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PHYSICAL NULL ZONES AND RADIATION REPRESENTATION*

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ABSTRACT

General conditions for reconstructing physical null radiation zones in single photon tree amplitudes are given. The systematic analysis has been carried out using invariant quantities. For arbitrary values of masses and charges these zones are always smaller than in the massless and equal charges case. As an application the radiative W boson decay into heavy quarks is studied. This process turns out to be a rather sensitive test of the current quark masses $m_q(M_W^2)$, as well as of the $\bar{q}qW$, $q\bar{q}\gamma$ and $W^+W^-\gamma$ vertices. This is due to the presence of a null line in the photon phase space with a location which strongly depends on m_q. A recently proposed radiation representation for single photon tree amplitudes is analyzed. Explicit examples are given for a number of cases including fermion and vector boson lines.

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1. Introduction

Recently Brodsky and Brown [1] generalizing previous results [2] have shown that in a large class of field theories including gauge theories, any tree amplitude for single photon emission vanishes identically in certain kinematical domains. In a systematic investigation of scattering and decay transitions which allow a null zone, a scheme must be developed to find when these null zones lie inside the physical region.

This analysis can be conveniently carried out in terms of invariant quantities. Within this framework we study the conditions for a null zone [1]

$$\frac{e_i}{p_i \cdot k} = \text{const.} \qquad (i = 1, \ldots, n)$$

where e_i, p_i label the charges and four-momenta in a tree graph with n external lines, k being the photon four-momentum.

When all the incoming and outgoing particles are massless and have identical charges, the null zone is always physical as already pointed out in ref. [1]. Turning on masses and different charges put constraints on the null zone, since certain combinations of external momenta must be spacelike. In terms of invariants used to describe the process, this is equivalent to the condition that each subenergy must be greater than some threshold while each momentum transfer must be smaller than some other threshold if we want a physical and nontrivial null zone.

In the massless case these tresholds are zero and for the general case the null zone is always smaller.

Following this analysis we found that the most promising process for detecting null zones is perhaps the radiative decay $W \to q\bar{q}\gamma$ [3] where we have discovered a screening of the decay into light quarks for certain kinematical regions. Indeed there is a line of zeros in the photon phase space unique to each quark generation, and for suitable values of the heavy quark masses, the light quark null line will cross the physical region for the W decay into a pair of heavy quarks. Thus the decay into $c\bar{s}$ and perhaps $t\bar{b}$, if the top mass is not too heavy, is dominant in the whole range of values of the photon variables where the zeros occur.

QCD effects have been neglected in these calculations since they are of order $\alpha_s(M_W^2)$.

When the t mass is very large, there are no physical values for the photon energy and angle where the production of light quark pairs is filtered. Nevertheless, the angular distribution for $W^+ \to t\bar{b}\gamma$ is extremely peculiar, due to the presence of zeros for all values of the photon energy. These zeros are strongly dependent on m_t, especially for hard photons. This process may therefore represent a direct test of the e. m. properties of charged vector bosons and quarks together with the value of the current quark masses $m_q(M_W^2)$.

Independently from the existence of physical null zones, any tree amplitude for single photon emission can be written in the so-called radiation representation [1] where the vanishing is manifest.

Using the techniques of ref. [6] we introduce in this context explicit polarization vectors for the photon which allow us to extend the previous examples of radiation representations for spinless particles to the more complex situation where spin as well as contact terms are present, including the case of internal charged vector bosons.

In all the cases analyzed, the final expressions are very simple, in accord with the general strategy outlined in ref. [6], and the vanishing on the null zone is automatically guaranteed.

The outline of the paper is as follows. In sect. 2 we describe the conditions for a physical null zone. In sect. 3 the radiative W decay into quarks is analyzed. In sect. 4 the radiation representation is explicitly given for a number of cases including internal line radiation. In the appendix we present the expression for the doubly differential decay rate for $W^+ \to t\bar{b}\gamma$.

2. Physical Null Zones

We consider the process $1+2 \to 3+\ldots+n+\gamma$. The following notations and conventions are used (see fig. 1). All the momenta are incoming and the metric is such that a timelike momentum squared is negative. Charge and momentum conservation imply

$$\sum_{i=1}^{n+1} p_i = 0 \; ; \; \sum_{i=1}^{n} e_i = 0 \; .$$

To describe the process $1+2 \to 3+\ldots+n+\gamma$, we introduce the invariants

$$s_{ij} = -\left(p_i + p_j\right)^2 = m_i^2 + m_j^2 - 2p_i \cdot p_j \qquad (i,j = 1, \ldots, n+1)$$

$$m_{n+1} = 0 \qquad (2.1)$$

The null zone for this process is given by the equations

$$p_{ij} \cdot k = 0 \quad \text{with} \quad p_{ij} = p_i - \frac{e_i}{e_j} p_j \qquad (j \neq i, \ell)$$

or (2.2)

$$e_j s_{i,n+1} - e_i s_{j,n+1} = e_j m_i^2 - e_i m_j^2 \quad .$$

A nontrivial (i.e., $k_0 \neq 0$) physical null zone exists provided that all the momenta are in the physical region and all the p_{ij} are spacelike. The second condition can be cast in the form

$$s_{ij} \geq s_{ij}^o \quad \text{for} \quad \frac{e_i}{e_j} > 0$$

$$s_{ij} \leq s_{ij}^o \quad \text{for} \quad \frac{e_i}{e_j} < 0 \qquad (2.3)$$

with

$$s_{ij}^o = \left(1 + \frac{e_j}{e_i}\right) m_i^2 + \left(1 + \frac{e_i}{e_j}\right) m_j^2 \quad .$$

