SLAC-PUB-3056
February 1983
(T)

GAUGED SUPERSYMMETRIC GENERALIZED NONLINEAR SIGMA MODELS FOR QUARKS AND LEPTONS*

CHONG-LEONG ONG
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We consider the idea of describing quarks and leptons, and their interactions, in terms of gauged supersymmetric generalized nonlinear sigma models. It is found that the best model is the one based on the Kahler manifold $E_7/SU(4) \times U(1)^4$, with an appropriate complex structure. The isotropy representation of the manifold is sufficient for embedding three families of quarks and leptons. We aslo discuss the problem of the ABJ-anomaly which so frequently occurs in such models.

Submitted to Physical Review Letters

* Work supported by the Department of Energy, contract DE-AC03-76SF00515.

(1) The observed spectrum of quarks and leptons has expanded significantly during the last ten years. According to their interactions, the fermions may be grouped into three identical families, with two predicted members, ν_τ and t-quark, yet to be confirmed by further experiments. In $SU(3)_C \times SU(2)_L \times U(1)$ unified gauge theory, each fermion family has the representation content $(\underline{3}, \underline{2}) + 2(\underline{3}^*, \underline{1}) + (\underline{1}, \underline{2}) + (\underline{1}, \underline{1})$, omitting the $U(1)$ charges. In $SU(5)$ grand unified gauge theory, each family is described as $\underline{5}^* + \underline{10}$. And in $SO(10)$ grand unified gauge theory, it is embedded in the $\underline{16}$ representation. An explanation for the repetition of the family structure is, however, still unavailable. Proceeding to conventional unified gauge theories with bigger gauge groups does not provide a satisfactory answer.[1]

(2) It is reasonable that the idea of compositeness may be useful somehow for explaining the quark-lepton spectrum. A main theoretical challenge here is that of reconciling the relatively small masses (and in some instances, possibly vanishing masses) of quarks and leptons with the large mass scale[2] characterizing the extra strong interactions of the preons (the hypothetical constituents of quarks and leptons). The only analogous situation known in particle physics is that of pions. They are composite and have relatively small masses; the masses are small because of pions' role as Goldstone particles.

Preon models in which massless composite fermions might be produced because of an unbroken chiral symmetry were, and are still, vigorously searched for after 't Hooft proposed the anomaly matching condition.[3] Disappointingly, a commonly acceptable model of this sort has yet to be found. Another interesting proposal[4] is to identify quarks and leptons as composite Goldstone fermions resulting from broken supersymmetries. This approach has the advantage of avoiding questions concerning the nature of preons and their interactions. It requires, however, too many supersymmetries, one for each two-component fermion.

(3) In this paper we shall consider gauged supersymmetric generalized nonlinear sigma models[5,6] (henceforth to be referred to simply as GSGN σ-models) for quarks and leptons. Accordingly, quarks and leptons may be identified either as supersymmetric partners of the scalar fields of the σ-models (henceforth to be referred to as σ-fermions[7]), or as gauge fermions; or as some combination of the two. Since the scalar fields may be thought of as Goldstone bosons, like pions, the σ-fermions may in this sense be regarded as composite.

In 3+1 dimensional space-time, N=1 supersymmetric generalized nonlinear σ-models[8] exist only for cases where the scalar fields take values on Kahlerian complex manifolds.[9] We shall focus our attention only on homogeneous Kahler manifolds expressible as G/H, where G is a compact, connected, simple Lie group and H a closed subgroup. In order for G/H to be Kahlerian, it is necessary and sufficient[10] that H be the centralizer of a torus of G. The hermitean symmetric spaces considered in Ref. 5 are cases of a 1-dimensional torus. H is the isotropy group of the manifold G/H. The scalar fields form a linear representation, usually reducible, of H, but a nonlinear realization of G. The σ-fermions possess the same isotropy representation content. Indeed the fields of σ-fermions span the fibres of the fibre bundle of which the Kahler manifold is the base.[8] In cases where the choice of G and H results in a reducible isotropy representation, inequivalent complex manifolds may be constructed.[11] The models are invariant with respect to transformations belonging to the group G.

When gauging supersymmetric generalized nonlinear σ-models based on the Kahler manifold G/H, one may choose to gauge the group G completely, or simply gauge any subgroup $S \subset G$. According to an argument[5] based on counting of degrees of freedom, the supersymmetry will be spontaneously broken if S is not completely contained in H. The same condition also enforces the breaking of gauge group S to the subgroup $S \cap H$.

