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OPERATOR PRODUCT EXPANSIONS IN THE MASSLESS SCHWINGER MODEL*

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ABSTRACT

The known operator solution of the massless Schwinger model is used to calculate exactly, in three operator product expansions, the coefficient functions of the first few operators of low dimension which contribute when vacuum matrix elements are to be taken. A comparison of the results provides a test of the procedure used by Shifman, Vainshtein, and Zakharov in their study of QCD. It is found that the shift in vacua does not affect the calculation of coefficient functions. The vacuum insertion approximation yields somewhat misleading estimates of vacuum expectation values of some composite operators; however, the standard method used to estimate the errors of vacuum insertion indicates that the approximation is unreliable in this model.

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1. INTRODUCTION

The operator product expansion (OPE) [1] has proven to be a useful theoretical tool in a wide variety of circumstances. Of particular note, Shifman, Vainshtein, and Zakharov (SVZ) [2] initiated a semi-phenomenological program of the study of resonance physics in charmonium spectroscopy and e^+e^- annihilation at lower Q^2, while others subsequently have applied their methodology to additional areas of hadronic structure [3].

Only matters which can be related to the vacuum expectation value (vev) of an operator product are potentially amenable to treatment by the SVZ prescription. Their analysis employs the OPE in its conventional form including terms of twist-4, and often twist-6. It is assumed that nonperturbative effects do not spoil the factorization exhibited by the OPE until terms of yet higher twist, or modify the coefficient functions of the low twist operators from the perturbative results. However, the nonperturbative features of QCD are presumed to give nonzero vev's to the local, perturbatively renormalized and normal-ordered operators in the expansion.

Typically, the operators which play a role are $\mathbf{1}$, $:\bar{\phi}\phi:$, $:F^2:$, $:\bar{\phi}\gamma^\mu\lambda^a_c\phi\,\bar{\phi}\gamma_\mu\lambda^a_c\phi:$, and $:\bar{\phi}\lambda^a_c\gamma^\mu\gamma^5\phi\,\bar{\phi}\lambda^a_c\gamma_\mu\gamma^5\phi:$, although other operators are important occasionally. [Hereafter, $|\Omega\rangle$ will denote the physical vacuum, as opposed to the perturbative vacuum, $|0\rangle$.] The quantity, $\langle\Omega|:\bar{\phi}\phi:|\Omega\rangle$, can be estimated from results in current algebra, provided that the quark masses are known. In the view of SVZ it would be ideal if the other nontrivial operator vev's could be estimated in terms of this one. In practice, $\langle\Omega|:F^2:|\Omega\rangle$ is extracted from the analysis of charmonium spectroscopy, and the vev's of the other fermionic operators are evaluated in terms of $\langle\Omega|:\bar{\phi}\phi:|\Omega\rangle$ by retaining the vacuum as the sole intermediate state, i.e. (see [2]):

$$\langle\Omega|:\bar{\phi}\Gamma_1\phi\,\bar{\phi}\Gamma_2\phi:|\Omega\rangle = \frac{1}{N^2}[Tr\,\Gamma_1\;Tr\,\Gamma_2 - Tr\,(\Gamma_1\Gamma_2)]\;\langle\Omega|:\bar{\phi}\phi:|\Omega\rangle^2 \quad . \tag{1.1}$$

In this manner a number of fundamental, nonperturbative quantities can be studied in QCD.

It is of interest to establish, though, whether or not the effect of nonperturbative elements on the structure of the OPE is in fact as assumed. The first treatment [4] of this matter was in a four-dimensional scalar field theory. The present paper addresses this issue in the context of massless QED in two dimensions [5].

The massless Schwinger model is an appropriate testing ground for a number of reasons. It is exactly soluble in Lorentz gauge with an explicit operator solution [6]. Cluster decomposition is violated when working in the perturbative vacuum, and one must shift vacua in order to restore it; the concomitant change in the operator solution on the physical subspace is known as well [6]. The failure of cluster decomposition reflects the confinement of electric charge in this model; similarly, the confinement of color electric charge in QCD is known to be related to the breakdown of cluster [7]. Schwinger's original solution for the Green's functions could be found, presumably, though the summation of the perturbative expansion to all orders. It is possible, then, given the complete operator solution, to compare the exact expression for the OPE of any two operators obtained through a full calculation on the physical subspace about the perturbative vacuum, with that obtained about the true vacuum of the confined gauge theory.

Of course, the only divergences which appear can be removed by normal ordering. Furthermore, the operators requiring subtractions do not acquire vacuum expectation values. Because of this, the issue of whether or not the subtractions required to normal order or renormalize perturbatively an operator are modified by non-trivial structure in the vacuum cannot be addressed here. The model is not asymptotically free (see however [8]); consequently, the problem of the convergence of the perturbative expansion of QCD, which poses an additional difficulty for the SVZ prescription, cannot be examined here realistically either. These matters, among others, have been studied recently in a different two–dimensional model [9].

There are just two topics which will be addressed here. The first is the question of whether or not the SVZ prescription correctly arrives at the coefficient functions in operator product expansions. Secondly, the calculation of physical vev's of fermion operators using the vacuum insertion procedure [see Eq. (1.1)] will be examined.

Section 2 contains a review of the operator solution both on the indefinite metric 'Hilbert' space and physical subspace following [6]. A number of gauge invariant composite operators are constructed in Section 3 on the physical subspace. Then, in Section 4 three different OPE's are computed using the operator solution, assuming that the final result is to be sandwiched between the vacuum state. Comparison are made among the various expansions before conclusions are given.

