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BEAM SIZE ENHANCEMENT DUE TO THE PRESENCE OF SEXTUPOLE MAGNETS IN A RING*

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A perturbation method[1] using the Green's function of the Fokker-Planck equation is applied to evaluate an enhancement of the vertical beam size of a bunch due to the presence in a storage ring of sextupolar magnetic fields. The size of the bunch is presented as the power series in sextupole strength parameter.

Consider the beam size in function of the machine tune. The curve will correctly describe the size enhancement only if tune is not too close to a resonance. In immediate vicinity of the resonance the beam size will deviate from correct value, since here breaks the assumption put into the base of the calculation. Namely, the perturbed distribution function will deviate strongly from its unperturbed value. The perturbation theory cannot be used near the resonance and other methods are needed to treat the problem. On the other hand, far from resonance the assumption is valid and the method gives an estimate of the perturbed beam size. A numerical example of the beam enhancement due to the presence of sextupole magnets in PEP storage ring is presented. Discussion of this in more detail can be found in Ref. 2.

To describe a particle motion in a storage ring we use the Courant-Snyder variables[3] u, ϕ ($u' \equiv du/d\phi$) for the horizontal and v, θ ($v' \equiv dv/d\theta$) for the vertical planes respectively. The sudden change in the particle velocity by a passage through a nonlinear magnet ('kick') in these variables is connected to the kick in variables x and y by the following relationship [cf. expressions (5.9) and (5.10) of Ref. 1]:

$$\tilde{F}_x(u,v) = \nu\sqrt{\beta_x}\; F_x[x(u), y(v)]\ , \quad \text{and} \tag{1}$$

$$\tilde{F}_y(u,v) = \tau\sqrt{\beta_y}\; F_y[x(u), y(v)]\ , \tag{2}$$

where ν and τ are horizontal and vertical tunes of the machine correspondingly.

For a sextupole $F_x = S(x^2 - y^2)$, $F_y = -2Sxy$, where S is the integrated strength of the magnet. We get in the variables u, v:

$$\tilde{F}_x = \nu(S^{(1)}u^2 - S^{(2)}v^2) \quad \text{and} \tag{3}$$

$$\tilde{F}_y = -2\tau S^{(2)}uv\ , \quad \text{where} \tag{4}$$

$$S^{(1)} = S\beta_x^{3/2} \quad \text{and} \tag{5}$$

$$S^{(2)} = S\beta_x^{1/2}\beta_y\ . \tag{6}$$

The perturbed distribution function $\psi = \psi_0 + \psi_1 + \psi_2$, where ψ_1 and ψ_2 are the first and the second order corrections for the distribution function,[1] allows us to calculate the perturbed vertical beam emittance E_y:

$$E_y = \int dV\ \psi(V)\,(v^2 + v'^2/\tau^2)/2 \tag{7}$$

Since the distribution function ψ is found in the form of a series expansion, the vertical beam emittance E_y is also an expansion. The zeroth order term of this series is the unperturbed beam emittance ϵ_y. It is easy to see that due to symmetry of the sextupole field, the first order term in E_y is zero. Hence

$$\frac{E_y}{\epsilon_y} = 1 + \Delta_2\ , \quad \text{where} \tag{8}$$

$$\begin{aligned}\Delta_2 = &\frac{1}{2\epsilon_y}\sum_{k<\ell}\int dV\,(v^2 + v'^2/\tau^2)\int dV_1\, G(V, V_1, s_{\ell k})\\ &\times \sum_{m<k}\left(\tilde{F}_x\frac{\partial}{\partial u'} + \tilde{F}_y\frac{\partial}{\partial v'}\right)_{V_1}\int dV_0\, G(V_1, V_0, s_{km})\\ &\times\left(\tilde{F}_x\frac{\partial\psi_0}{\partial u'} + \tilde{F}_y\frac{\partial\psi_0}{\partial v'}\right)_{V_0}\ .\end{aligned} \tag{9}$$