Both supersymmetry and gauge symmetry would remain intact if S $\subset$ H. If we denote the scalar boson fields and σ-fermion fields as ϕ^i and χ^i respectively, and the gauge boson fields and gauge fermion fields as $A_\mu^{(a)}$ and $\lambda^{(a)}$ respectively, then the Lagrangian density can be expressed in the general form[6]

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4} F_{\mu\nu}^{(a)} F_{\mu\nu}^{(a)} - i\lambda^{(a)} \sigma^\mu D_\mu \bar{\lambda}^{(a)} - \frac{1}{2} e^2 D^{(a)2} \\ &- g_{ij^*} D_\mu \phi^i D^\mu \phi^{*j} - \frac{1}{2} i g_{ij^*} \chi^i \sigma^\mu D_\mu \bar{\chi}^j \\ &- \frac{1}{2} i g_{ij^*} \bar{\chi}^j \bar{\sigma}^\mu D_\mu \chi^i + \frac{1}{4} R_{ik^*j\ell^*} (\chi^i \chi^j)(\bar{\chi}^k \bar{\chi}^\ell) \\ &+ e\sqrt{2}\, g_{ij^*} (V^{(a)i} \bar{\chi}^j \bar{\lambda}^{(a)} + V^{*(a)j} \chi^i \lambda^{(a)}) \end{aligned}$$

where g_{ij^*} and $R_{ik^*j\ell^*}$ are respectively the metric tensor and curvature tensor of the Kahler manifold G/H, $V^{(a)i}$ are the Killing vectors, $D^{(a)}$ are real functions defined on the manifold such that

$$g_{ij^*} V^{*(a)j} = i \frac{\partial D^{(a)}}{\partial \phi^i} \quad ,$$

and the covariant derivatives are

$$\begin{aligned} D_\mu \phi^i &= \partial_\mu \phi^i - e A_\mu^{(a)} V^{(a)i} \\ D_\mu \chi^i &= \partial_\mu \chi^i + g^{i\ell^*} \frac{\partial g_{j\ell^*}}{\partial \phi^k} D_\mu \phi^j \chi^k - e A_\mu^{(a)} \frac{\partial V^{(a)i}}{\partial \phi^j} \chi^j \\ D_\mu \lambda^{(a)} &= \partial_\mu \lambda^{(a)} + e f^{abc} A_\mu^{(b)} \lambda^{(c)} \\ F_{\mu\nu}^{(a)} &= \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} + e f^{abc} A_\mu^{(b)} A_\nu^{(c)} \ . \end{aligned}$$

Of course the gauge coupling constant e may take a different value for each factor of S when S is a product of simple Lie groups and/or $U(1)$.

(4) Let us now estimate the lower bound on the dimension of group G such that a GSGN σ-model based on Kahler manifold G/H may contain all the quarks and leptons, and unbroken color and electromagnetism gauge symmetries. We are convinced that there are at least three families of quarks and leptons, which amount to $3 \times (5 + 10) = 45$ two-component Weyl spinors. In the context of supersymmetric generalized nonlinear σ-model, each two-component Weyl spinors has a complex scalar boson field as its supersymmetric partner. Each of the complex scalar fields is regarded as a complex coordinate on the Kahler manifold G/H and thus corresponds to two generators of G. On the other hand, there is an unbroken gauge group which includes at least the $SU(3)_C \times U(1)_{em}$ of chromodynamics and electromagnetism, which amounts to 8+1 = 9 unbroken group generators. In summary, the dimension of G should be greater than $(2 \times 45 + 9) = 99$. More specifically we have the following results: (i) For $G = SU(n)$, dim $G = n^2 - 1$, rank $G = n - 1$. The number of off-diagonal generators is $(n^2 - 1) - (n - 1) \geq 99 - 2 - 1 = 96$, therefore $n \geq 11$. $SU(11)$ has dimension 120 and rank 10. (ii) For $G = SO(2n)$, dim $G = n(2n - 1)$, rank $G = n$. One obtains $n \geq 8$. $SO(16)$ has dimension 120 and rank 8. (iii) For $G = SO(2n+1)$, dim $G = n(2n+1)$, rank $G = n$. One obtains $n \geq 7$. $SO(15)$ has dimension 105 and rank 7. (iv) For $G = Sp(2n)$, dim $G = n(2n+1)$, rank $G = n$. One obtains $n \geq 7$. $Sp(14)$ has dimension 105 and rank 7. (v) E_7 and E_8 are the only two exceptional Lie groups with dimension greater than 99. E_7 has dimension 133 and rank 7. E_8 has dimension 248 and rank 8.