2. REVIEW OF THE OPERATOR SOLUTION

For completeness the operator solution of the Schwinger model will be reviewed in this section following [6]. The conventions and notation of [6] and [10] will be adhered to as closely as possible.

Schwinger originally used functional methods to arrive at expressions for Green's functions in Lorentz gauge. The standard perturbative expansion of the Green's functions could be generated from his initial formulation by usual techniques. His solution then must correspond to the summation of the perturbative expansion. The vacuum state in this treatment will be called therefore the 'perturbative' vacuum, for want of a better terminology. As will be reviewed below some Green's functions calculated in the perturbative vacuum fail to cluster, and it will be necessary to shift vacua to the 'physical' vacuum.

Of particular interest are Green's functions of fermion fields alone. ([11] contains an explicit calculation of them.) A fermion Green's function is expressible in terms of exponentials of differences of free massive and massless scalar Green's functions, multiplying the appropriate Green's function of a free massless fermion. Lowenstein and Swieca [6] identified an 'operator fit' to the 'renormalized' Wightman functions. The renormalization was a finite rescaling of the unrenormalized, interacting fermion field, $\hat{\phi}$, to form the field, ϕ. It will prove to be convenient here to work with the more standard field, $\hat{\phi}$.

The operator fit of [6] which reproduces the unrenormalized fermion Wightman functions is

$$\begin{aligned}\hat{\phi}(x) = exp\left[\frac{1}{2}B(0)\right]\phi(x) = exp\left[\frac{1}{2}B(0)\right]\ exp\left\{i\sqrt{\pi}\ \gamma^5\left[\tilde{\eta}^+(x)+\tilde{\Sigma}^+(x)\right]\right\}\ \psi(x) \\ \times\ exp\left\{i\sqrt{\pi}\ \gamma^5\left[\tilde{\eta}^-(x)+\tilde{\Sigma}^-(x)\right]\right\}\ ,\end{aligned} \tag{2.1}$$

where ψ is a free zero mass fermion field; $\tilde{\Sigma} = \tilde{\Sigma}^+ + \tilde{\Sigma}^-$ is a free scalar field with mass $m = e/\sqrt{\pi}$:

$$\langle 0|\,\tilde{\Sigma}(x)\,\tilde{\Sigma}(y)|0\rangle = -i\Delta^-(x-y)\ , \tag{2.2}$$

$$\tilde{\Sigma}^-\,|0\rangle = 0\ , \tag{2.3}$$

see [12] for the evaluation and asymptotics of $\Delta^-(x)$; and $\tilde{\eta} = \tilde{\eta}^+ + \tilde{\eta}^-$ is a zero mass field quantized with indefinite metric:

$$\langle 0|\,\tilde{\eta}(x)\,\tilde{\eta}(y)|0\rangle = iD^-(x-y)\ . \tag{2.4}$$

$$\begin{aligned}D^-(x) &= \frac{i}{2\pi}\int d^2p\ \delta(p^2)\ \theta(p^o)\left[e^{-ip\cdot x} - \theta(\kappa - p^o)\right] \\ &= \frac{1}{4\pi i}\ \ell n\ (-\mu^2x^2 - i\epsilon x^o)\end{aligned} \tag{2.5}$$

is the infrared regularized two-point function of $\tilde{\eta}$, with $\mu = \kappa\ exp\,[-\Gamma'(1)]$, [10].

$$B(x) = i\pi\,[\Delta^-(x) - D^-(x)] \tag{2.6}$$

$$B(z) \underset{\substack{z\to 0\\ z^2\neq 0}}{\longrightarrow} \frac{1}{4}\left[\ell n\left(\frac{m^2}{4\mu^2}\right) + 2\gamma_0 - \frac{m^2 z^2}{4}\ \ell n\ \left(\frac{m^2|z^2|}{4}\right)\right] + O(z^2)\ . \tag{2.7}$$

Any physical quantity must be independent of μ, the arbitrary infrared cutoff.

From the Dirac equation,

$$A^\mu(x) = -\ \frac{1}{m}\ \epsilon^{\mu\nu}\ \partial_\nu\,[\tilde{\Sigma}(x) + \tilde{\eta}\,(x)]\ \ . \tag{2.8}$$

By substitution from (2.1),

$$\begin{aligned}\hat{\phi}\,(x)\,\hat{\phi}^\dagger\,(y) =& exp\,[B(0) - \gamma_x^5\,\gamma_y^5\,B(x-y)]\\ &:exp\,[i\,\sqrt{\pi}\,\{\gamma_x^5[\tilde{\eta}\,(x) + \tilde{\Sigma}\,(x)] - \gamma_y^5[\tilde{\eta}\,(y) + \tilde{\Sigma}\,(y)]\}] : \psi(x)\ \psi^\dagger(y)\end{aligned} \tag{2.9}$$