It follows that all the incoming and outgoing charges must have the same sign for a physical null zone. Indeed for i = 1, j = 2 the physical region is bounded by

$$s_{12} \geq s_{th} = \left(m_1 + m_2\right)^2 \quad .$$

But

$$s_{12}^o - s_{th} = \left(\rho^{-\frac{1}{2}} m_1 - \rho^{\frac{1}{2}} m_2\right)^2 > 0 \quad \text{for} \quad \rho = \frac{e_1}{e_2} > 0 \quad ,$$

$$s_{12}^o - s_{th} = -\left(\chi^{-\frac{1}{2}} m_1 + \chi^{\frac{1}{2}} m_2\right)^2 < 0 \quad \text{for} \quad \chi = -\rho > 0 \quad .$$

Thus $e_1/e_2 > 0$. For i,j > 2 we may go to the c.m.s. of particles i and j, deriving the condition $e_i/e_j > 0$. Thus charge conservation gives [1]

$$\text{sign } e_1 = \text{sign } e_2 = - \text{ sign } e_i^{in} = \text{sign } e_i^{out} \qquad (i = 3, \ldots, n) \quad .$$

Before giving the general prescription to construct the boundaries of the physical null zone we consider as an example the case n = 3. First we do not make any choice of independent variables and use the constraints as auxiliary conditions. We define

$$s = s_{12}, \quad t = s_{13}, \quad u = s_{23} \quad .$$

Working for simplicity in the equal mass case, we get a null zone described by

$$e_2 u - e_1 t = \left(e_2 - e_1\right) m^2 , \quad e_3 u - e_1 s = \left(e_3 - e_1\right) m^2 \quad . \qquad (2.4)$$

Using $s + t + u = 3m^2$, we get

$$s + (1 + \rho) t = (1 + 2\rho) m^2 \qquad \rho = \frac{e_2}{e_1} \quad . \qquad (2.5)$$

When the mass is neglected, the angular distribution for the process has a zero independent from the energy. Equations (2.3) give

$$s \geq s^o = \frac{(1+\rho)^2}{\rho} m^2 , \quad t \leq t^o = - \frac{\rho^2}{1+\rho} m^2 , \quad u \leq u^o = - \frac{1}{\rho(1+\rho)} m^2 \quad . \qquad (2.6)$$

Momentum conservation implies that on the 1-dimensional null zone described by eq. (2.5) $s \geq s^o$ is the only independent constraint. The physical region for the n = 3 process is constructed by requiring [4]

$$G_2 = G\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} < 0 \quad ; \qquad G_3 = G\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} < 0 \qquad (2.7)$$

where

$$G\begin{pmatrix} 1 & \ldots & \ell \\ 1 & \ldots & \ell \end{pmatrix} = \det\left(p_i \cdot p_j\right) \qquad (i,j = 1, \ldots \ \ell) \quad .$$

The condition $G_2 < 0$ simply gives $s > 4m^2$; the region where $G_3 < 0$ also is shown in fig. 2. In view of the complexity for $n > 3$, we may restrict our analysis to the null zone. In this case we only require

$$G_3\Big|_{NZ} < 0 \quad .$$

If one of the particles is neutral, the same arguments apply. When $e_3 = 0$ the null zone is always unphysical because $e_1 = -e_2$. When $e_1 = 0$ or $e_2 = 0$ we discover that an infinite energy is needed to create a physical null zone.

For n arbitrary there are $3n - 7$ independent invariants. The null zone is a $(2n - 5)$-dimensional surface which is physical and nontrivial when all the momenta are within the physical region and the constraints [eq. (2.3)] are imposed. If we do not make any choice of independent variables the null zone will be described by the equations

$$\rho_i s_{1,n+1} - s_{i,n+1} = \rho_i m_1^2 - m_i^2 \,, \quad \rho_i = \frac{e_i}{e_1} \quad (i = 2, \ldots, n) \,. \tag{2.8}$$

The constraints are now

$$G_2\Big|_{NZ} < 0 \,, \qquad G_3\Big|_{NZ} < 0 \,, \qquad G_4\Big|_{NZ} < 0$$

$$G\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = G\begin{pmatrix} 1 & 2 & 3 & 4 & 6 \\ 1 & 2 & 3 & 4 & 6 \end{pmatrix} = G\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \ldots = 0$$

$$s_{12} \geq s_{12}^o$$

$$s_{1i} \leq s_{1i}^o \,, \qquad s_{2i} \leq s_{2i}^o \qquad (i = 3, \ldots, n)$$

$$s_{ij} \geq s_{ij}^o \qquad\qquad (i,j = 3, \ldots, n) \quad . \tag{2.9}$$

For $n = 4$ we choose as independent variables s_{12}, s_{14}, s_{15}, s_{34} and s_{45}. The s_{ij} fulfill the identity

$$\sum_{j \neq i} s_{ij} = M^2 - 3m_i^2 \qquad M^2 = \sum_i m_i^2 \quad .$$

The solution is

$$\begin{aligned}
s_{13} &= M^2 - 3m_1^2 - s_{12} - s_{14} - s_{15} \\
s_{23} &= M^2 - 3\left(m_2^2 + m_3^2\right) + s_{14} + s_{15} + s_{45} \\
s_{24} &= M^2 - 3m_4^2 - s_{14} - s_{34} - s_{45} \\
s_{25} &= -M^2 + 3\left(m_3^2 + m_4^2\right) - s_{12} - s_{15} + s_{34} \\
s_{35} &= 2M^2 - 3\left(m_3^2 + m_4^2\right) + s_{12} - s_{34} - s_{45} \quad .
\end{aligned}$$

