SLAC-PUB-3048
January 1983
(I)

IONIZATION STATISTICS AND DIFFUSION: ANALYTICAL ESTIMATE OF THEIR CONTRIBUTION TO SPATIAL RESOLUTION OF DRIFT CHAMBERS*

Giora J. Tarnopolsky
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

The spatial resolution of a drift chamber often is the foremost design parameter. The calculation described here — a design tool — permits us to estimate the contributions of ionization statistics and diffusion to the spatial resolution when actually sampling the drift-pulse waveform. Useful formulae are derived for the cylindrical and jet-chamber cell geometries.

* Work supported by the Department of Energy, contract DE-AC03-76SF00515.

Presented at the 1983 Wirechamber Conference, Vienna, Austria, February 15-18, 1983

1. Introduction

The spatial resolution of a drift chamber often is the foremost design parameter. Measurements on a prototype cell are the best indicators of the resolution ultimately achievable in a given set up. However, these measurements demand considerable effort when many alternative designs are considered. Simulation programs [1] thus become design tools of choice. These programs consider many relevant effects such as the random ionization deposition, the drift and diffusion of primary and secondary ionization, the signal amplification and collection around the sense wire, and the electronic shaping and processing of the signal. This rather incomplete list of the elements included in the Monte Carlo programs indicates their size. As with any other Monte Carlo, it is desirable to verify the validity of these simulations by transparent methods and for a wide range of parameters' values.

The statistical laws governing ionization statistics and diffusion, namely, Poisson and Gaussian distributions, respectively, are amenable to analytic calculation. An analytic evaluation brings forth the relative magnitude of effects, permits rapid evaluation of a given configuration, and, most importantly, yields a result that, on average, correct Monte Carlo simulations must reproduce. My calculation takes account of the cell geometry, of ionization statistics, and of diffusion. The assumptions of the derivation are:

i) A sense wire collects ionization from a track segment of length a.

ii) The particle moves in a plane perpendicular to the sense wire. Its impact parameter is b.

iii) A minimum ionizing particle produces, on average, μ ionization clusters per unit length. The actual number of primary ionizations in the length a obeys Poisson statistics, the probability of having n such clusters being

$$\frac{e^{-\mu a}(\mu a)^n}{n!} \quad .$$

iv) Each cluster has a single electron.

v) The electric field is high enough to saturate the drift velocity in the gas, which has a constant value v.

vi) The electronic readout is a waveform sampler that provides sufficient information to characterize the original drift pulse.

2. Heuristic Derivation for Radial Charge Collection

For the time being, we disregard the randomness of the charge distribution, replacing it by a continuous uniform distribution of charge of density μ, as shown in fig. 1. This ionization is collected between the times $t' = b/v$ and $t'' = [b^2 + (a/2)^2]^{\frac{1}{2}}/v$. The charge collected between the times t and $t + dt$ is

$$Q(t)dt = 2\mu\, dx = 2\mu \frac{v^2 t}{\sqrt{v^2t^2 - b^2}}\, dt$$

$$\equiv 2\mu\, f(t)\, dt \qquad\qquad t' \le t \le t'' \quad .$$

The identity defines the function f(t); for impact parameters $b > (a/2)$, Q(t) is a sharply peaked function, with a long tail extending up to the latest accessible time. The distribution Q(t) is used next to calculate the physical values of interest, namely, the average time of arrival $\bar{t}$, the variance s^2 of the distribution, and the spread $\sigma_{\bar{t}}$ of the average time of arrival; but before deriving these quantities, I shall show that, when including the random ionization deposition, the probability distribution of charge arrival at the wire coincides with the (deterministic) function Q(t). Later in section 3 we compute $\bar{t}$, s^2 and $\sigma_{\bar{t}}$.

