambda\right)|\ell-1,\lambda>
\end{aligned}
\tag{4.36}
$$

It is clear that the ansatz (4.29) is a sufficient condition for representing Z, $\bar{Z}$, $L_\pm$, and L_0 on the same Hilbert space. It may not be a necessary one. However, any solution other than the one given above necessarily involves a spectrum which has degenerate $U(1)$ eigenvalues. To see this let us assume that we can represent the algebra on a set of non-degenerate states $|\ell>$, $\ell=\nu+n$ with $0<\nu\leq 1$ and n an integer. Since the operator $\bar{Z}$ carries one unit of ℓ, we necessarily have

$$
\bar{Z}\,|\ell>= C_\ell|\ell+1> \tag{4.37}
$$

Hermiticity requires

$$
<\ell+1|\,\bar{Z}\,|\ell>=<\ell|Z|\ell+1>^*
$$

or

$$
Z|\ell>= C^*_{\ell-1}|\ell-1> \quad . \tag{4.38}
$$

Commutativity, Eq. (4.30), requires

$$
Z\,\bar{Z}\,|\ell>= \bar{Z}\,Z|\ell>
$$

or

$$
|C_\ell|^2=|C_{\ell-1}|^2\equiv|C|^2 \tag{4.39}
$$

Thus $|C_\ell|$ is independent of ℓ. Since operation by $Z\,\bar{Z}$ reduces to multiplication by a constant, this operator commutes with $L_\pm$ and it follows from the commutation relations (3.4) and (3.5) that $|C| = 1$; i.e. the unit circle is the only $SU(1,1)$ invariant circle for the variable Z. We may choose a phase convention such that

$$C_\ell = 1 \tag{4.40}$$

We now define

$$-iL_-|\ell> = d_\ell|\ell> \tag{4.41}$$

and from (3.5)

$$-iL_-\,\bar{Z}\,|\ell> = -i\,\bar{Z}\,L_-|\ell> - |\ell> \tag{4.42}$$

we obtain

$$d_{\ell+1} = d_\ell - 1$$

which implies

$$d_\ell = d - \ell \tag{4.43}$$

Since we have assumed a non-degenerate $U(1)$ spectrum, the states $|\ell>$ either belong to an irreducible representation of $SU(1,1)$ of the discrete, principal or supplementary series, or a superpositon of irreducible representations of the discrete series which have no common ℓ-values. Then equating the values of the matrix element

$$< \ell| - L_+L_-|\ell> = |d-\ell|^2 = \ell(\ell-1) - \nu(\nu-1) \tag{4.44}$$

as obtained from (4.12) and from (4.43) we find

$$Re\ d = \frac{1}{2} \quad , \quad d = \frac{1}{2} + i\lambda \qquad |d|^2 = \frac{1}{4} + \lambda^2 = (1-\nu)\nu \tag{4.45}$$

which correspond to the principal series: $\nu = \frac{1}{2} + i\lambda$.

Thus, while we have not shown that the representation (4.34) of the algebra (3.1), (3.4), (3.5) and (4.30) is unique, it appears to be a minimal one in the sense that it is irreducible under $SU(1,1)$ (except for $\lambda = 0$ in (4.45) in which case it splits into the two discrete representations $\ell \geq \frac{1}{2}$ and $\ell \leq -\frac{1}{2}$), while any other representation will necessarily be reducible under $SU(1,1)$.

5. Unitary Realizations of the Superinvariance Algebra

Once we have obtained a realization of the algebra defined in Eqs. (3.1), (3.4), (3.5) and (4.31), the full superinvariance algebra is defined by Eqs. (3.1) and (3.20-3.22). If $|\ell, \nu, \lambda>$ is a state of $U(1)$ eigenvalue ℓ, Casimir $-\nu(\nu-1)$ and helicity λ, we represent the operators defined in Eq. (3.20) by:

$$\begin{aligned} P_\mu^{n,m}|\ell,\nu,\lambda> &= P_\mu|\ell+m-n,\nu,\lambda> \\ Q^{n,m}|\ell,\nu,\lambda> &= Q|\ell+m-n,\nu,\lambda> \end{aligned} \tag{5.1}$$