Here $V = (u, u', v, v')$ and $V_0 = (u_0, u_0', v_0, v_0')$, are points in a four-dimensional phase space of the transverse motion,

$$\psi_0 = exp\left\{-\frac{u^2}{2\epsilon_x} - \frac{u'^2}{2\epsilon_x\nu^2} - \frac{v^2}{2\epsilon_y} - \frac{v'^2}{2\epsilon_y\tau^2}\right\}\Big/ (2\pi)^2\epsilon_x\nu\ \epsilon_y\tau \tag{10}$$

and

$$\begin{aligned}G(V, V_0, s_{km}) = &G_u(u, u', \phi_k | u_0, u_0', \phi_m)\\ &\times G_v(v, v', \theta_k | v_0, v_0', \theta_m)\end{aligned} \tag{11}$$

is the Green's function as it is discussed in Ref. 1. It is simplier to perform the space integrations first over V, then over V_1 and at last over V_0. The integral of v^2 over V is the second Green's function moment $P_2 = p_0 + p_1^2v_1^2 + p_2^2v_1'^2 + 2p_3v_1v_1'$, which has been evaluated in Appendix B of part I[1] [see formulae (B.12) through (B.15) for coefficients $p_i(\theta)$]. The second Green's function moment (of v'^2) $Q_2 = q_0 + q_1^2v_1^2 + q_2^2v_1'^2 + 2q_3v_1v_1'$ is found in Appendix B of Part II[4] [see formulae (B.10) through (B.13) for coefficients $q_i(\theta)$]. Since neither P_2, nor Q_2 depend on u_1', only the term containing $\tilde{F}_y$ contributes to the integral over V_1. In addition to this, only terms in P_2 and Q_2 which depend on v_1' contribute to the value of the integral. Since $\tilde{F}_y(u_1, v_1)$ is proportional to u_1v_1 [compare expression (4)], the integral over V_1 is the product of the first moment of G_u (P_1 in the notation of Ref. 1) and the sum of two second moments of G_v ($\dot{P}_2$ and P_2 in the same notation):

$$P_1 = \tilde{p}_1u_0 + \tilde{p}_2u_0'\ , \quad \text{where} \tag{12}$$

$$\tilde{p}_1 = e^{-\alpha\phi}\left(cos\ \nu\phi + \frac{\alpha}{\nu}\ sin\ \nu\phi\right)\ , \tag{13}$$

$$\tilde{p}_2 = e^{-\alpha\phi}\ sin\ \nu\phi/\nu\ . \tag{14}$$

Here ϕ stands for $\phi_k - \phi_m$. Further,

$$P_2 = p_0 + p_1^2v_0^2 + p_2^2v_0'^2 + 2p_3v_0v_0' \quad \text{and} \tag{15}$$

$$\dot{P}_2 = r_0 + r_1v_0^2 + r_2v_0'^2 + 2r_3v_0v_0'\ . \tag{16}$$

Expressions for coefficients $p_i(\theta)$ and $r_i(\theta)$, where $\theta = \theta_k - \theta_m$, may be found in Appendix B, Part I.[1]

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The last integration in (3.4) over V_0 is very simple. The result is:

$$\Delta_2 = 2\sum_{k<\ell} S_k^{(2)}\Big\{\tau^2 f_2 \sum_{m<k}\Big[2\epsilon_x S_m^{(2)} e^{-\alpha\phi-2\delta\theta} \times \cos 2\tau\theta\Big(\cos\nu\phi + \frac{\alpha}{\nu}\sin\nu\phi\Big) - \epsilon_y S_m^{(2)} e^{-\alpha\phi-2\delta\theta}\sin\nu\phi\Big(\sin 2\tau\theta + \frac{\delta}{\tau} - \frac{\delta}{\tau}\cos 2\tau\theta\Big)\Big] + \tau f_3 \sum_{m<k}\Big[\Big(\epsilon_y S_m^{(2)} - \epsilon_x S_m^{(1)}\Big) e^{-\alpha\phi}\sin\nu\phi + \epsilon_y S_m^{(2)} e^{-\alpha\phi-2\delta\theta}\sin\nu\phi\Big(1+\cos 2\tau\theta + 2\frac{\delta}{\tau}\sin 2\tau\theta\Big) + 2\epsilon_x S_m^{(2)} e^{-\alpha\phi-2\delta\theta}\Big(\cos\nu\phi + \frac{\alpha}{\nu}\sin\nu\phi\Big) \times \Big(\sin 2\tau\theta + \frac{\delta}{\tau} - \frac{\delta}{\tau}\cos 2\tau\theta\Big)\Big]\Big\} \quad . \tag{17}$$