3. Charge Distribution with Random Ionization

The probability of a single ionization event, in which the electron is emitted between x and x + dx, is given by the product of three terms: 1) the probability of no ionization from 0 to x; 2) the probability of producing one electron between x and x + dx; and 3) the probability of no ionization from x + dx up to a:

$$P_1(x)dx = P[0;(0,x)] \cdot P[1;(x,x+dx)] \cdot P[0;(x+dx,a)]$$

where, $P_1(x)dx$ is the desired probability, and $P[n;(x',x'')]$ is the probability of producing n electrons in the interval (x',x'').

$$P_1(x)dx = e^{-\mu x} \quad e^{-\mu dx} \quad \mu dx \quad e^{-\mu(a-x-dx)} = e^{-\mu a} \quad \mu dx \quad .$$

The charge distribution at the wire, in time, due to this event, is given by its probability times a geometric factor relating time of arrival to emission coordinates:

$$Q_1(t)dt = \mu e^{-\mu a}\, dt \int_0^a \delta\left(t - \frac{\sqrt{b^2 + \left(x - \frac{a}{2}\right)^2}}{v}\right) dx \quad ,$$

$$= e^{-\mu a}\; 2\mu f(t)\, dt \quad , \qquad t' \leq t \leq t'' \quad .$$

δ is Dirac's function.

The contribution to the charge distribution from n ionization events, $Q_n(t)$, can be calculated similarly. First, we compute the probability of having emitted n electrons at the coordinates $0 \leq x_1 \leq x_2 < \ldots < x_n \leq a$:

$$P_n(x_1, x_2, \ldots, x_n) \prod_{i=1}^{n} dx_i =$$

$$e^{-\mu x_1} \cdot e^{-\mu dx_1}\, \mu dx_1 \cdot e^{-\mu(x_2 - x_1 - dx_1)} \cdot e^{-\mu dx_2}\, \mu dx_2 \ldots$$

$$\cdot\; e^{-\mu(x_n - x_{n-1} - dx_{n-1})} \cdot e^{-\mu dx_n}\, \mu dx_n \cdot e^{-\mu(a - x_n - dx_n)}$$

$$= \mu^n e^{-\mu a} \prod_{i=1}^{n} dx_i \quad .$$

Now, we get the temporal charge distribution at the wire, using the appropriate geometric factors:

$$Q_n(t)dt = \mu^n e^{-\mu a}\, dt \int_0^{x_1} dx_1' \int_{x_1}^{x_2} dx_2' \int_{x_2}^{x_3} dx_3' \ldots \int_{x_{n-1}}^{x_n} dx_n'$$

$$\cdot \left[\delta\left(t - \frac{\sqrt{b^2 + \left(x_1' - \frac{a}{2}\right)^2}}{v}\right) + \ldots + \delta\left(t - \frac{\sqrt{b^2 + \left(x_n' - \frac{a}{2}\right)^2}}{v}\right)\right]$$

$$= \mu^n e^{-\mu a} \frac{dt}{n!} \int_0^a \int_0^a \ldots \int_0^a \prod_{i=1}^{n} dx_i' \cdot \left[\sum \delta\left(t - \frac{\sqrt{b^2 + \left(x_i' - \frac{a}{2}\right)^2}}{v}\right)\right]$$

In the first expression, the sequence $x_1 < x_2 < \ldots < x_n$ is preserved. The factor n! in the second expression accounts for the permutations of the n coordinates x_i introduced by releasing all integration limits to [0,a]. By straightforward integration,

$$Q_n(t)dt = \frac{e^{-\mu a} \; (\mu a)^{n-1}}{(n-1)!} \cdot 2\mu \, f(t) \, dt \quad .$$

The probability distribution of charge at the wire can be calculated now by adding all probability contributions,

$$Q(t)dt = \sum_{n=1}^{\infty} Q_n(t)dt = 2\mu \, f(t) \, dt \, , \qquad t' \leq t \leq t'' \quad .$$

The probability distribution of change arrival is thus equal to that of an equivalent continuous uniform charge deposition.

Now we complete the study of the radial charge collection. The average time of arrival $\bar{t}$ is given by

$$\bar{t} = \frac{b}{v} \left[\frac{1}{2} \sqrt{1 + \left(\frac{a}{2b}\right)^2} + \frac{b}{a} \; \ell n \left(\frac{a}{2b} + \sqrt{1 + \left(\frac{a}{2b}\right)^2} \right) \right] .$$

Limits of this expression, for very small and for very large impact parameters, are

$$\bar{t} \xrightarrow[b \ll \frac{a}{2}]{} \frac{a}{4v} + \frac{b^2}{va} \; \ell n \left(\frac{a}{b} \right)$$

$$\bar{t} \xrightarrow[b \gg \frac{a}{2}]{} \frac{b}{v} \left[1 + \frac{1}{6} \left(\frac{a}{2b}\right)^2 - \frac{1}{40} \left(\frac{a}{2b}\right)^4 + \ldots \right] .$$