Note that because Z is restricted to the unit circle, the doubly infinite set of operators $(n, m = 0, 1, 2, \dots, \infty)$ is reduced to a simply infinite set

$$\begin{aligned} P_\mu^{n,m} &\rightarrow P_\mu^a \\ & \qquad\qquad a = m - n = 0, \pm 1, \pm 2, \dots \pm \infty \\ Q^{n,m} &\rightarrow Q^a \end{aligned}$$

These representations will be characterized by infinite series of supermultiplets characterized by a fixed value of λ_{max} and $U(1)$ eigenvalues

$$\begin{aligned} \ell_n(\lambda) &= \ell_n(\lambda_{max}) + \frac{\lambda_{max} - \lambda}{2} \\ \ell_n(\lambda_{max}) &= \ell_0(\lambda_{max}) + n \ ; \quad n = 0, \pm 1, \dots \pm \infty \\ 0 < \ell_0&(\lambda_{max}) \leq 1 \end{aligned} \tag{5.2}$$

The representation relevant to attempts[11–18] to connect supergravity with conventional gauge theories is a superposition of the two representations which contain the zero-mass-shell projection of the supercurrent multiplet, namely

$$\begin{aligned} \lambda_{max} &= \frac{3}{2} \quad ; \quad \ell_0(\lambda_{max}) = -\frac{1}{4} \\ \lambda_{max} &= 0 \quad ; \quad \ell_0(\lambda_{max}) = -1 \end{aligned} \tag{5.3}$$

We cannot conclude that there may not be other, possibly more interesting representations, for example with an infinite spectrum of helicities. However, as emphasized in Ref. 39, the operators $P_\mu^{n,m}$ can be represented if and only if Z and $\bar{Z}$ can be represented, in which case representations of both $P_\mu^{n,m}$ and $Q^{n,m}$ of the above structure follow immediately.

6. Extension to Higher Extended Supergravity Theories

From the point of view of the non-compact symmetry and its compatibility with supersymmetry the higher supergravity theories ($N = 5, 6, 8$) have essentially the same structure as the $N = 4$ theory. In these higher theories the scalar fields transform non-linearly under the non-compact global invariance group like cosets $SU(5,1)/U(5)$, $SO(12)^*/(U(6)$ and $E_{7(7)}/SU(8)$, respectively. One can choose a gauge in which the scalar fields parameterizing these coset spaces are represented by some matrix fields z satisfying a constraint of the form

$$(I - z^\dagger z) > 0$$

This implies that, as in the $N = 4$ theory, the scalar fields Y take their values in a bounded homogeneous domain and they undergo a generalized linear fractional transformation under the non-compact invariance group that maps the domain into itself. This non-compact "automorphism" group of the domain corresponds to a combined G_{gl} and an induced H_{loc} transformation just as in the $N = 4$ case. In the case $N = 5$ and $N = 6$ the corresponding domains have a complex structure and it is well-known that the holomorphic discrete series unitary representations of the non-compact groups $SU(5,1)$ and $SO(12)^*$ can be constructed over the Hilbert spaces of analytic functions of the complex variables (scalar fields) which take values on the respective domains.[31,32,45,46] In the $N = 8$ case the corresponding domain does not have a complex structure, which is a reflection of the fact that $E_{7(7)}/SU(8)$ is not a hermitian symmetric space.[47] Therefore the extension of the above construction to this case may involve some novel features.

A systematic study of possible representations of the $N = 8$ superinvariance algebra in terms of general classes of irreducible representations of $E_{7(7)}$ has not yet been done, even to the limited, and not entirely conclusive, extent of the analysis of the $N = 4$ case presented in this paper. Similarly to the $N = 4$ case, we can construct oscillator-like representations and/or induced representations in the 70-dimensional space of the physical scalars z of the theory. The oscillator-like representations are equivalent[32,46] to those obtainable from the Hilbert spaces of analytic functions of the z_i. In References 32 and 49 it was shown how to construct the oscillator-like unitary representations of the superinvariance algebras of all ESGTs ($N = 4 - 8$) using the oscillator-like unitary representations of the corresponding non-compact groups in a coherent state basis labelled by the scalar fields. The coherent state representations of the non-compact groups $SO(12)^*$ and $E_{7(7)}$ of the $N = 6$ and 8 ESGTs are reducible.[31] Thus in the unitary realization of the superinvariance algebra of the