Equations (2.8) become

$$\rho_i s_{15} - s_{i5} = \rho_i m_1^2 - m_i^2 \qquad i = 2,3,4 \quad .$$

The most severe cuts on the null zone are for arbitrary charges and masses but to illustrate this example we simply choose the equal charge, equal mass case. Arbitrarily selecting $i = 3,4$ we get

$$s_{15} = m^2 - s_- \, , \qquad s_{45} = m^2 + s_- \tag{2.10}$$

where we have introduced $s_\pm = \frac{1}{2}\left(s_{12} \pm s_{34}\right)$. The constraints are

$$s_+ \geq |s_-| \, , \qquad s_{14} \leq 0 \, , \qquad s_+ + s_{14} \geq 0 \quad . \tag{2.11}$$

Next we impose on the null zone

$$G_2 = G\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} < 0 \ , \quad G_3 = G\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} < 0 \ , \quad G_4 = G\begin{pmatrix} 1 & 2 & 4 & 5 \\ 1 & 2 & 4 & 5 \end{pmatrix} < 0 \quad , \qquad (2.12)$$

p_3 being the momentum fixed by conservation. On the null zone

$$p_1 \cdot p_2 = m^2 - \frac{1}{2}\left(s_+ + s_-\right) \ , \quad p_1 \cdot p_4 = m^2 - \frac{1}{2}\ s_{14} \ , \quad p_{15} = \frac{1}{2}\, s_-$$

$$p_2 \cdot p_4 = \frac{1}{2}\left(s_{14} + s_+\right) \ , \quad p_{25} = -p_{45} = \frac{1}{2}\, s_- \quad .$$

$G_2^{NZ} < 0$ again gives $s_{12} = s_+ + s_- > 4m^2$. The remaining two conditions are cubic equations in any s_{14}, s_+ plane corresponding to a fixed value of s_-. In the massless limit, where we know that constraints (2.12) are sufficient

$$G_3^{NZ} = \frac{1}{4}\ s_{14}\left(s_{14} + s_+\right)\left(s_+ + s_-\right)$$

$$G_4^{NZ} = \frac{1}{16}\ s_-^2 \left(s_-^2 + 4s_{14}^2 + 4s_{14}s_+\right) \quad .$$

Thus the null zone, which is always physical and nontrivial, corresponds to

$$s_+ \geq s_- \ , \quad G_4^{NZ} < 0 \quad .$$

If one or more particles are neutral, we may have a physical null zone with all the invariants finite if $m_i = 0$ for all the $e_i = 0$. This is equivalent to

$$s_{i,n+1} = 0 \quad \text{if} \quad e_i = 0 \quad .$$

$p_i \cdot k = 0$ is unphysical for a timelike p_i.

For $1 + 2 \to 3 + \gamma$ where for example $e_2 = 0$ the null zone is given by

$$e_3 s_{14} - e_1 s_{34} = e_3 m_1^2 - e_1 m_3^2$$

$$s_{24} = 0 \qquad\qquad k \equiv p_4 \quad .$$

Using $s_{14} = 4$, $s_{34} = s$, $s_{24} = t$ together with $e_1 + e_3 = 0$, the two equations reduce to $t = 0$ for $m_2 = 0$ and no further constraint on s arises.

The conditions $s_{i,n+1} = 0$, where i denotes any neutral particle are a consequence of the theorem [1] which requires neutral particles to be massless and propagating along the photon direction. For the case $n = 4$ with $e_2 = m_2 = 0$ the null zone is described by

$$e_i s_{15} - e_1 s_{i5} = e_i m_1^2 - e_1 m_i^2 \qquad i = 3,4 \quad ;$$

plus the condition

$$p_{2\mu} \propto k_\mu \quad .$$

We may go to the rest frame of particle 1 with the photon in the z direction: the massless and neutral particle 2 must have zero components in the x,y plane or

$$G\begin{pmatrix}1 & 5 & 2\\ 1 & 5 & 2\end{pmatrix} = 0 \ , \quad G\begin{pmatrix}1 & 5\\ 1 & 2\end{pmatrix} = p_1 \cdot p_2 \left[- G\begin{pmatrix}1 & 5\\ 1 & 5\end{pmatrix}\right]^{1/2} .$$

If these conditions are satisfied

$$p_2 = \frac{G\begin{pmatrix}1 & 5\\ 1 & 2\end{pmatrix}}{G\begin{pmatrix}1 & 5\\ 1 & 5\end{pmatrix}}\, k \quad .$$

Moreover,

$$G\begin{pmatrix} 1 & 5 & 2 & 4 \\ 1 & 5 & 2 & 4 \end{pmatrix} < 0$$

and p_3 is fixed by momentum conservation. In this way neutral particles can be included as well by only requiring some determinants to be zero.

3. Example of Null Zone: W Radiative Decay

The first and most celebrated zero was discovered in the angular distribution for $u\bar{d} \to W + \gamma$ [5]. Here we want to discuss a related process, namely the radiative W decay [3]. Notations are given in figure 3. Moreover, $e_- - e_+ = \pm 1$.