Here θ denotes $\theta_k - \theta_m$, $\phi = \phi_k - \phi_m$, and

$$f_2 = e^{-2\delta\tilde\theta}\Big(1 - \frac{\delta}{\tau}\sin 2\tau\tilde\theta\Big)\Big/\tau^2 \quad , \tag{18}$$

$$f_3 = \delta e^{-2\delta\tilde\theta}(1-\cos 2\tau\tilde\theta)\Big/\tau^2 \quad , \tag{19}$$

where $\tilde\theta$ stands for $\theta_\ell - \theta_k$.

The subscript (k or m) in the notation of sextupole magnets numbers them in the order in which they are seen by a bunch.

Consider now the general case of n arbitrarily positioned sextupoles in the ring. Taking as an example the typical sum over m of cosine terms in expression (17), we present it in the following form:

$$R_1^+ = \sum_{m>k} S_m^{(2)} \exp[-\alpha(\phi_m-\phi_k) - 2\delta(\theta_m-\theta_k)] \times \cos[2\tau(\theta_m-\theta_k) + \nu(\phi_m-\phi_k)]$$
$$= S_1^{(2)}\Big[\frac{1}{2} + \sum_{m=1}^{\infty}\exp(-2\pi\alpha m - 4\pi\delta m)\cos(4\pi\tau m + 2\pi\nu m)\Big] + S_2^{(2)}\exp[-\alpha(\phi_2-\phi_1) - 2\delta(\theta_2-\theta_1)] \times \sum_{m=0}^{\infty}\exp(-2\pi\alpha m - 4\pi\delta m) \times \cos[4\pi\tau m + 2\pi\nu m + 2\tau(\theta_2-\theta_1) + \nu(\phi_2-\phi_1)] + \ldots + S_n^{(2)}\exp[-\alpha(\phi_n-\phi_1) - 2\delta(\theta_n-\theta_1)] \times \sum_{m=0}^{\infty}\exp(-2\pi\alpha m - 4\pi\delta m) \times \cos[4\pi\tau m + 2\pi\nu m + 2\tau(\theta_n-\theta_1) + \nu(\phi_n-\phi_1)] \quad . \tag{20}$$

Here quite arbitrarily the starting sextupole is numbered as S_1. In the first term of expression (20) only half pulse at 'time" zero is taken into account in accord with the summation rule developed in Ref. 4. Performing some algebra we get:

$$R_1^+ = \frac{\pi(2\delta+\alpha)}{1-\cos 2\pi(2\tau+\nu)}\sum_{i=1}^{n} S_i^{(2)} \times \cos[2\tau(\theta_i-\theta_1) + \nu(\phi_i-\phi_1)] + \frac{1}{2\sin\pi(2\tau+\nu)}\sum_{i=2}^{n} S_i^{(2)} \times \exp[-\alpha(\phi_i-\phi_1) - 2\delta(\theta_i-\theta_1)] \times \sin[\pi(2\tau+\nu) - 2\tau(\theta_i-\theta_1) - \nu(\phi_i-\phi_1)] \quad . \tag{21}$$

There are in general n different sums R_j^+, $j = 1,2,\ldots,n$, similar to R_1^+. They appear, when one consider similar sums starting from the sextupole, positioned at j-th place in the ring. Any sum evaluated for the sextupole shifted by $n+1$ positions from the j-th one is equal to the j-th sum:

$$R_{j+n}^+ = R_j^+ \quad . \tag{22}$$

Exponents in each of the second sum in expression (21) are very close to 1, since α and δ are usually small. Hence, the exponential factors may be expanded and all the terms of the order $(\alpha\Delta\phi)^2$ and $(\delta\Delta\theta)^2$ and higher may be omitted.