(5) We shall now show that amongst the lowest rank candidates for G, namely, $SO(15)$, $Sp(14)$, and E_7, only E_7 leads to a Kahler manifold G/H with the desired isotropy representation content. The method to be employed is one of analyzing the representation content, with respect to a $SU(5)$ subgroup of G, of the positive root

systems of the Lie algebras. Relevant mathematical theorems concerning positive root systems and their connection to complex structures on the manifold G/H can be found in Ref. 11.

The root system[12] of $SO(15)$ Lie algebra consists of $\pm e_i$, and $\pm e_i \pm e_j$ with $1 \leq i \neq j \leq 7$. The e_i's are the basis vectors of the Euclidean space containing the roots. With respect to the ordering $e_1 > e_2 > \ldots > e_7 > 0$, the positive roots are e_i, and $e_i \pm e_j$ with $1 \leq i < j \leq 7$. The roots $e_i - e_j$ with $1 \leq i < j \leq 5$ form the positive root system for a $SU(5)$. The remaining ones have the following representation content with respect to the $SU(5)$, namely, $\{e_i, 1 \leq i \leq 5\} \leftrightarrow \underline{5}$, $\{e_6\} \leftrightarrow \underline{1}$, $\{e_7\} \leftrightarrow \underline{1}$, $\{e_i + e_j, 1 \leq i < j \leq 5\} \leftrightarrow \underline{10}$, $\{e_i \pm e_6, 1 \leq i \leq 5\} \leftrightarrow \underline{5}$, $\{e_i \pm e_7, 1 \leq i \leq 5\} \leftrightarrow \underline{5}$, and $\{e_6 \pm e_7\} \leftrightarrow \underline{1}$. Obviously this list does not fit the known pattern, namely, $3 \times (\underline{5}^* + \underline{10})$, for quarks and leptons. Choosing a different ordering may only change some $\underline{5}$'s to $\underline{5}^*$'s, and/or $\underline{10}$ to $\underline{10}^*$, and therefore does not help.

The root system of $Sp(14)$ Lie algebra consists of $\pm 2e_i$, and $\pm e_i \pm e_j$ with $1 \leq i \neq j \leq 7$. Again take the roots $e_i - e_j$, with $1 \leq i < j \leq 5$, as the positive root system for a $SU(5)$. The remaining positive roots have the representation content $(4 \times \underline{1} + 4 \times \underline{5} + \underline{15})$ with respect to the $SU(5)$ according to the ordering $e_1 > e_2 > \ldots > e_7 > 0$.

The root system of E_7 Lie algebra consists of $\pm e_i \pm e_j$ with $1 \leq i \neq j \leq 6$, $\pm\sqrt{2}\ e_7$, and $(1/2)(\pm e_1 \pm e_2 \pm \ldots \pm e_6) \pm (1/\sqrt{2})e_7$ with even number of plus signs in the bracket. With respect to the ordering $(-e_6) > e_7 > e_1 > e_2 > \ldots > e_5 > 0$, the positive roots are $e_i \pm e_j$ with $1 \leq i < j \leq 5$, $-e_6 \pm e_i$ with $1 \leq i \leq 5$, $\sqrt{2}\ e_7$, and $-(1/2)e_6 \pm (1/\sqrt{2})e_7 + (1/2)(\pm e_1 \pm e_2 \pm \ldots \pm e_5)$ with even number of plus signs in the bracket. The roots $e_i - e_j$ with $1 \leq i < j \leq 5$ form the positive root system for a $SU(5)$. The remaining ones have the representation content $\Gamma = 3 \times (\underline{1} +$

$\underline{5}^* + \underline{10}) + \underline{5}$ with respect to the $SU(5)$. Group theoretically this representation content is sufficient for the observed leptons and quarks. The Kahler manifold with such an isotropy representation is $E_7/SU(5) \times U(1)^3$ with a complex structure corresponding to the above choice of positive root system. However the representation is not ABJ-anomaly[13] free with respect to the unbroken gauge group $SU(3)_C \times U(1)_{em}$, assuming the standard embedding $SU(3)_C \times U(1) \subset SU(4) \subset SU(5)$. In order to avoid this problem we are led to a bigger Kahler manifold, namely $E_7/SU(4) \times U(1)^4$ with positive root system chosen according to the ordering $(-e_6) > e_7 > e_5 > e_1 > e_2 > e_3 > e_4 > 0$. [The positive roots of the $SU(4)$ are identified as $e_i - e_j$ with $1 \leq i < j \leq 4$.] The isotropy representation is $\Gamma + \underline{4}^*$.