Three diagrams contribute to this process and a 1-dimensional null zone exists even for massive fermions, as we explicitly show in the next section. Using the results of section 2 we find a null zone described by the equation

$$e_- s \pm t = e_- M_W^2 \pm m_-^2 \tag{3.1}$$

where

$$s = -\left(p_+ + p_-\right)^2 , \quad t = -\left(Q - p_+\right)^2 , \quad u = -\left(Q - p_-\right)^2 , \; p_\pm^2 = -m_\pm^2 , \; Q^2 = -M_W^2 .$$

The photon phase space is easily reconstructed if we go to the W rest frame with p_- along the z axis. There

$$E_- = \frac{1}{2} \, \frac{s + t - m_+^2}{M_W} \, , \quad E = \frac{M_W^2 - s}{2M_W} \quad . \tag{3.2}$$

Moreover, if we denote by θ the photon polar angle, the curve $\cos^2\theta = 1$ is given by

$$m_-^2\left(M_W^2 - s\right)^2 + \left(s + t - m_+^2\right)\left(m_-^2 - t\right)\left(M_W^2 - s\right) + M_W^2\left(m_-^2 - t\right)^2 = 0 \quad . \tag{3.3}$$

The physical region is given in fig. 4. For $m_\pm \cong 0$ the boundaries are given by $s = t = 0$ and $s + t = M_W^2$. First we consider the decay $W^- \to \bar{\nu}\ell\gamma$ which is equivalent to $e_+ = m_+ = 0$. The null zone is now

$$t - s = m_\ell^2 - M_W^2 \,, \qquad k = Kp_+ \quad , \qquad (e_- = -1)$$

where K is a given constant. However $k = Kp_+$ gives $u = 0$, which combined with the first equation has a solution $s = M_W^2$ and $t = m_\ell^2$. As a consequence the leptonic radiative decay of the W only possesses an unphysical null zone. On the other hand $W \to q\bar{q}\gamma$ may have a physical null zone and different quark families produce a zero in the amplitude at different photon configurations.

In particular we may think of a situation where the null line for light quarks crosses the physical region of heavy quarks, making this decay optimal for heavy quarks detection.

We consider the W^+ decay where $e_- = 2/3$ and $e_+ = -1/3$. The $u,\bar{d}$ null zone is given by

$$2s + 3t = 2M_W^2 + 3m_u^2 \quad . \tag{3.4}$$

If a pair of heavy quarks is produced with a photon in the same configuration (i.e., E_γ and θ are the same) then the line

$$2s + 3t = 2M_W^2 + 3\left(m_u^2 - m_d^2 + m_s^2\right) \tag{3.5}$$

in the $c,\bar{s}$ phase space corresonds to a suppression of $W^+ \to ud\gamma$. The same happens in the $t,\bar{b}$ phase space along the line

$$2s + 3t = 2M_W^2 + 3\left(m_u^2 - m_d^2 + m_b^2\right) \quad . \tag{3.6}$$

The range of values for θ and E_γ where we may detect heavy quarks with a camplete screening of light pairs is given by the intersection of the previous lines with the relative physical region.

This analysis is affected by the fact that we observe jets and not single quarks. A crucial point is the value of θ, the angle between the photon and the charge 2/3 quark which must be large. Otherwise, the uncertainty related to a measurement of the photon-jet angle would spoil the argument. QCD corrections to the subprocess are of order $\alpha_s(M_W^2)/\pi$ and to this accuracy we still expect a screening effect of light quarks. We stress here that an experiment exists which is very sensitive to $m_q(M_W^2)$ and in some cases makes the production of light quarks highly suppressed in comparison with heavy quarks.

For the case $W^+ \to t\bar{b}\gamma$ with $m_u = m_d = 0$, we may compute the intersections of the null line [eq. (3.6)] with the boundary of the physical region. Thus

$$s_{min}\left(m_t\right) \le s \le s_{max}\left(m_t\right) \quad \text{for} \quad t = \frac{2}{3}\left(M_W^2 - s\right) + m_b^2 \quad .$$

On the null line $\cos\theta_{\gamma t}^L$ is a given function of s

$$\cos\theta_{\gamma t}^L = \Delta^{-1}\left[1 - 2M_W^2 \frac{2\left(M_W^2 - s\right) + 3\left(m_b^2 - m_t^2\right)}{\left(M_W^2 - s\right)\left(2M_W^2 + s\right)}\right]$$

with

$$\Delta^2 = 1 - 36 \frac{m_t^2 M_W^2}{\left(2M_W^2 + s\right)^2} \quad .$$

From this formula we compute the behaviour of $\cos\theta^L_{\gamma t}$ as a function of the unknown t mass. The results are shown in Table 1 using $M_W = 80$ GeV and $m_b = 5$ GeV. For $m_t \gtrsim 30$ GeV there is no intersection between the light quark null line and the physical region for $W^+ \to t\bar{b}\gamma$. As expected, $\cos\theta^L_{\gamma t}$ is an increasing function of s with a slope which is larger for large values of m_t where $s_{max} - s_{min}$ goes to zero. Therefore the events in the central region of $\theta^L_{\gamma t}$ lie in an interval of s which is a rapidly decreasing function of m_t.

For table 1 we also report $\cos\theta^H_{\gamma t}$, where $\theta^H_{\gamma t}$ is the angle where the decay rate for $W^+ \to t\bar{b}\gamma$ is zero. Notice that for $m_t \leq 30$ GeV the values of $\theta^L_{\gamma t}$ and $\theta^H_{\gamma t}$ at a given value of the photon energy are well separated. As a consequence of this the screening is effective.