The properly defined current is

$$\begin{aligned}j^\mu(x) =& -\lim_{\substack{\epsilon\to 0\\ \epsilon^2\neq 0}}\ Tr\Big[\gamma^o\ \gamma^\mu\Big\{\hat{\phi}\,(x)\,\hat{\phi}^\dagger\,(x+\epsilon) - \langle 0|\,\hat{\phi}\,(x)\,\hat{\phi}^\dagger\,(x+\epsilon)|0\rangle\\ &\times[1 - ie\ \epsilon^\nu A_\nu(x)]\Big\}\Big]\ =\ j_L^\mu(x) - \frac{1}{\sqrt{\pi}}\ \epsilon^{\mu\nu}\ \partial_\nu\ \tilde{\Sigma}(x)\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}j_L^\mu(x) \equiv& :\bar{\psi}\,(x)\ \gamma^\mu\ \psi(x) : -\frac{1}{\sqrt{\pi}}\ \epsilon^{\mu\nu}\ \partial_\nu\ \tilde{\eta}\,(x)\\ =& \ j_f^\mu(x) - \frac{1}{\sqrt{\pi}}\ \epsilon^{\mu\nu}\ \partial_\nu\ \tilde{\eta}\,(x)\ \ .\end{aligned} \tag{2.11}$$

$j_L^\mu(x)$ creates zero norm states from the vacuum, satisfies the wave equation, and Maxwell's equations are violated by a term proportional to it,

$$\partial_\nu F^{\mu\nu}(x) = -e[j^\mu(x) - j_L^\mu(x)]\ \ . \tag{2.12}$$

It is appropriate, then, in analogy with the Gupta-Bleuler formalism, to specify the physical subspace, $\mathcal{H}_{\rm phys}$, by the condition

$$j_L^{\mu -}|\Psi\rangle = 0 \quad , \quad \forall \ |\Psi\rangle \in \mathcal{H}_{\rm phys} \ . \tag{2.13}$$

The task of extracting the Wightman function on the physical subspace from Schwinger's original solution is formidable. It is more convenient to construct operators which identically satisfy Maxwell's equations, and then to construct their Wightman functions. Lowenstein and Swieca found a one parameter family of such solutions. For the purposes of this article it is sufficient to use only the solution specified by $\alpha = \sqrt{\pi}$. The interacting field is

$$\phi^{\sqrt{\pi}}(x) = exp\left[\frac{i\pi}{2}(Q+\tilde{Q}) + i\chi^{+}(x)\right]\psi(x) \ \exp\left[i\chi^{-}(x)\right] \tag{2.14}$$

with

$$\chi^{\pm}(x) = \sqrt{\pi}\, j^{\pm}(x) + \gamma^5\left[\sqrt{\pi}\,\tilde{j}^{\pm}(x) + \sqrt{\pi}\,\tilde{\Sigma}^{\pm}(x)\right] \tag{2.15}$$

where $j, (\tilde{j})$, is the potential (pseudopotential), of the free current with an infrared cutoff ([6] and [10]), e.g.

$$\frac{1}{\sqrt{\pi}}\ \partial^{\mu} j(x) = j_f^{\mu}(x) \ . \tag{2.16}$$

The motivation for (2.14) is the following. The physical subspace can be constructed by applying Wightman polynomials of $F^{\mu\nu}(x)$, $j_L^{\mu}(x)$, and $exp[i\sqrt{\pi}\ \ \eta^{+}(x)]\phi(x)$ $exp[i\sqrt{\pi}\ \ \eta^{-}(x)]$, with $\epsilon^{\mu\nu}\partial_{\nu}\tilde{\eta}(x) = \partial^{\mu}\eta(x)$, as these commute with $j_L^{\rho}(x)$. In the quotient space $\mathcal{H}_{\rm phys}/\mathcal{H}_0$, where $\mathcal{H}_0$ is the space of all null vectors in $\mathcal{H}_{\rm phys}, j_L^{\mu}$ vanishes or $\eta(x) = j(x)$ and $\tilde{\eta}(x) = \tilde{j}(x)$. The expression for $\phi^{\sqrt{\pi}}(x)$ follows accordingly, up to the Klein transformation, the phase $exp[(i\pi/2)(Q+\tilde{Q})]$. The introduction of it will prove to be convenient a little later. The Dirac equation for $\phi^{\sqrt{\pi}}$ gives

$$A_{\mu}^{\sqrt{\pi}}(x) = -\frac{1}{m}\ \epsilon_{\mu\nu}\ \partial^{\nu}\ \tilde{\Sigma}(x) \ . \tag{2.17}$$

The Wightman functions of $\phi^{\sqrt{\pi}}$ are

$$\langle 0|\phi^{\sqrt{\pi}}(x_1)\ldots\phi^{\sqrt{\pi}}(x_n)\phi^{\dagger\sqrt{\pi}}(y_1)\ldots\phi^{\dagger\sqrt{\pi}}(y_n)|0\rangle = exp[iG(x,y)]\ W^{(n)}_{\sqrt{\pi}}(x,y) \quad , \tag{2.18}$$

where

$$\begin{aligned} G(x,y) =\pi\Bigg\{ & \sum_{j<k}\Big[\gamma^5_{x_j}\gamma^5_{x_k}\Delta^-(x_j-x_k)+\gamma^5_{y_j}\gamma^5_{y_k}\Delta^-(y_j-y_k)\Big] \\ & -\sum_{j,k}\gamma^5_{x_j}\gamma^5_{y_k}\Delta^-(x_j-y_k)-\frac{1}{2}\sum_{k=1}^{n}\ k\Big(\gamma^5_{x_k}-\gamma^5_{y_{n+1-k}}\Big)\Bigg\} \end{aligned} \tag{2.19}$$

and $W^{(n)}_{\sqrt{\pi}}$ are the Wightman functions of the Thirring model with $\alpha=\beta=\sqrt{\pi}$, in the notation of [10]. $W^{(n)}_{\sqrt{\pi}}$ are in fact just constants (see [10]):

$$exp\Bigg[-\frac{i\pi}{2}\sum_{k=1}^{n}\ k\Big(\gamma^5_{x_k}-\gamma^5_{y_{n+1-k}}\Big)\Bigg]W^{(n)}_{\sqrt{\pi}}(x,y)=\Big(\frac{\mu}{2\pi}\Big)^n \quad . \tag{2.20}$$

The Klein transformation cancelled certain phases in $W_{\sqrt{\pi}}$.