Let us introduce notations:

$$A_j^+ = \sum_{i=1}^{n} S_{j+i-1}^{(2)}\cos[2\tau(\theta_{j+i-1}-\theta_j) + \nu(\phi_{j+i-1}-\phi_j)] \tag{23}$$

$$B_j^+ = \sum_{i=2}^{n} S_{j+i-1}^{(2)} \times \sin[\pi(2\tau+\nu) - 2\tau(\theta_{j+i-1}-\theta_j) - \nu(\phi_{j+i-1}-\phi_j)] \tag{24}$$

$$C_j^+ = \sum_{i=1}^{n} S_{j+i-1}^{(2)}\Big[2(\theta_{j+i-1}-\theta_j) + \frac{\alpha}{\delta}(\phi_{j+i-1}-\phi_j)\Big] \times \sin[\pi(2\tau+\nu) - 2\tau(\theta_{j+i-1}-\theta_j) - \nu(\phi_{j+i-1}-\phi_j)] \quad (j=1,2,\ldots,n) \quad . \tag{25}$$

These values have the same periodic property (22) as R_j^+. The latter can be now written as follows:

$$R_j^+ = \frac{\pi(2\delta+\alpha)}{2\sin^2\pi(2\tau+\nu)}A_j^+ + \frac{1}{2\sin\pi(2\tau+\nu)}B_j^+ - \frac{\delta}{2\sin\pi(2\tau+\nu)}C_j^+ \quad . \tag{26}$$

It is easy to find now the double sum over k and m of the considered term:

$$\sum_{k>\ell} S_k^{(2)}\exp[-2\delta(\theta_k-\theta_\ell)]R_k^+ = \frac{(1+\alpha/2\delta)}{4\sin^2\pi(2\tau+\nu)}a^+ - \frac{1}{8\pi\sin\pi(2\tau+\nu)}c^+ \tag{27}$$

where a new notations are introduced:

$$a^+ = \sum_{j=1}^{n} S_j^{(2)} A_j^+ \tag{28}$$

$$c^+ = \sum_{j=1}^{n} S_j^{(2)} C_j^+ \quad . \tag{29}$$

Notice the absence in expression (27) of the term proportional to $b^+ = \sum_{j=1}^{n} S_j^{(2)} B_j^+$. Indeed, it is easy to check, that this quantity is equal to zero.

Introduce now similar notations for all other types of terms:

$$a^- = \sum_{j=1}^{n} S_j^{(2)} A_j^- \quad , \tag{30}$$

$$c^- = \sum_{j=1}^{n} S_j^{(2)} C_j^- \quad \text{where} \tag{31}$$

$$A_j^- = \sum_{i=1}^{n} S_{j+i-1}^{(2)} \cos\left[2\tau(\theta_{j+i-1} - \theta_j) - \nu(\phi_{j+i-1} - \phi_j)\right] \ , \tag{32}$$

$$C_j^- = \sum_{i=1}^{n} S_{j+i-1}^{(2)} \left[2(\theta_{j+i-1} - \theta_j) + \frac{\alpha}{\delta}(\phi_{j+i-1} - \phi_j)\right] \times \sin[\pi(2\tau - \nu) - 2\tau(\theta_{j+i-1} - \theta_j) + \nu(\phi_{j+i-1} - \phi_j)] \tag{33}$$

$$a_{1,2}^o = \sum_{j=1}^{n} S_j^{(2)} A_j^{(1,2)} \quad , \quad \text{where} \tag{34}$$

$$A_j^{(1,2)} = \sum_{i=1}^{n} S_{j+i-1}^{(1,2)} \cos \nu(\phi_j - \phi_{j+i-1}) \ . \tag{35}$$

The terms of $\bar{c}_{1,2}^o$ types are of higher order of magnitude in α and do not enter the final result.