If $m_t > 30$ GeV (we are especially interested in values of m_t too large for using an e^+e^- machine) we can fit the data for $W^+ \to t\bar{b}\gamma$ to the theoretical prediction and obtain a combined test for the W and quark anomalous magnetic moment and the value of the current t mass. Indeed the distribution $d^2\Gamma/dsdt$ is very sensitive to m_t due to the presence of an angular zero for any fixed photon energy. In table 2 we give the values t^o and $\cos\theta^o_{\gamma t}$ at which the zeros occur for different values of $E\gamma$ and m_t. According to the results of section 2, the physical null zone is further restricted by

$$s \geq \frac{3}{2}\ m_t^2 + 3\,m_b^2 \quad , \qquad t \leq \frac{2}{3}\ M_W^2 - 2\,m_b^2$$

We are now in a position to compare the sequence of zeros for different values of m_t. When E_γ is small we find that $\theta^o_{\gamma t}(E_\gamma, m_t)$ and $\theta^o_{\gamma t}(E_\gamma, m_t')$ are very close even for large $\Delta m_t = m_t' - m_t$. However, for

E_γ large, when the two quarks tend to be collinear with the photon in the backward direction, the same difference becomes appreciable. This difference is perhaps within experimental feasibility. Moreover, the t quark momentum is still large in the region of the zero, typically of O(5-10 GeV) if $m_t \approx$ 40-50 GeV, reducing the uncertainty connected with the jet direction. For large values of m_t the t-jet direction should be measured with good precision from the hadronic decay modes. Finally we compute the doubly differential decay rate which is given by

$$\frac{d^2\Gamma}{dsdt} = \frac{G_F\alpha}{32\pi^2}\,\frac{\tilde{\Gamma}}{M_W} \quad , \qquad \tilde{\Gamma} = \frac{16}{9}\, h^2\, H\left(\tilde{s},\ \tilde{t},\ \tilde{M}_W^2\right)$$

where

$$h^2 = \frac{m_t^2}{N^2}\left(2s + 3t - 2M_W^2 - 3m_t^2\right)^2 \ , \quad N^2 = \left(m_t^2 - s\right)\left(m_t^2 - t\right)u - m_t^2 u^2$$

and the b quark mass has been neglected. H is a function of the scaled variables $\tilde{s} = s/m_t^2$, ... The expression for H is very lengthy and will be given in the appendix. In fig. 7, results are presented for m_t = 40 GeV and s = 0.4 M_W^2, 0.5 M_W^2. Different values for m_t can be obtained by a simple rescaling in the function H.

The variable t is restricted by kinematical cuts once s is fixed. For large t the u-channel starts to dominate and $d^2\Gamma$ increases very rapidly.

4. Examples of Radiation Representations

One of the important consequences of the theorem of ref. 1 is the so-called radiation representation for the amplitude relative to a given process with one external photon. This result is equivalent to the existence of a null zone, which however can be unphysical. In the radiation representation the amplitude is expressed as a sum of terms, each of which is automatically zero on the null zone. According to ref. 1, this representation for an arbitrary diagram can be reduced to the special case of a vertex by means of a quasi-vertex expansion.

Then for an arbitrary vertex with n lines, the amplitude is expressed as

$$\sum_{i=2}^{n-1} p_i k \left(\frac{e_1}{p_1 \cdot k} - \frac{e_i}{p_i \cdot k}\right)\left(\frac{j_n}{p_n \cdot k} - \frac{j_i}{p_i \cdot k}\right) \quad ,$$

j_i being the product of the current for photon emission by the ith leg and the remainder of the amplitude [1].

In the following we derive examples for the case where fermions and vector bosons are present.

Null zones and radiation representations are independent from photon polarization but in order to deal with particles with spin we found it convenient to introduce explicit photon polarization vectors. According to ref. [6] when a photon is radiated by a fermion line, we use $\varepsilon_\mu^\pm$ given by

$$\varepsilon_\mu^\pm = \frac{1}{\sqrt{2}} \left(\varepsilon_\mu^\parallel \pm i\varepsilon_\mu^\perp\right)$$

$$\varepsilon_\mu^\parallel = \frac{i}{N}\left[\left(p_+ \cdot k\right)p_{-\mu} - \left(p_- \cdot k\right)p_{+\mu}\right] \quad , \qquad \varepsilon_\mu^\perp = \frac{1}{N} \; \varepsilon_{\mu\alpha\beta\nu}\, p_+^\alpha\, p_-^\beta\, k^\nu \tag{4.1}$$

$$N^2 = 2\left(p_+ \cdot p_-\right)\left(p_+ \cdot k\right)\left(p_- \cdot k\right)$$

where p_+, p_- and k denote the outgoing antiparticle, particle and photon momenta. Fermion masses have been neglected.

First we consider a n = 3 process, namely W decay. The three diagrams of fig. 5 contribute. The amplitude may be cast in the form

$$A = (2\pi)^4 i \; \frac{g}{2\sqrt{2}} \sum_{i=1,3} A_i$$

$$A_1 = \frac{e_+}{2p_+\cdot k} \left(2\varepsilon\cdot p_+ J_0 + \bar{u}\,\not{\eta}\,\not{k}\,\not{\varepsilon}\,\gamma_+ \,v\right)$$

$$A_2 = -\frac{e_-}{2p_-\cdot k} \left(2\varepsilon\cdot p_- J_0 - \bar{u}\,\not{\eta}\,\not{k}\,\not{\varepsilon}\,\gamma_+ \,v + 2k\cdot\eta\,\bar{u}\,\not{\varepsilon}\,\gamma_+ v - 2\varepsilon\cdot\eta\,\bar{u}\,\not{k}\,\gamma_+ v\right) \tag{4.2}$$

$$A_3 = -\frac{e_W}{2Q\cdot k} \left(2\varepsilon\cdot Q\,J_0 + 2k\cdot\eta\,\bar{u}\,\not{\varepsilon}\,\gamma_+ v - 2\varepsilon\cdot\eta\,\bar{u}\,\not{k}\,\gamma_+ v\right)$$

where

$$J_0 = \bar{u}\,\not{\eta}\,\gamma_+ v \quad , \qquad \gamma_+ = 1 + \gamma_5$$

and η^μ is the vector boson polarization.