The variance s^2 of the time distribution is given by

$$s^2 = \left(\frac{b}{v}\right)^2 \left\{ \frac{1}{4}\left[3 + \frac{1}{3}\left(\frac{a}{2b}\right)^2\right] - \left(\frac{b}{a}\right)^2 \ell n^2 \left(\frac{a}{2b} + \sqrt{1 + \left(\frac{a}{2b}\right)^2}\right) \right.$$
$$\left. - \frac{b}{a}\sqrt{1 + \left(\frac{a}{2b}\right)^2} \ \ell n \left(\frac{a}{2b} + \sqrt{1 + \left(\frac{a}{2b}\right)^2}\right) \right\} .$$

Finally, the root mean square variation of the average time of arrival is the square root of the variance divided by the average number of ionization events,

$$\sigma_{\bar{t}} = \sqrt{\frac{s^2}{\mu a}} .$$

This is the expected error in the measurement of the average time of arrival, due to ionization statistics. In a cylindrical cell of radius r, $a = 2(r^2 - b^2)^{\frac{1}{2}}$.

The derivation above shows how to generalize the formulae to other geometries. The shape of the electrons' trajectories is contained in the function f(t),

$$f(t) = \int_0^a \delta[t - g(x)]\, dx$$

where g(x) gives the transit time of an electron emitted at the coordinate x and collected at time t. Consider, for instance, the cell shown in fig. 2, where the radial field around the wire extends up to a radius r, becoming a parallel field thereafter. In this case,

$$g(x) = \begin{cases} \dfrac{b + r - \sqrt{r^2 - \left(x - \frac{a}{2}\right)^2}}{v} & \left|x - \frac{a}{2}\right| < r, b > r \\[2ex] \dfrac{b + \left|x - \frac{a}{2}\right|}{v} & \left|x - \frac{a}{2}\right| > r, b > r \end{cases}$$

and the f function becomes:

$$f(t) = \begin{cases} \dfrac{(b + r - vt)\,v}{\sqrt{r^2 - (vt - b - r)^2}} & \frac{b}{v} \le t \le \frac{b+r}{v}, \quad b > r \\[3ex] v & \frac{b+r}{v} \le t \le \frac{b + \frac{a}{2}}{v}, \quad b > r \end{cases}$$

Henceforth, I set $r = a/2$ and $b > r$. The corresponding charge distribution has a large spike at the earliest possible times, and a long tail characteristic of this geometry. In a system capable of sampling the pulse shape, it is advantageous to truncate the pulse after a substantial fraction α of the total charge has been collected. The cutoff time t_c is given by

$$t_c - \frac{b}{v} = \frac{a}{2v}\left(1 - \sqrt{1 - \alpha^2}\right) .$$

Half the charge is collected in 17% of the total pulse length. The average time of arrival $\bar{t}_\alpha$ for the truncated distribution is

$$\bar{t}_\alpha = \frac{b}{v} + \frac{a}{2v}\left(1 - \frac{\sqrt{1 - \alpha^2}}{2}\right) + \frac{a}{4v}\,\frac{\left(\text{arc}\cos\alpha - \frac{\pi}{2}\right)}{\alpha} .$$

The variance of the truncated distribution is

$$s_\alpha^2 = \frac{a^2}{4v^2}\left[\frac{3 - \alpha^2}{3} - \frac{1}{\alpha^2}\left(\frac{\pi}{4} + \frac{\alpha\sqrt{1 - \alpha^2}}{2} - \frac{1}{2}\,\text{arc}\cos\alpha\right)^2\right]$$

with the limit $s_1^2 = (a^2/4v^2)[(2/3) - (\pi/4)^2] = 1.245 \times 10^{-2}\ (a/v)^2$. Finally, the root mean square deviation of $\bar{t}_\alpha$ is

$$\sigma_{\bar{t}_\alpha} = \sqrt{\frac{s_\alpha^2}{\alpha\mu a}} .$$

In this geometry, $\sigma_{\bar{t}_\alpha}$ is independent of the impact parameter for $b > r$.

4. Results and Discussion

As an electron drifts with velocity v in the gas, for a time t, along a direction y, it will not in general be at the coordinate $y_o = vt$. Due to the collisions with gas molecules, its position will obey a Gaussian distribution $\exp\{-(y - y_o)^2/2\sigma^{o2} vt\}$. Here σ^{o2} gives the mean square coordinate fluctuation per unit length. As each electron diffuses independently of all others, the effect of diffusion is to broaden $f(t)$ incoherently.