In a similar way one can calculate also the sine type terms in expression (17). In this case one needs to introduce quantities

$$d^\pm = \sum_{j=1}^{n} S_j^{(2)} D_j^\pm \quad \text{and} \tag{36}$$

$$d_{1,2}^o = \sum_{j=1}^{n} S_j^{(2)} D_j^{(1,2)} \quad , \quad \text{where} \tag{37}$$

$$D_j^\pm = \sum_{i=1}^{n} S_{j+i-1}^{(2)} \times \cos\left[\pi(2\tau \pm \nu) - 2\tau(\theta_{j+i-1} - \theta_j) \mp \nu(\phi_{j+i-1} - \phi_j)\right] \tag{38}$$

and

$$D_j^{(1,2)} = \sum_{i=2}^{n} S_{j+i-1}^{(1,2)} \cos\left[\pi\nu - \nu(\phi_{j+i-1} - \phi_j)\right] \ . \tag{39}$$

In terms of the quantities a, c and d the final result for Δ_2 is:

$$\Delta_2 = \frac{1}{4\pi}\left\{\frac{2\pi(\epsilon_x + \epsilon_y/2)(1 + \alpha/2\delta)}{\sin^2\pi(2\tau + \nu)} a^+ - \frac{\epsilon_x + \epsilon_y/2}{\sin \pi(2\tau + \nu)} c^+ + \frac{2\pi(\epsilon_x - \epsilon_y/2)(1 + \alpha/2\delta)}{\sin^2\pi(2\tau - \nu)} a^- - \frac{\epsilon_x - \epsilon_y/2}{\sin \pi(2\tau - \nu)} c^- + \frac{(\epsilon_x + \epsilon_x \frac{\alpha\tau}{\nu\delta} + \epsilon_y)}{\tau \sin \pi(2\tau + \nu)} d^+ + \frac{(\epsilon_x - \epsilon_x \frac{\alpha\tau}{\nu\delta} - \epsilon_y)}{\tau \sin \pi(2\tau - \nu)} d^- + \frac{\epsilon_y d_2^o}{\tau \sin \pi\nu} - \frac{\epsilon_x d_1^o}{\tau \sin \pi\nu}\right\} \ . \tag{40}$$

For a ring with M identical superperiods expression (40) is invariant under the following transformation:

$$\Delta_2(\nu, \alpha, \tau, \delta, n) = \Delta_2\left(\frac{\nu}{M}, \frac{\alpha}{M}, \frac{\tau}{M}\, \frac{\delta}{M}, \frac{n}{M}\right) \ . \tag{41}$$

The quantities $a^{\pm,o}$, $c^\pm$ and $d^{\pm,o}$ can be named the (sextupole) distribution factors. For a given ring only the distribution factors change, when parameters of the distribution of the sextupole magnets, i.e. their number, strengths and positions in the ring, are changed. In general there are eight distribution factors altogether.

Here is the result of the evaluation of the vertical beam size growth for the current PEP configuration (assuming optimal coupling): $\nu = 21.25$, $\beta_y^* = 0.11m$, $\tau = 18.19$, $\alpha/\delta = 1.0$, $\beta_x^* = 3.0m$, $\sqrt{\epsilon_y/\epsilon_x} = 0.19$, $\sigma_z = 0.58mm$.

The calculation by formula (40) yields:

$$\frac{\Delta\epsilon_y}{\epsilon_y} = 1.3\% \ . \tag{42}$$

The beam emittance enhancement of the same magnitude is found also for other points of the tune diagram which are not too close to the resonance lines $2\tau \pm \nu =$ integer and $\nu =$ integer.

The beam increase $\Delta\epsilon_y/\epsilon_y$ goes as cube of the beta function value at the sextupole positions. Hence its value might be much larger for a larger size ring. The invariance of the result (42) under the transformation (41) has been confirmed by the numerical calculation.

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