According to Eqs. (2.18)-(2.20) one can write

$$\phi^{\sqrt{\pi}}(x)=exp\Big[i\sqrt{\pi}\ \gamma^5\tilde{\Sigma}^+(x)\Big]\Big(\frac{\mu}{2\pi}\Big)^{1/2}\sigma(x)exp\Big[i\sqrt{\pi}\ \gamma^5\tilde{\Sigma}^-(x)\Big] \quad , \tag{2.21}$$

where

$$\begin{aligned} \sigma(x)=\sqrt{\frac{2\pi}{\mu}}\ & exp\Big\{\frac{i\pi}{2}(Q+\tilde{Q})+i\sqrt{\pi}\Big[j^+(x)+\gamma^5\tilde{j}^+(x)\Big]\Big\}\psi(x) \\ & \times exp\Big\{i\sqrt{\pi}\Big[j^-(x)+\gamma^5\tilde{j}^-(x)\Big]\Big\} \end{aligned} \tag{2.22}$$

must satisfy

$$\begin{aligned} & \langle 0|\sigma_1(x_1)\ldots\sigma_1(x_m)\sigma_2(x_{m+1})\ldots\sigma_2(x_n)\sigma_1^\dagger(y_1)\ldots \\ & \qquad \sigma_1^\dagger(y_m)\sigma_2^\dagger(y_{m+1})\ldots\sigma_2^\dagger(y_n)|0\rangle = 1 \quad , \end{aligned} \tag{2.23}$$

$$[\sigma(x),\sigma(y)] \ = \ [\sigma(x),\sigma^\dagger(y)] \ = \ 0 \quad , \tag{2.24}$$

and all other Wightman functions of σ vanish. Effectively $\sigma(x)$ is independent of x. Equation (2.23) demonstrates the failure of cluster decomposition in the perturbative vacuum. It can be restored by shifting to the physical vacuum

$$|\theta_1\theta_2\rangle = \frac{1}{2\pi} \sum_{n_1,n_2=-\infty}^{\infty} e^{i(n_1\theta_1+n_2\theta_2)}(\sigma_1)^{n_1}(\sigma_2)^{n_2}|0\rangle \ . \tag{2.25}$$

Here $\sigma_i^{-1} = \sigma_i^\dagger$. On the perturbative vacuum $\sigma_1\sigma_1^\dagger = \sigma_2\sigma_2^\dagger = 1$; while on the physical vacuum more generally σ_i is a c-number, $\sigma_i = \exp(-i\theta_i)$. $\sigma(x)$ was the natural candidate for an operator which would create a charged state. The confinement of electric charge appears, in the physical subspace, as the transformation of $\sigma(x)$ from an operator to a c-number on the physical vacuum.

In the next section the primary concern will be the construction of gauge-invariant composite operators. For that purpose it is convenient to define, in the physical subspace, the bilocal gauge-invariant operator corresponding to the formal expression

$$\hat{T}(x,y) \simeq \hat{\phi}(x) exp\left[ie\int_x^y A^\mu(t)dt_\mu\right]\hat{\phi}^\dagger(y) \ . \tag{2.26}$$

The nonformal definition, analogous to Eq. (2.9), is

$$\hat{T}(x,y) = exp\left[B(0) - \gamma_x^5\gamma_y^5 B(x-y)\right] exp\left[iK^+(x,y)\right]\psi(x)\psi^\dagger(y) exp\left[iK^-(x,y)\right] \tag{2.27}$$

where

$$K(x,y) = e\int_x^y A_\mu(t)dt^\mu + \chi(x) - \chi(y) \tag{2.28}$$

and $\chi(x)$ is given by Eq. (2.15). $\hat{T}(x,y)$ differs from $T(x,y)$ as defined in [6] by the factors involving B. Some rearrangement gives

$$\begin{aligned} \hat{T}(x,y) = \hat{M}(x,y)\, exp\left[-i\sqrt{\pi}\int_x^y \epsilon^{\mu\nu}\partial_\nu\tilde{\Sigma}^+(t)dt_\mu\right]\phi^{\sqrt{\pi}}(x)\ \phi^{\dagger\sqrt{\pi}}(y) \\ \times\, exp\left[-i\sqrt{\pi}\int_x^y \epsilon^{\mu\nu}\partial_\nu\tilde{\Sigma}^-(t)dt_\mu\right] \end{aligned} \tag{2.29}$$

where Eq. (2.14) has been used:

$$\hat{M}(x,y) = exp\,[B(0) - \gamma_x^5\gamma_y^5 B(x-y)]\;\; exp\Big\{-i\pi\Big[(1+\gamma_x^5\gamma_y^5)D^-(x-y) - \gamma_x^5\gamma_y^5\Delta^-(x-y) + (\gamma_x^5+\gamma_y^5)\,\tilde{D}^-\,(x-y) - \frac{1}{2}\,(\gamma_x^5-\gamma_y^5)\Big]\Big\}\;. \tag{2.30}$$