$N = 8$ ESGT the representations of $E_{7(7)}$ that occur for a given helicity are always reducible.[32,49]

An alternative technique was used in Ref. 39 where it was shown that it is possible to construct a class of representations of the superinvariance algebra for the most interesting case of $N = 8$ by making an ansatz[44] similar to (4.29) relating the non-compact generators Y_i and the operators Z_i as an operator equation valid on the Hilbert space of those representations. Their structure is similar to those of the $SU(1,1)$ representations displayed in Section 5. The non-compact generators Y_i can be represented as before by differential operators whose precise form depends on the $SU(8)$ transformation properties of the inducing representation, similarly to the η-dependence in Eqs. (4.4). Representations constructed in this way are generally not irreducible under $E_{7(7)}$. For the special class of representations constructed in Ref. 39, this is reflected in the fact that states which transform according to irreducible representations of $SU(8)$ are not eigenstates of the $E_{7(7)}$ Casimir operator, except for $SU(8)$ singlets which have eigenvalues of the form:

$$K = N\nu(69 - \nu) \quad ; \quad \nu = \frac{69}{2} + i\lambda \quad , \quad N = \frac{3}{23} \tag{6.1}$$

When the Casimir operator is represented as a differential operator in the 70-dimensional space of the asymptotic scalar field operators Z_i, one finds solutions to the eigenvalue equations which are $SU(8)$ invariant and whose functional form is again a hypergeometric function multiplied by a monomial. The behavior of the monomial on the invariant surface $\sum Z_i^2 = N$ which bounds the 70-dimensional volume over which the scalar field variables are defined[39] is again dictated by the value of the Casimir:

$$(1-\frac{1}{N}Z_i^2)^\nu \tag{6.2}$$

When square integrated over the invariant metric

$$d^{70}Z\left(1-\frac{1}{N}\sum_i Z_i^2\right)^{70} \tag{6.3}$$

the functions corresponding to the eigenvalues (6.1) have divergence properties similar to those for the principal series of $SU(1,1)$, indicating that solutions obtained using the ansatz of the type in Eqs. (4.30) have particular properties which are independent of the choice of group.

The representations found using the above technique are in fact of the type conjectured in Ref. 13 where it was shown that such representations allow unwanted particles to acquire group invariant masses once the $SU(8)$ invariance is broken to an invariance under a subgroup no larger than $SU(6)$ (with possibly a simple supersymmetry surviving as an additional invariance of the theory). Unfortunately these group theoretic considerations are insufficient to determine what, if any, set of bound states should remain massless, which is relevant to the more important question of whether the bound state spectrum conjectured[11–18] as arising from $N=8$ supergravity can indeed lead to a realistic effective gauge theory of present energy interactions.

Induced representations which are irreducible under $E_{7(7)}$ can be constructed on spaces smaller than the 70-dimensional one of the scalar fields.[48] We have examined briefly examples of such representations and found that if each helicity state of a given supermultiplet is assigned to an irreducible representation of $E_{7(7)}$ the states generated by operating successively with the Y_i do not fall into supermultiplets. This may indicate that we cannot represent the full algebra using irreducible representations of this type, although one might be able to find[50] some set of such irreducible $E_{7(7)}$ representations which would be able to represent the algebra, possibly involving an infinite spectrum of spins. In addition the Z_i can be represented in this case only if they satisfy additional constraints which restrict them to the appropriate smaller dimensional space. This implies constraints among the generalized P's and Q's which would have to be consistent with the $N = 8$ superinvariance algebra. Such a realization would further mean that the 8 ordinary supersymmetry generators do not transform linearly under $SU(8)$ but only under some subgroup of $SU(8)$. A full investigation of these questions clearly requires some new mathematical techniques for studying infinite algebras of the type extracted from the invariance groups of extended supergravity theories.

ACKNOWLEDGEMENTS

We have enjoyed many discussions with Murray Gell-Mann and Raymond Stora, and we thank Anne Davis for instructive conversations. Two of us (M.K.G. and B.Z.) would like to acknowledge the hospitality extended to us at Fermilab and at the Aspen Center for Physics where part of this work was done.