(6) Similar analysis rules out the cases where the group $G = SU(n)$, $SO(2n)$, $SO(2n+1)$, or $Sp(2n)$, for any n. Since $E_8 \supset E_7$, there is no problem in finding a Kahler manifold of the form E_8/H such that the isotropy representation can accommodate the $3 \times (\underline{5}^* + \underline{10})$ of quarks and leptons, e.g., that of $H = SU(4) \times U(1)^5$. The dimension of E_8 is, however, almost twice that of E_7. Thus we conclude that $E_7/SU(4) \times U(1)^4$ with isotropy representation $\Gamma + \underline{4}^*$ is the best Kahler manifold for constructing a GSGN σ-model for quarks and leptons.

(7) What subgroup $S \subset G$ shall we gauge? We have to confine ourselves to an S which should include at least the $SU(3)_C \times U(1)_{em}$ gauge group if the weak interactions are not regarded as mediated by $SU(2)_L$ gauge bosons of Weinberg-Salam-Glashow model, and to an S including at least $SU(3)_C \times SU(2)_L \times U(1)$ otherwise. In the extreme case where $S = SU(3)_C \times U(1)_{em}$, the gauge group S and the isotropy group H will align themselves such that $S \subset H$, so the gauge symmetry and supersymmetry are not broken. More specifically one may choose the embedding $S = SU(3)_C \times U(1)_{em} \subset SU(4) \subset H = SU(4) \times U(1)^4$, with the standard fractional electric charges for quarks

and the charged lepton being regarded as the fourth color. This GSGN σ-model, with Kahler manifold $E_7/SU(4) \times U(1)^4$ and gauge group $SU(3)_C \times U(1)_{em}$, is free of ABJ-anomaly. It consists of massless $SU(3)_C \times U(1)$ gauge bosons and gauge fermions, and massless complex scalar bosons and σ-fermions with representation content $\Gamma + \underline{4}^*$.

On the other extreme, one may choose the gauge group $S = G = E_7$. Then supersymmetry is inevitably broken while the gauge group E_7 breaks into $SU(4) \times U(1)^4$. All the scalar bosons are absorbed by the gauge fields associated with the broken generators of E_7 and the σ-fermions combines with appropriate gauge fermions to form massive four-component fermions with representation content $\Gamma + \underline{4}^*$. The remaining, massless particles are the $SU(4) \times U(1)^4$ gauge bosons and gauge fermions, and the left over two-component gauge fermions of the representation content $\Gamma + \underline{4}^*$. The massless fermions give a nonvanishing contribution to ABJ-anomaly with respect to the $U(1)^4$ gauge group.

In between the two extremes, the one with which we are most concerned is $S = SU(3)_C \times SU(2)_L \times U(1)$. The alignment between S and H is such that $S \cap H = SU(3)_C \times U(1)_{em} \times U(1)_Z$, where the $U(1)_Z$ is the $U(1)$ corresponding to the Z-boson of Weinberg-Salam-Glashaw model. Supersymmetry is broken while the gauge group $SU(3)_C \times SU(2)_L \times U(1)$ breaks down to $SU(3)_C \times U(1)_{em} \times U(1)_Z$. The massive particles include two vector bosons, namely the usual $W_\mu^\pm$, and one fermion. The massless sector consists of $SU(3)_C \times U(1)_{em} \times U(1)_Z$ gauge bosons and gauge fermions, one left-over gauge fermion with charges identical to that of a left-handed electron, and scalar bosons and σ-fermions with representation content $\Gamma + \underline{3}^*$. The model has a nonvanishing ABJ-anomaly with respect to the unbroken $U(1)_Z$ gauge group.