Using the definition (4.1) and the relation

$$\not{\varepsilon}^{\pm} = -\frac{i}{2\sqrt{2}\,N} \left(\not{k}\,\not{p}_-\,\not{p}_+\,\gamma_\pm - \not{p}_-\,\not{p}_+\,\not{k}\,\gamma_\mp \mp 2p_+\cdot p_-\,\not{k}\,\gamma^5\right)$$

we get

$$A(\lambda) = -(2\pi)^4\,\frac{1}{4}\,g \left\{ \frac{p_+\,p_-}{N} \left[e_+ + e_- - e_W\,\frac{k\,(p_+ - p_-)}{k\,(p_+ + p_-)} \right] J_0 \right.$$
$$+ \frac{1-\lambda}{N}\left(\frac{e_+}{p_+\cdot k} + \frac{e_-}{p_-\cdot k}\right)\, p_+\cdot k\left[(p_-\cdot k)J_0 - p_-\cdot\eta J_1\right] - \frac{\lambda}{N}\,(p_+\cdot p_-)(k\cdot\eta)\left(\frac{e_-}{p_-\,k} + \frac{e_W}{Q\,k}\right) J_1$$
$$\left. - i\sqrt{2}\;\varepsilon^\lambda\cdot\eta\left(\frac{e_-}{p_-\cdot k} + \frac{e_W}{Q\cdot k}\right) J_1 \right\} \qquad (\lambda = \pm 1) \tag{4.3}$$

where

$$J_1 = \bar{u}\,\not{k}\,\gamma_+\,v$$

$A(\lambda)$ is automatically zero on the null zone. We observe that the $\not k \gamma^5$ term which appears in the expression for $\not\epsilon^{\pm}$ cannot be neglected in this example. The cancellation is already manifest in eq. (4.2) where the fermion masses have disappeared after some manipulations on the γ matrices; hence the null zone exists for any set of masses but its location in the s,t plane is mass dependent.

Next we discuss the radiation representation for a generic four fermion process, which may take place both in the s and t channels via the exchange of a vector boson. Notations are given in fig. 6. $F(p_i,e_i)$ denotes a fermion with momentum p_i and charge e_i flowing in the p_i direction. Six different processes are simulated in fig. 6. They are

1) $F_a(p_2,e_a)+\bar{F}_a(p_1,e_a)\to F_b(-p_3,e_b)+\bar{F}_b(-p_4,e_b)+\gamma(k)$

2) $F_a(p_2,e_a)+\bar{F}_b(p_1,e_b)\to F_c(-p_3,e_c)+\bar{F}_d(-p_4,e_d)+\gamma(k) \quad e_a-e_b=e_c-e_d$

3) $F_a(p_2,e_a)+F_b(p_4,e_b)\to F_a(-p_1,e_a)+F_b(-p_3,e_b)+\gamma(k)$

4) $F_a(p_2,e_a)+\bar{F}_b(p_3,e_b)\to F_a(-p_1,e_a)+\bar{F}_b(-p_4,e_b)+\gamma(k)$

5) $F_a(p_2,e_a)+F_b(p_4,e_b)\to F_c(-p_1,e_c)+F_d(-p_3,e_d)+\gamma(k) \quad e_a-e_c=e_b-e_d$

6) $F_a(p_2,e_a)+\bar{F}_b(p_3,e_b)\to F_c(-p_1,e_c)+\bar{F}_d(-p_4,e_d)+\gamma(k) \quad e_a+e_c=e_b+e_d$.

We introduce the spinors $U_{i\pm}$ [6]

$$\frac{1}{2}\gamma_{\pm}U_{i\pm}=U_{i\pm} \quad , \qquad \bar{U}_{i\pm}\frac{1}{2}\gamma_{\mp}=\bar{U}_{i\pm}$$

where the U_i are arbitrary spinors. Thus

$$U_\pm = u_\pm \quad \text{for incoming particle} \quad ,$$

$$\bar{U}_\pm = \bar{u}_\pm \quad \text{for outgoing particle} \quad ,$$

$$U_\pm = v_\mp \quad \text{for outgoing antiparticle} \quad ,$$

$$\bar{U}_\pm = \bar{v}_\mp \quad \text{for incoming antiparticle} \quad ;$$

also

$$[\Gamma]^\pm_L = \bar{U}_{1\pm}\,\Gamma\,U_{2\pm} \quad , \qquad [\Gamma]^\pm_R = \bar{U}_{3\pm}\,\Gamma\,U_{4\pm} \quad ,$$

where Γ is an arbitrary string of γ matrices and the label i refers to the momentum carried by the fermion.