REFERENCES

1. E. Cremmer and B. Julia, Phys. Lett. 80B (1978) 48; Nucl. Phys. B159 (1979) 141.
2. E. Cremmer, S. Ferrara and J. Scherk, Phys. Lett. 74B (1978) 61.
3. M. K. Gaillard and B. Zumino, Nucl. Phys. B193 (1981) 221.
4. A. D'Adda, P. DiVecchia and M. Lüscher, Nucl. Phys. B146 (1978) 63; B152 (1979) 125; E. Witten, Nucl. Phys. B149 (1979) 285.
5. Sigma models on non-compact coset spaces were formulated in K. Cahill, Phys. Rev. D 18 (1978) 2930; E. Cremmer and J. Scherk, Phys. Lett. 74B (1978) 341; B. Julia and J. F. Luciani, Phys. Lett. 90B (1980) 270.
6. E. R. Nissimov and S. J. Pacheva, Comp. Rendus de ℓ'Acad. Bulgare des Sciences 43 (1979) 1475.
7. E. R. Nissimov and S. J. Pacheva, Lett. Math. Phys. 5 (1981) 67, 333.
8. M. Gell-Mann, Talk at the 1977 Washington Meeting of the American Physical Society, unpublished.
9. M. Gell-Mann, Talk at the Aspen Workshop on Octonionic Quantum Mechanics (1978), unpublished.
10. T. Curtright and P.G.O. Freund, "Supergravity," ed. P. van Nieuwenhuizen (North Holland, Amsterdam, 1979) p. 197.
11. J. Ellis, M. K. Gaillard, L. Maiani and B. Zumino, "Unification of the Fundamental Interactions," ed. by S. Ferrara, J. Ellis and P. van Nieuwenhuizen (Plenum Press, N.Y., 1980) p. 69.
12. J. Ellis, M. K. Gaillard and B. Zumino, Phys. Lett. 94B (1980) 343.

13. J. Ellis, M. K. Gaillard and B. Zumino, Acta Physica Polonica B13 (1982) 253.

14. B. Zumino, 1980 Madison Interntional Conference on High Energy Physics, ed. L. Durand and L. G. Pondrom (A.I.P., N.Y., 1981) p. 964. The idea that explicit invariance under the non-compact group might be realized linearly on the bound states first emerged during discussions involving W. A. Bardeen and E. Rabinovici.

15. B. Zumino, "Superspace and Supergravity," ed. S. W. Hawking and M. Rŏcek (Cambridge University Press, Cambridge, 1981) p. 423; M. K. Gaillard in "Unified Theories of Elementary Particles, Critical Assessment and Prospects," ed. P. Breitenlohner and H.-P. Dürr, Lecture Notes in Physics 160 (Springer-Verlag, Berlin, 1982) p. 1.

16. J. P. Derendinger, S. Ferrara and C. A. Savoy, CERN preprint TH-3025 (1981), unpublished; and Nucl. Phys. B188 (1981) 77.

17. J. E. Kim and H. S. Song, Phys. Rev. D 25 (1982) 2996.

18. J. Ellis, Talk at the Second Europhysics Study Conference on Unification of the Fundamental Particle Interactions, Erice, Sicily 6-14 October (1981), CERN preprint TH-3206 (1981).

19. B. DeWit and H. Nicolai, Nucl. Phys. B188 (1981) 98, and B208 (1982) 323.

20. P. Frampton, Phys. Rev. Lett. 46 (1981) 881.

21. G. Altarelli, N. Cabibbo, L. Maiani, Nucl. Phys. B206 (1982) 397.

22. M. Günaydin, Talk at the Second Europhysics Study Conference on Unification of the Fundamental Particle Interactions, Erice, Sicily 6-14 October (1981), CERN preprint TH-3222 (1981).

23. W. A. Bardeen, B. W. Lee and R. E. Shrock, Phys. Rev. D 14 (1976) 985; H. Haber, I. Hinchliffe and E. Rabinovici, Nucl. Phys. B172 (1980) 458.