In this discussion, the space resolution σ has two terms: the contribution from ionization statistics and that from diffusion. These terms add in quadrature. For a measurement of the average time of arrival, σ is given by

$$\sigma = \sqrt{\frac{v^2 s_\alpha^2 + \sigma^{o2}\,\bar{t}_\alpha\,v}{\alpha\mu a}} .$$

Notice that σ does not explicitly depend on the drift velocity, since both $v^2 s_\alpha^2$ and $v\,\bar{t}_\alpha$ are determined solely by the cell geometry. In practice, the drift velocity affects the resolution due to its dependence on the electric field and its impact on the required speed of the readout electronics. These facts restrict the validity of these expressions accordingly. The diffusion parameter σ^o also depends on the operating conditions, and an appropriate value must be used.

A cell of the central drift chamber of the SLD [2] detector is shown in fig. 3. The cell is described in detail in the proposal. The sense-wire interval is 8 mm, and guard wires are interspersed between the sense ones. The maximum drift distance is about 15 mm. In the calculation the ionization collection pattern is approximated as shown in fig. 2. A sense wire collects charge from a track length of 8 mm. The charge drifts towards the wire along parallel drift lines for track impact parameters larger than a given radius around the wire. Within this circle, the charge drifts radially. Results are shown in fig. 4; two examples have been worked out — corresponding to 50% and 66% of total charge collected. In the latter case, the ionization statistics limit for impact parameters beyond the radial collection region is about 75 μm. By including the diffusion contribution, the limiting resolution varies from about 85 μm at 4 mm impact parameters to about 115 μm at the furthest point in the cell. These values have been obtained with $\sigma^o = 280\ \mu m/\sqrt{cm}$.

The resolution shown for impact parameters $b < r$ corresponds to the radial charge collection formulae, and for $b \approx r$ the two curves have been joined smoothly.

In conclusion, formulae have been derived that permit us to estimate the resolution of a drift cell when measuring the temporal charge distribution arriving at the wire. The approximations used restrict the validity of the calculation to cases where the electron drift velocity is nearly constant and relatively low, as in a time expansion chamber [3], or when waveform sampling electronics can sample the charge at rates in excess of 100 MHz. Within these limitations, our simple formulae apply. Measurements from a prototype cell under construction will be used to assess the range of validity of our calcuations. The formulae presented here complement the well-known [4,5] expressions that apply to the case of leading edge threshold discrimination.

References

[1] J. Va'vra and L. Roberts, "Computer Simulation of Drift Chamber Pulse Shapes," to be published.

[2] SLD, SLAC Letter of Intent SLC9, September 30, 1982.

[3] A. H. Walenta, IEEE Trans. Nucl. Sci. NS-26, 73 (1979).

[4] V. Palladino and B. Sadoulet, Nucl. Instrum. Methods 128, 323-335 (1975).

[5] H. Cramér, Mathematical Methods of Statistics, Princeton University Press (1946), Chap. 28.