Since we are dealing with processes where the photon can be radiated by two different fermion lines (denoted by L and R), we have to introduce two sets of polarization vectors, $\varepsilon^\pm_L$ and $\varepsilon^\pm_R$. However, the arguments given in ref. 6 show how they are connected.

$$\begin{aligned} \varepsilon^\pm_L &= e^{i\varphi_\pm}\,\varepsilon^\pm_R + \lambda_\pm\, k \\ e^{i\varphi_\pm} &= N_L\, N_R\,(\alpha \pm \beta) \end{aligned} \tag{4.4}$$

where $N_{L,R}$ is the usual normalization factor and

$$\begin{aligned} \alpha = {} & (p_1\cdot k)(p_4\cdot k)(p_2\cdot p_3) + (p_3\cdot k)(p_2\cdot k)(p_1\cdot p_4) \\ & - (p_4\cdot k)(p_2\cdot k)(p_1\cdot p_3) - (p_3\cdot k)(p_1\cdot k)(p_2\cdot p_4) \end{aligned} \tag{4.5}$$

$$\beta = \varepsilon_{\mu\alpha\beta\nu}\, p_4^\alpha\, p_3^\beta (p_2 - p_1)^\mu\, k^\nu\, k\cdot(p_1 + p_2) \tag{4.6}$$

This way of expressing $e^{i\varphi_\pm}$ is particularly convenient because for all the neutral channels $k\cdot(p_1+p_2) = 0$ is one of the null zone equations. There $e^{i\varphi_\pm} = N_L N_R \alpha$. For all the processes where $e_{int} = 0$, the amplitude is split into

$$A_L^\pm = (2\pi)^4 \frac{2i}{\Delta_L} \left\{ g_+ [\gamma^\mu]_R^+ + g_- [\gamma^\mu]_R^- \right\} \bar{U}_1 \left\{ e_1 \, L^\pm(p_1) \, \gamma^\mu (g_+\gamma_+ + g_-\gamma_-) - e_2 \, \gamma^\mu (g_+\gamma_+ + g_-\gamma_-) \, R^\pm(p_2) \right\} U_2$$

and (4.7)

$$A_R^\pm = (2\pi)^4 \frac{2i}{\Delta_R} \, e^{i\varphi_\pm} \, \bar{U}_3 \left\{ e_3 \, L^\pm(p_3) \, \gamma^\mu (g_+\gamma_+ + g_-\gamma_-) - e_4 \, \gamma^\mu (g_+\gamma_+ + g_-\gamma_-) \, R^\pm(p_4) \right\} U_4 \left\{ g_+ [\gamma^\mu]_L^+ + g_- [\gamma^\mu]_L^- \right\}$$

where

$$\Delta_L = 2p_3\cdot p_4 + M^2 , \qquad \Delta_R = 2p_1\cdot p_2 + M^2$$

$$L^\pm(p_1) = \not{\epsilon}_L^\pm \, \frac{\not{p}_1 + \not{k}}{2p_1\cdot k} , \quad R^\pm(p_2) = \frac{\not{p}_2 + k}{2p_2\cdot k} \, \not{\epsilon}_L^\pm \quad \ldots \quad . \tag{4.8}$$

The current-current terms will be denoted by J^{ij}

$$J^{ij} = [\gamma^\mu]_R^i \, [\gamma^\mu]_L^j \qquad (i,j = +,-) \tag{4.9}$$

for $e_{int} = 0$ the $\not{k}\,\gamma^5$ term in $\not{\epsilon}^\pm$ can be dropped and the action of $L^\pm, R^\pm$ becomes particularly simple [6]. The final expression can be further simplified using [7],

$$\left[\not{p}_i\right]_R^{\pm} \left[\not{p}_j\right]_L^{\pm} = p_i \cdot p_j \; J^{\pm\pm} \quad \text{for } i = 1,\ j = 4 \text{ or } i = 2,\ j = 3$$

$$\left[\not{p}_i\right]_R^{\pm} \left[\not{p}_j\right]_L^{\mp} = p_i \cdot p_j \; J^{\pm\mp} \quad \text{for } i = 1,\ j = 3 \text{ or } i = 2,\ j = 4 \,. \tag{4.10}$$

Finally we introduce

$$C_L = \frac{1}{N_L \Delta_L} \,, \qquad C_R^{\pm} = \frac{e^{i\varphi_{\pm}}}{N_R \Delta_R}$$

$$K^{ij}\left(p_\ell, p_m\right) = - p_\ell \cdot p_m \; J^{ij} + \left[\not{p}_\ell\right]_R^i \left[\not{p}_m\right]_L^j \qquad (i,j = +- \;;\; \ell, m = 1, \ldots, 4) \,. \tag{4.11}$$

It follows

$$\begin{aligned}
A^+(+,+) &= (2\pi)^4 \, 4\sqrt{2} \, ig_+^2 \left(e_1 C_L + e_3 C_R^+\right) K^{++}\left(p_2, p_4\right) \\
A^+(+,-) &= (2\pi)^4 \, 4\sqrt{2} \, ig_+ g_- \left(e_2 C_L + e_3 C_R^+\right) K^{+-}\left(p_1, p_4\right) \\
A^+(-,+) &= (2\pi)^4 \, 4\sqrt{2} \, ig_+ g_- \left(e_1 C_L + e_4 C_R^+\right) K^{-+}\left(p_2, p_3\right) \\
A^+(-,-) &= (2\pi)^4 \, 4\sqrt{2} \, ig_-^2 \left(e_2 C_L + e_4 C_R^+\right) K^{--}\left(p_1, p_3\right) \,.
\end{aligned} \tag{4.12}$$