24. S. Coleman, Comm. Math. Phys. 31 (1973) 259.

25. M. Günaydin and C. Saçlioğlu, Phys. Lett. 108B (1982) 169.

26. V. S. Makhankov and O. K. Pashaev, Dubna preprints JINR E2-81-264 and 540 (1981).

27. Y. Cohen and E. Rabinovici, CERN preprint TH-3502 (1983).

28. A. C. Davis, A. J. Macfarlane and J. W. van Holten, Princeton preprint "Composite Gauge Bosons in Sigma Models and Supergravity" (1983), see also CERN preprint TH-3405 (1982).

29. M. T. Grisaru and H. J. Schnitzer, Phys. Lett. 107B (1981) 196 and Nucl. Phys. B204 (1982) 267.

30. M. Green and J. Schwarz, Nucl. Phys. B181 (1981) 502 and B198 (1982) 252.

31. M. Günaydin and C. Saçlioğlu, Comm. Math. Phys. 87 (1982) 159.

32. M. Günaydin, Talk at the "XI International Colloquium on Group Theoretical Methods in Physics," Istanbul, Turkey, 23-28 August (1982), to appear in the proceedings of the colloquium; Ecole Normale Supérieure preprint LPTENS 83/5 (1983).

33. R. Moody, Bull. Am. Math. Soc. 73 (1967) 217 and J. of Alg. 10 (1968) 211; V. Kac, Math. U.S.S.R. Izvestiya Ser. Math. 32 (1968) 1271.

34. E. Cremmer and J. Scherk, Nucl. Phys. B127 (1977) 259.

35. A. Das, Phys. Rev. D 15 (1977) 2805.

36. B. Zumino, Phys. Lett. 87B (1979) 203.

37. J. Bagger and E. Witten, Princeton Univ. preprint "Matter Couplings in $N = 2$ Supergravity" (1983).

38. I. B. Frenkel and V. G. Kac, Inv. Math. 62 (1980) 23.

39. M. K. Gaillard, U.C. Berkeley preprint UCB-PTH-82/18 (1982).

40. "Dual Theory," ed. M. Jacob (North Holland, Amsterdam, 1974).

41. Unitary representations of $SU(1,1)$ were first given by V. Bargmann, Ann. Math. 48 (1947) 568; Comm. Pure and Appl. Math. 14 (1961) 187. For a novel reformulation see I. M. Gelfand, M. I. Graev and N. Ya. Vilenkin, Vol. 5 (Academic Press, New York, 1966). The operator formulation of the representation theory of Gelfand et al. was given in F. Gürsey, "Representations of Some Non-compact Groups Related to the Poincaré Group," Yale Univ. mimeographed notes (1971); I. Bars and F. Gürsey, J. of Math. Phys. 13 (1972) 131; F. Gürsey and S. Orfanidis, Phys. Rev. D 7 (1973) 2414. See also A. O. Barut and C. Fronsdal, Proc. Roy. Soc. London A287 (1965) 532; A. O. Barut, Proc. 1966 Boulder Summer Institute, p. 125.

42. L. C. Biedenharn, Ann. Phys. 39 (1966) 1; 47 (1968) 205.

43. W. Schmid in "Representation Theory of Lie Groups," London Math. Soc. Lecture Note Series 34 (Cambridge University Press, Cambridge, 1979).

44. Similar constructions were also known to Y. Dothan, M. Gell-Mann and Y. Ne'eman (unpublished); see also R. Hermann, "Lie Groups for Physicists" (Benjamin, New York, 1966) and Y. Dothan and Y. Ne'eman in "Symmetry Groups," ed. F. Dyson (Benjamin, New York, 1966).

45. A. W. Knapp, "Bounded Symmetric Domains and Holomorphic Discrete Series" (Marcel Dekker, Paris, 1972).

46. I. Bars and M. Günaydin, CERN preprint TH-3350 (1982) to be published in Comm. Math. Phys.

47. J. A. Wolf, J. Math. and Mech., 13 (1964) 489.

48. M. Gell-Mann, private communication. For an outline of Gell-Mann's method of constructing unitary representations of non-compact groups on the coset spaces of their maximal compact subgroups see R. Hermann's book in Ref. 44.

49. E. Cremmer and M. Günaydin, in preparation (1983); M. Günaydin, Talk at the Annecy Meeting on Supersymmetry and Supergravity, 11-13 January 1983.

50. M. Gell-Mann and R. Slansky, in progress (1983).