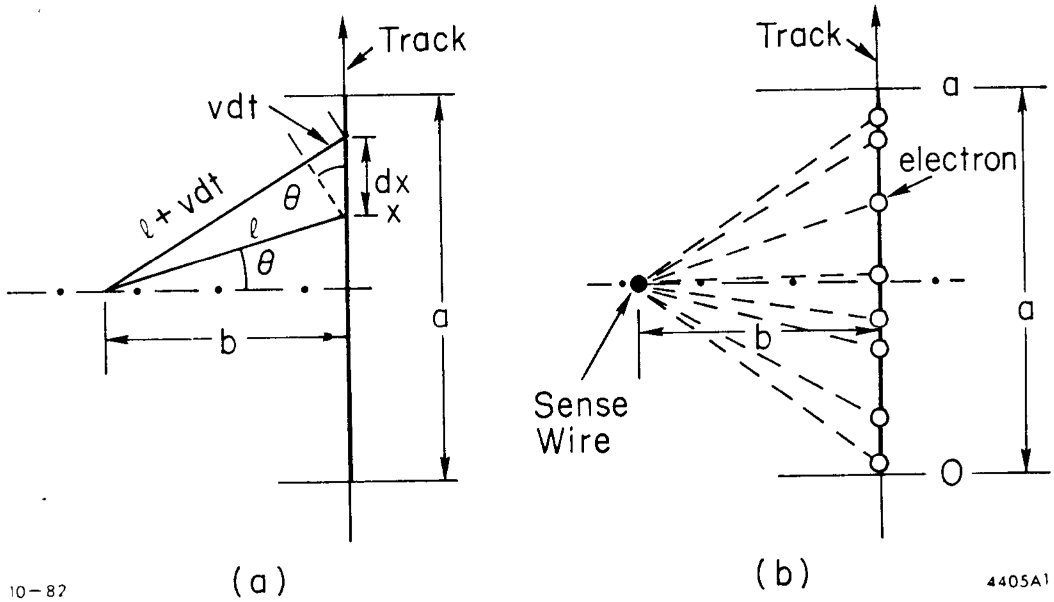
Figure Captions

Fig. 1. Schematics of ionization collection in a radial field. The electrons drift towards the sense wires in straight lines. In fig. 1(a) the random distribution of charge along the track is approximated by a continuous line charge density. The coordinate x is measured from 0 along the track.

Fig. 2. Charge collection geometry with radial field up to distance r from the wire and parallel field thereafter.

Fig. 3. A cell of the central drift chamber of the SLD detector. The distance between nearest neighbor wires is 4 mm, and the average cell width is 30 mm.

Fig. 4. Resolution as a function of impact parameter in the SLD geometry. Each pair of curves corresponds to different pulse truncations. The ionization statistics contribution, and that plus the diffusion contribution are shown separately.



Track
vdt
$\ell + vdt$
ℓ
θ
θ
dx
x
a
b
10-82
(a)
Track
a
electron
a
Sense Wire
b
O
(b)
4405A1