Substituting Eq. (2.21) into Eq. (2.29) and making further simplifications gives

$$\hat{T}(x,y) = \frac{\mu}{2\pi}\,\sigma\,\sigma^*\;\hat{N}(x,y)\; : exp\Big\{i\sqrt{\pi}\Big[-\int_x^y \epsilon^{\mu\nu}\partial_\nu\,\tilde{\Sigma}(t)dt_\mu + \gamma_x^5\,\tilde{\Sigma}(x) - \gamma_y^5\,\tilde{\Sigma}(y)\Big]\Big\} : \tag{2.31}$$

where

$$\hat{N}(x,y) = exp\,[B(0) - \gamma_x^5\gamma_y^5 B(x-y)]\;\; exp\Big\{-i\pi\Big[(1+\gamma_x^5\gamma_y^5)D^-(x-y) + (\gamma_x^5+\gamma_y^5)\,\tilde{D}^-\,(x-y) - \frac{1}{2}\,(\gamma_x^5-\gamma_y^5)\Big]\Big\}\;. \tag{2.32}$$

In this form, the shift of vacua can easily be taken into account.

In summary, Eqs. (2.17) and (2.31) provide a succinct statement of the operator solution on the physical subspace. The shift from the perturbative to the physical vacuum is accomplished by changing σ in eq. (2.31)from an operator to a c-number.

3. COMPOSITE OPERATORS

The principle used to construct composite operators will be Zimmerman's definition of normal product operators applied beyond the context of perturbative theory. A Green's function with the normal product composite operator $N[O(x)]$ inserted on it is the finite part of the same Green's function with the operator insertion of $O(x)$ [13]. The operator solution is written in terms of free fields. Often the normal product of two operators will be simply the Wick product of the operators as expressed in terms of free fields. However, chirality–changing operators are the exponentials of free fields and a little care must be exercised in the definition of normal products of such operators.

The composite operators which are easiest to treat are those which are constructed out of $F^{\mu\nu}$ alone. Those of interest are ${}^*F$ and F^2. ${}^*F$ requires no subtractions and is simply defined as

$$
{}^*F(x) = 2\epsilon^{\mu\nu}\partial_\mu A_\nu^{\sqrt{\pi}}(x) = 2m\,\tilde{\Sigma}(x) \tag{3.1}
$$

$F^2(x)$ does require a c-number subtraction in order that all Green's functions with it as an insertion are finite.

$$
N[F^2(x)] = -2m^2 : \tilde{\Sigma}^2(x) : \quad . \tag{3.2}
$$

Gauge-invariant composite operators with fermion fields are readily constructed from $\hat{T}(x,y)$ on the physical subspace, using either eq. (2.27) or (2.31) as appropriate. Consider first the chirality-preserving operators. From Eq. (2.31)

$$
\begin{aligned}
Tr\Big[\gamma^o\ \gamma^\mu\ \hat{T}(x,y)\Big] &= \frac{\mu}{2\pi}\ exp\,[B(0) - B(x-y)] \\
Tr\Big[\gamma^o\ \gamma^\mu\ &exp\Big\{-i2\pi[D^-(x-y) + \gamma^5\tilde{D}^-(x-y)]\Big\} \\
&:exp\left\{i\sqrt{\pi}\left[-\int_x^y \epsilon^{\nu\rho}\partial_\rho\tilde{\Sigma}(t)dt_\nu + \gamma^5(\tilde{\Sigma}(x) - \tilde{\Sigma}(y))\right]\right\}:\Big] \ .
\end{aligned} \tag{3.3}
$$

Using results from [10],

$$
exp\Big\{-i2\pi\Big[D^-(\xi) + \gamma^5\tilde{D}^-(\xi)\Big]\Big\} = -\frac{i}{\mu}\ \frac{\not{\xi}\gamma^o}{\xi^2 - i\epsilon\xi^o} \quad . \tag{3.4}
$$

As in [6] one can define the current on the physical subspace about the shifted vacuum to be

$$
\begin{aligned}
J^\mu(x) &= -\lim_{\substack{\epsilon\to 0\\ \epsilon^2\neq 0}} Tr\,\{\gamma^o\gamma^\mu\,[\hat{T}(x+\epsilon,x) - \langle 0|\,\hat{T}(x+\epsilon,x)|0\rangle]\} \\
&= -\frac{1}{\sqrt{\pi}}\ \epsilon^{\mu\nu}\ \partial_\nu\ \tilde{\Sigma}(x) \ .
\end{aligned} \tag{3.5}
$$

Note that the unphysical, longitudinal part of the current, j_L^μ, has disappeared. Using eq. (2.27) the vector current has the same form on the physical subspace about the unshifted vacuum as well.

Equations. (3.2) and (3.5) indicate that there are curious relations among operators in the exact solution. Consider for example

$$N[J^\mu(x)J_\mu(x)] = -\frac{1}{\pi} : \partial^\mu \, \tilde{\Sigma}(x) \, \partial_\mu \, \tilde{\Sigma}(x) : = \frac{1}{2\pi} \, N[F^2(x)] \tag{3.6}$$

where a total derivative operator has been dropped, and the equation of motion for $\tilde{\Sigma}$ has been used. Because of such identites there is some essentially inconsequential arbitrariness in the expressions of OPE's.