(8) It is clear from the above analysis that it is difficult to avoid ABJ-anomaly in GSGN σ-models. This is partly a reflection of the restrictiveness of the models. The effect of ABJ-anomaly on GSGN σ-models is still to be investigated. In conventional unified gauge theories, ABJ-anomaly could cause incompatibility among the requirements of gauge invariance, unitarity, and renormalizability; and is therefore absolutely intolerable.[14] Nonlinear σ-models are well known to be nonrenormalizable; hopefully the abdication of renormalizability, together with the additional structures of GSGN σ-models, would soften the effect of ABJ-anomaly, when present. A reasonable effect we would expect is that the corresponding gauge symmetry be broken. Thus, for example, the $U(1)_Z$ gauge group mentioned above may be broken, and the Z-boson may derive a mass.

(9) Another interesting effect is that of radiative corrections to the masses of massless scalar bosons and fermions. These corrections have to be calculated before any precocious attempt to compare the models with phenomenology in a detailed manner.

(10) The GSGN σ-model based on the Kahler manifold $E_7/SU(4) \times U(1)^4$ runs into difficulty with the goal of grand unification of strong, weak, and electromagnetic interactions. If we were to gauge, say, an SU(5) group rather than $SU(3)_C \times SU(2)_L \times U(1)$, the resulting unbroken gauge group would be $SU(4) \times U(1)$. And the $W_\mu^\pm$ bosons would be of the same mass scale as the baryon-number-changing gauge boson Y_μ.

One conceivable way to incorporate the Georgi-Glashow SU(5) grand unified theory of strong, weak, and electromagnetic interactions is to return to the Kahler manifold $E_7/SU(5) \times U(1)^3$, and cancel the ABJ-anomaly of the isotropy representation Γ, discussed in (5), by introducing an extra chiral superfield with, say, a $\underline{5}^*$ representation content with respect to the SU(5), and proper U(1) charges. For this minimally enlarged system one can gauge at least the SU(5) subgroup of E_7 without breaking the supersymmetry and the SU(5) gauge group iteself. Alternatively, it may even be possible to gauge a bigger subgroup of E_7 so that the supersymmetry is broken and the unbroken gauge group is SU(5) multiplying a torus; both cases are free of ABJ-anomaly with respect to the unbroken gauge groups. Extended GSGN σ-models of this type are under further study.

ACKNOWLEDGEMENTS

This work was supported by the Department of Energy, contract number DE-AC03-76SF00515. I am grateful to H. Samelson for hours of patient discussions and illuminating explanations. I would also like to thank S. S. Chern for a discussion, M. Peskin and H. Quinn for reading the manuscript, and M. Soldate for help in gammar.

REFERENCES

1. For a recent review, see H. Georgi, Harvard University preprint HUTP-82/A049.
2. For a more detailed discussion of this point, see S. J. Brodsky and S. D. Drell, Phys. Rev. D 22, 2236 (1980).
3. G. 't Hooft, in Recent Developments in Gauge Theories, Proceedings of the NATO Advanced Study Institute, Cargese, 1979, edited by G. 't Hooft et al. (Plenum, New York, 1980), p. 125.
4. W. A. Bardeen and V. Visnjic, Nucl. Phys. *B*194, 422 (1982); B. de Wit and D. Z. Freedman, Phys. Rev. Lett. 35, 827 (1975).
5. C. L. Ong, Phys. Rev. D 27, 911 (1983).
6. J. Bagger and E. Witten, Phys. Lett. 118*B*, 103 (1982).

7. W. Buchmuller, R. D. Peccei and T. Yanagida, Max Planck Institute Report MPI-PAE/PTh 80/82, call them quasi Nambu-Goldstone fermions. See also A. J. Buras and W. Slominski, MPI-PAE/PTh 5/83.
8. B. Zumino, Phys. Lett. 87*B*, 203 (1979); L. Alvarez-Gaume and D. Z. Freedman, Commun. Math. Phys. 80, 443 (1981).
9. S. S. Chern, Complex Manifolds Without Potential Theory (Springer-Verlag, New York, 1979); R. O. Wells, Differential Analysis on Complex Manifolds (Springer-Verlag, New York, 1980); K. Yano, Differential Geometry on Complex and Almost Complex Spaces (The Macmillan Co., New York, 1965).
10. A. Borel, Proceedings of the National Academy of Sciences 40, 1147 (1954).
11. A. Borel and F. Hirzebruch, Am. J. Math. 80, 458 (1958).
12. R. Gilmore, Lie Groups, Lie Algebras, and Some of Their Applications (John Wiley and Sons, New York, 1974), p. 292.
13. S. L. Adler, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60*A*, 47 (1969).
14. D. J. Gross and R. Jackiw, Phys. Rev. D 6, 477 (1972).