Fig. 1

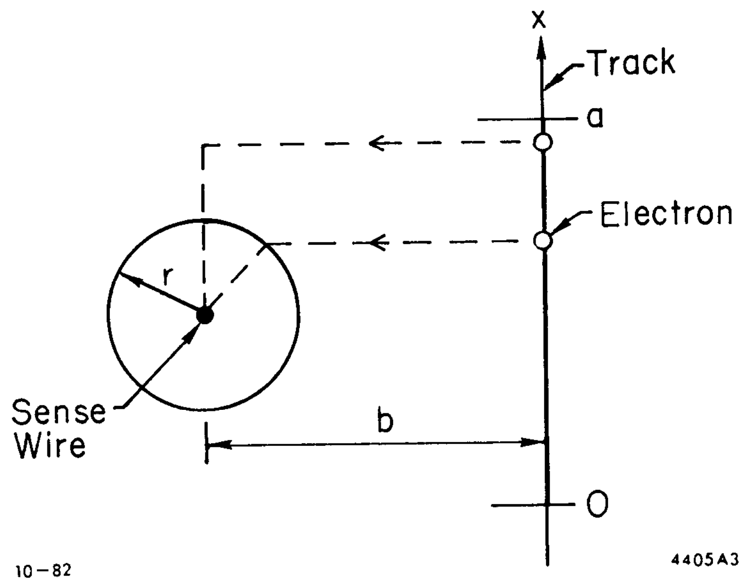


Fig. 2

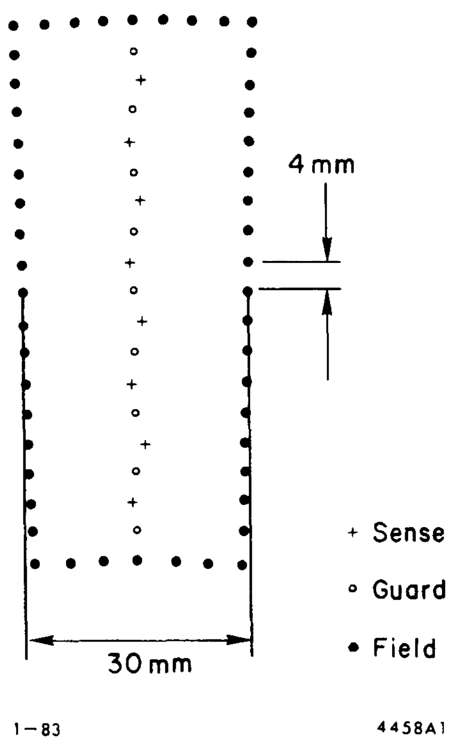
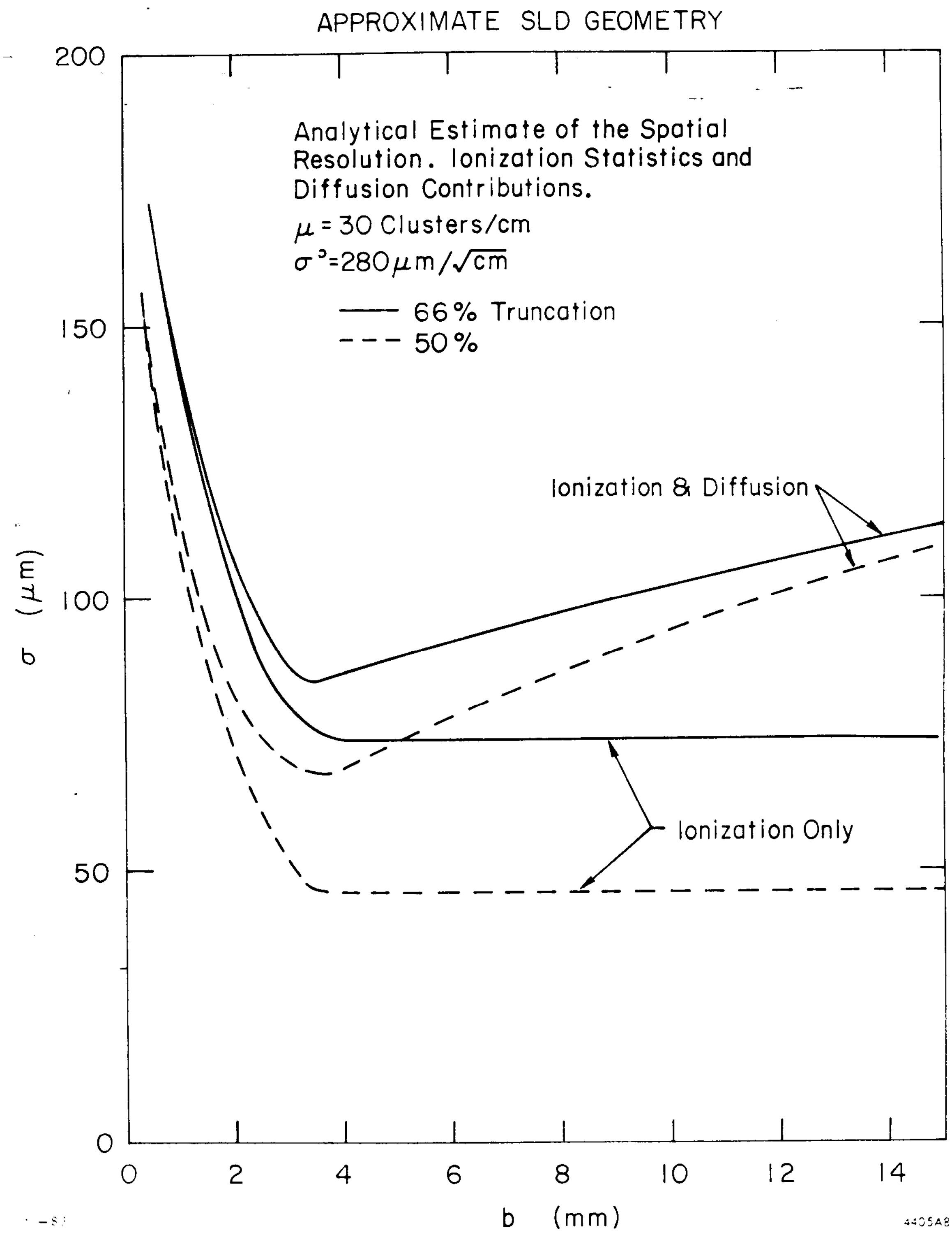


Fig. 3



APPROXIMATE SLD GEOMETRY
Analytical Estimate of the Spatial Resolution. Ionization Statistics and Diffusion Contributions.
μ = 30 Clusters/cm
σ°=280μm/$\sqrt{cm}$
66% Truncation
50%
Ionization & Diffusion
Ionization Only
200
150
100
50
0
0
2
4
6
8
10
12
14
σ (μm)
b (mm)
4405A8

Fig. 4