Digressing a bit, there is a natural definition of the stress-energy tensor in the physical subspace. Written in terms of the interacting fields, the symmetric stress energy tensor in the indefinite metric Hilbert space is

$$t^{\mu\nu} = t^{\mu\nu}_{e\gamma} + t^{\mu\nu}_{\gamma} \tag{3.7}$$

where

$$t^{\mu\nu}_{\gamma} = -F^{\mu\lambda} \, F^{\nu}_{\lambda} + \frac{g^{\mu\nu}}{4} \, F^2 \quad , \tag{3.8}$$

$$t^{\mu\nu}_{e\gamma} = \frac{i}{4}\left\{\hat{\phi}^\dagger \, \gamma^o\gamma^\mu(\vec{\partial}^\nu - ieA^\nu)\,\hat{\phi} - \left[(\vec{\partial}^\nu + ieA^\nu)\,\hat{\phi}^\dagger\right]\gamma^o\gamma^\mu\,\hat{\phi}\right\} + (\mu \leftrightarrow \nu) \quad . \tag{3.9}$$

In two dimensions

$$t^{\mu\nu}_{\gamma} = -\frac{1}{4} \, g^{\mu\nu} \, F^2 \quad . \tag{3.10}$$

Based on the correspondence seen in Eq. (2.26),

$$\left[\hat{\phi}^\dagger \, \gamma^o\gamma^\mu(\partial^\nu - ieA^\nu)\,\hat{\phi}\right](x) \Rightarrow - \lim_{\substack{y\to x \\ (x-y)^2 \neq 0}} \partial^\nu_x \, Tr\left[\gamma^o\gamma^\mu \, \hat{T}(x,y)\right] \quad , \tag{3.11}$$

where the symbol, $\Rightarrow$, denotes equality on the physical subspace. The normal-ordered stress energy tensor on the physical subspace can then be written as

$$T^{\mu\nu}(x) = \frac{1}{2}\, m^2 g^{\mu\nu} : \tilde{\Sigma}^2(x) : -\frac{i}{4} \lim_{\substack{y \to x \\ (x-y)^2 \neq 0}} \Big[(\partial_x^\nu - \partial_y^\nu) \times Tr\{\gamma^o \gamma^\mu [\hat{T}(x,y) - \langle 0|\, \hat{T}(x,y)|0\rangle]\} + (\nu \leftrightarrow \mu) \Big] \ . \tag{3.12}$$

Define $\xi = x - y$. On the physical vacuum,

$$\lim_{\substack{\xi \to 0 \\ \xi^2 \neq 0}} \partial_x^\nu\, Tr\{\gamma^o \gamma^\mu [\hat{T}(x,y) - \langle 0|\, \hat{T}(x,y)|0\rangle]\} = \lim_{\substack{\xi \to 0 \\ \xi^2 \neq 0}} \frac{exp[B(0) - B(\xi)]}{2\pi i} Tr\Bigg[\gamma^\mu \Big[\partial_x^\nu \Big(\frac{\not{\xi}}{\xi^2 - i\epsilon\xi^o} \Big) \Big] \Big\{ : expi\sqrt{\pi} \Big[\int_y^x \epsilon^{\rho\sigma} \partial_\sigma \tilde{\Sigma}(t) dt_\rho - \gamma^5 (\tilde{\Sigma}(x) - \tilde{\Sigma}(y)) \Big] : -1 \Big\} + \gamma^\mu \Big(\frac{\not{\xi}}{\xi^2 - i\epsilon\xi^o} \Big)\, i\sqrt{\pi} : \Big\{ \partial_x^\nu \Big[\int_y^x \epsilon^{\rho\sigma} \partial_\sigma \tilde{\Sigma}(t) dt_\rho - \gamma^5 (\tilde{\Sigma}(x) - \tilde{\Sigma}(y)) \Big] \Big\} \times exp\ i\sqrt{\pi} \Big[\int_y^x \epsilon^{\tau u} \partial_u \tilde{\Sigma}(t) dt_\tau - \gamma^5 (\tilde{\Sigma}(x) - \tilde{\Sigma}(y)) \Big] : \Bigg] \ . \tag{3.13}$$

Note that the nontrivial dependence on B dropped out after using Eqs. (2.7) and (3.7). The singularity of order $1/\xi^2$ has been removed by the subtraction of the vacuum expectation value. After frequent use of the identity

$$\epsilon_{\nu\rho}\, \epsilon^{\mu\sigma} = g_\nu^\sigma g_\rho^\mu - g_\nu^\mu g_\rho^\sigma \ , \tag{3.14}$$

one finds that the terms of order $1/\xi$ cancel, as do the direction-dependent terms proportional to $\xi^\mu \xi^\nu / \xi^2$. The result is

$$\lim_{\substack{\xi \to 0 \\ \xi^2 \neq 0}} \partial_x^\nu\, Tr\{\gamma^o \gamma^\mu [\hat{T}(x,y) - \langle 0|\, \hat{T}(x,y)|0\rangle]\} = i : \Big[(\partial^\mu \tilde{\Sigma})(\partial^\nu \tilde{\Sigma}) - \frac{1}{2}\, g^{\mu\nu} (\partial_\rho \tilde{\Sigma})(\partial^\rho \tilde{\Sigma}) \Big](x) : \ , \tag{3.15}$$

a direction-independent, covariant result. Finally,

$$T^{\mu\nu}(x) =: \left\{(\partial^{\mu}\tilde{\Sigma})(\partial^{\nu}\tilde{\Sigma}) - \frac{1}{2}\, g^{\mu\nu}\left[(\partial_{\rho}\tilde{\Sigma})(\partial^{\rho}\tilde{\Sigma}) - m^2\tilde{\Sigma}^2\right]\right\}(x) : \quad . \tag{3.16}$$

This is just the stress energy tensor of a free, massive scalar field, and exhibits explicitly the known physical spectrum [14].

The simplest chirality-changing operators are given by

$$D_{\pm}(x) = -\lim_{\substack{\xi\to 0\\ \xi^2\neq 0}} Tr\left[\gamma^o\left(\frac{1\pm\gamma^5}{2}\right)\hat{T}(x,y)\right] \quad , \tag{3.17}$$

or

$$D_{+}(x) = exp[2B(0)]\left(\frac{\mu}{2\pi}\right)\sigma_2\sigma_1^* : exp\left[i2\sqrt{\pi}\ \tilde{\Sigma}(x)\right] : \quad , \tag{3.18}$$

$$D_{-}(x) = exp[2B(0)]\left(\frac{\mu}{2\pi}\right)\sigma_1\sigma_2^* : exp\left[-i2\sqrt{\pi}\ \tilde{\Sigma}(x)\right] : \quad . \tag{3.19}$$

The difference between $\hat{T}(x,y)$ and $T(x,y)$, as defined in [6], resulted in the factor of $exp[2B(0)]$ which makes these operators independent of μ.

4. OPE'S AND VACUUM INSERTION

The three OPE's to be examined are those for a pair of vector currents, and two combinations of chirality–changing operators defined in the previous section. In each case, the OPE will be calculated on the physical subspace about both the perturbative and physical vacua. Then the vacuum insertion procedure will be criticized.

The clearest perspective to adopt when calculating the OPE of $O_1(x)O_2(0)$ is to imagine inserting $O_1(x)O_2(0)$ into a Green's function, taking $x \to 0$. Note that on the physical subspace all operators can be written in terms of σ and $\tilde{\Sigma}(x)$. In the end it is to be understood that the OPE's will be sandwiched between vacuum states. Consequently, only spin-0 operators will be kept, and all total derivative operators will be dropped.

Consider first the OPE of a pair of vector currents. From eq. (3.5) and the discussion thereafter

$$J^{\mu}(x) = -\frac{1}{\sqrt{\pi}}\, \epsilon^{\mu\nu}\partial_{\nu}\tilde{\Sigma}(x) \ . \tag{4.1}$$

The calculation on the physical subspace is quite easy and obviously independent of the choice of vacuum.

$$J^{\mu}(x)\ J^{\nu}(0) = \frac{i}{\pi}(-g^{\mu\nu}\Box + \partial^{\mu}\partial^{\nu})\Delta^{-}(x) + N[J^{\mu}(x)\ J^{\nu}(0)] \tag{4.2}$$

where

$$N[J^{\mu}(x)\ J^{\nu}(0)] = \frac{1}{\pi}\epsilon^{\mu\rho}\,\epsilon^{\nu\sigma}\ :\ \partial_{\rho}\tilde{\Sigma}(x)\,\partial_{\sigma}\tilde{\Sigma}(0)\ :\ . \tag{4.3}$$

The bilocal normal product can be expanded to give

$$N[J^{\mu}(x)\ J^{\nu}(0)] = N[J^{\mu}(x)\ J^{\nu}(0)] + \frac{1}{2}x_{\rho}x_{\sigma}\ \ N[(\partial^{\rho}\ \partial^{\sigma}\ J^{\mu}(0))J^{\nu}(0)] \tag{4.4}$$

The local normal products actually have the interpretation indicated by the notation. The spin-0 components of the operators on the right hand side of (4.4) can be projected out as usual.

$$\begin{aligned} N[J^{\mu}(x)\ J^{\nu}(0)] =& \frac{1}{2}g^{\mu\nu}\ \ N[J^{\alpha}(0)\ J_{\alpha}(0)] \\ & -\frac{m^2}{16}(3\ g^{\mu\nu}\ x^2 - 2\ x^{\mu}\ x^{\nu})N[J^{\alpha}(0)\Box\ \ J_{\alpha}(0)]\ +\ \ldots \end{aligned} \tag{4.5}$$

In fact, from the exact solution

$$N[J^{\alpha}(0)\Box\ \ J_{\alpha}(0)] = -m^2\ \ N[J^{\alpha}(0)\ J_{\alpha}(0)] \tag{4.6}$$

though this is not particularly important.

The OPE's of chirality-breaking operators contain a little more structure. The important point to realize when working about the perturbative vacuum is that there are no short–distance singularities in the product of two or more σ's, as eq. (2.23) indicates. Therefore, the only contributions to coefficient functions arise from reordering exponentials of the $\tilde{\Sigma}$ field when necessary to form finite operators. This fact guarantees that the OPE's calculated about the perturbative and physical vacua will agree completely. Explicitly,

$$
\begin{aligned}
D_+(x)\, D_-(0) &= \frac{m^2}{16\pi^2}\, e^{2\gamma_0}\, e^{-4\pi i \Delta^-(x)} \; : e^{i2\sqrt{\pi}[\tilde{\Sigma}(x)-\tilde{\Sigma}(0)]} : \\
&= \frac{m^2}{16\pi^2}\, e^{2\gamma_0}\; e^{-4\pi i \Delta^-(x)} \\
&\quad \left\{1 + x^2\Big[-i\frac{\sqrt{\pi}}{2} m^2 \tilde{\Sigma}(0) - \pi \; : \partial^\mu \; \tilde{\Sigma}(0)\, \partial_\mu \; \tilde{\Sigma}(0) : \Big] + \;\ldots\right\}
\end{aligned}
\tag{4.7}
$$

From eq. (3.5)

$$
: \partial^\mu \; \tilde{\Sigma}(0) \partial_\mu \; \tilde{\Sigma} \; : = -\pi \; N[J^\mu(x)\; J_\mu(0)] = 4\;\pi\; N[D_+(0)\;\; D_-(0)]\;, \tag{4.8}
$$

where the last equality follows by a Fierz transformation. Equation (4.7) becomes

$$
\begin{aligned}
D_+(x) D_-(0) &= -\frac{m^2 x^2}{4}\; e^{2\gamma_0}\; e^{-4\pi i \Delta^-(x)} \\
&\quad \left\{ -\frac{1}{4\pi^2 x^2} + \frac{i\sqrt{\pi}}{16\pi^2} m^* F(0) + N[D_+(0) D_-(0)] + \;\ldots \right\}
\end{aligned}
\tag{4.9}
$$

In this form one can easily check that the free–field expansion is recovered as m or e vanishes.

The final expansion examined is

$$
D_+(x)\; D_+(0) = (\sigma_1^*\; \sigma_2)^2\; \frac{m^2}{16\pi^2}\; e^{2\gamma_0} \;\; : e^{2\sqrt{\pi} i \tilde{\Sigma}(x)} : \; : e^{2\sqrt{\pi} i \tilde{\Sigma}(0)} : \tag{4.10}
$$

It is tempting to Wick–order the two exponentials; however, the product is not singular in the limit $x \to 0$ because $e^{4\pi i \Delta^-(x)}$ is not. The correct way to express the product of exponentials in terms of local normal–product operators is simply to Taylor expand the first exponential about $x = 0$. Note that the first two terms in the expansion vanish when multiplied by : $e^{2\sqrt{\pi} i \tilde{\Sigma}(0)}$: For example,

$$
\begin{aligned}
&2\pi i \ : e^{2\sqrt{\pi} i \tilde{\Sigma}(0)} \, \partial_\mu \ \tilde{\Sigma}(0) : : e^{2\sqrt{\pi} i \tilde{\Sigma}(0)} : \\
&= \lim_{x \to 0} \ \partial_\mu \left\{ e^{4\pi i \Delta^-(x)} \ : e^{2\sqrt{\pi} i [\tilde{\Sigma}(x) + \tilde{\Sigma}(0)]} : \right\} = 0 \quad .
\end{aligned}
\tag{4.11}
$$

This is a reflection of Fermi statistics; the fermion field has only two–components so that $\bar{\phi}(1 + \gamma_5)\phi\,\bar{\phi}(1 + \gamma_5)\phi(0)$ and $\bar{\phi}(1 + \gamma_5)\phi \ \partial_\mu \bar{\phi}(1 + \gamma_5)\phi(0)$ must vanish. The remaining terms in the expansion are simply interpreted.

$$
D_+(x) \ D_+(0) = \frac{1}{2} x_\mu x_\nu N[D_+(0)\partial^\mu \partial^\nu D_+(0)] + \ldots \tag{4.12}
$$

The shift in vacua did not affect the three OPE's examined above. The estimation of the vev of the operator $N[J_\mu(0)J^\mu(0)] = -4N[D_+(0)D_-(0)]$ by vacuum insertion is not precisely correct, however. In the exact solution this operator's vev is zero, while vacuum insertion estimates

$$
\langle \Omega | : \bar{\phi}\,\gamma_\mu \phi\,\bar{\phi}\,\gamma^\mu \phi : | \Omega \rangle \ \approx -\langle \Omega | \,\bar{\phi}\,\phi | \Omega \rangle^2 \tag{4.13}
$$

Following SVZ [2] one may try to estimate the corrections to the vacuum insertion procedure by calculating the contribution from a single particle intermediate state. For simplicity, consider the vacuum with $\theta_1 - \theta_2 = 0$. Including the correction,

$$
\langle \Omega | N[J_\mu J^\mu] | \Omega \rangle \approx -\langle \Omega | \,\bar{\phi}\,\phi | \Omega \rangle^2 \left[1 + 2(1 + e^{-2\gamma_0}) \int_0^\Lambda \frac{dp}{\sqrt{p^2 + m^2}} + \ldots \right], \tag{4.14}
$$

where a cutoff has been introduced to regulate the logarithmic ultraviolet divergence. SVZ encounter a similar divergence in the corresponding calculation in QCD and choose a cutoff of approximately 1 GeV, the scale at which precocious scaling sets in. The criterion in the case at hand is somewhat arbitrary, but one expects that $\frac{\Lambda}{m} \approx 1$.

Then, the correction is as large as the first term and approximation would not be expected to work.

5. CONCLUSION

The operator product expansions were unchanged by vacuum shift. Vacuum insertion is not particularly accurate; however, the procedure used by SVZ to estimate the errors would have indicated that the approximation could not be trusted in this instance. The underlying assumptions of the SVZ program prove to be justified in this model, though it may well be too simple a model to provide a serious testing ground.

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