$A^-(i,j)$ is

$$\begin{aligned}
A^-(+,+) &= (2\pi)^4 \, 4\sqrt{2} \, ig_+^2 \left(e_2 C_L + e_4 C_R^-\right) K^{++}\left(p_1, p_3\right) \\
A^-(+,-) &= (2\pi)^4 \, 4\sqrt{2} \, ig_+ g_- \left(e_1 C_L + e_4 C_R^-\right) K^{+-}\left(p_2, p_3\right) \\
A^-(-,+) &= (2\pi)^4 \, 4\sqrt{2} \, ig_+ g_- \left(e_2 C_L + e_3 C_R^-\right) K^{-+}\left(p_1, p_4\right) \\
A^-(-,-) &= (2\pi)^4 \, 4\sqrt{2} \, ig_-^2 \left(e_1 C_L + e_3 C_R^-\right) K^{--}\left(p_2, p_4\right) \,.
\end{aligned} \tag{4.13}$$

From these formulas all the helicity amplitudes for the $e_{int} = 0$ process can be reconstructed. [As an illustration, $A^+(+,+)$ in the case of $e^+e^- \to \mu^+\mu^-\gamma$ denotes the amplitude for $e^+(-) + e^-(+) \to \mu^+(-) + \mu^-(+) + \gamma(+)$.]

Charge conservation for $e_{int} = 0$ implies $e_1 = e_2$ and $e_3 = e_4$. The null zone is described by the equations

$$p_2 \cdot k = - p_1 \cdot k = - \rho p_3 \cdot k = \rho p_4 \cdot k \quad ; \quad \rho = \frac{e_1}{e_3} = \frac{e_2}{e_4} \quad . \tag{4.14}$$

On the null zone we can easily prove that

$$\begin{aligned} p_1 \cdot p_2 &= p_3 \cdot p_4 \quad , \quad \alpha = - \frac{1}{\rho} N_L^2 \\ N_R &= \frac{1}{\rho} N_L \quad , \quad \beta = 0 \end{aligned} \tag{4.15a}$$

Thus

$$\Delta_R = \Delta_L \quad , \quad e^{i\varphi_\pm} = - 1 \tag{4.15b}$$

and

$$e_1 C_L + e_3 C_R^\pm = 0 \text{ or } A^i(j,\ell) = 0 \qquad (i,j,\ell = +,-) \quad . \tag{4.16}$$

When $e_{int} \neq 0$ there is an additional diagram

$$A^\pm = A_L'^\pm + A_R'^\pm + A_{int}^\pm$$

The prime at $A_{L,R}$ means that they differ from the previous ones for the inclusion of $\not{k}\gamma^5$ terms in $\not{\epsilon}^\pm$.

For the neutral current exchange the cancellation takes place between terms of the L fermion line and terms of the R one. Now the situation is more complicated and the amplitude A_{int} must be decomposed into two parts which cancel separately the contribution from A_L and A_R [1].

Using

$$\frac{1}{\Delta_L \Delta_R} = - \frac{1}{2k \cdot \left(p_1 + p_2\right)} \frac{1}{\Delta_L} - \frac{1}{2k \cdot \left(p_3 + p_4\right)} \frac{1}{\Delta_R} \quad .$$

We get after some manipulation

$$\frac{A'_L}{(2\pi)^4} = \frac{4i}{\Delta_L} g^2\left(e_1 \frac{p_1\cdot\varepsilon}{p_1\cdot k} - e_2 \frac{p_2\cdot\varepsilon}{p_2\cdot k}\right) J^{++} + \frac{2i}{\Delta_L} g^2\left(\frac{e_1}{p_1\cdot k} + \frac{e_2}{p_2\cdot k}\right)\left[\gamma^\mu\right]^+_R \left[\not\varepsilon \not k \gamma^\mu\right]^+_L$$

$$- \frac{4i}{\Delta_L} g^2 \frac{e_2}{p_2\cdot k}\left\{[\not k]^+_R [\not\varepsilon]^+_L - [\not\varepsilon]^+_R [\not k]^+_L\right\}$$

$$\frac{A'_R}{(2\pi)^4} = \frac{4i}{\Delta_R} g^2\left(e_3 \frac{p_3\cdot\varepsilon}{p_3\cdot k} - e_4 \frac{p_4\cdot\varepsilon}{p_4\cdot k}\right) J^{++} + \frac{2i}{\Delta_R} g^2\left(\frac{e_3}{p_3\cdot k} + \frac{e_4}{p_4\cdot k}\right)\left[\not\varepsilon \not k \gamma^\mu\right]^+_R \left[\gamma^\mu\right]^+_L$$

$$- \frac{4i}{\Delta_R} g^2 \frac{e_4}{p_4\cdot k}\left\{[\not\varepsilon]^+_R [\not k]^+_L - [\not k]^+_R [\not\varepsilon]^+_L\right\} \tag{4.17}$$

$$\frac{A_{int}}{(2\pi)^4} = \frac{4i}{\Delta_L} \frac{g^2 e_5}{k.\left(p_1 + p_2\right)} \left[\gamma^\nu\right]^+_R \left[\gamma^\mu\right]^+_L \left\{\varepsilon_\mu k_\nu - \varepsilon_\nu k_\mu + \delta_{\mu\nu}\, \varepsilon\cdot\left(p_1 + p_2\right)\right\}$$

$$+ \frac{4i}{\Delta_R} \frac{g^2 e_5}{k\cdot\left(p_3 + p_4\right)} \left[\gamma^\nu\right]^+_R \left[\gamma^\mu\right]^+_L \left\{\varepsilon_\mu k_\nu - \varepsilon_\nu k_\mu - \delta_{\mu\nu}\, \varepsilon\cdot\left(p_3 + p_4\right)\right\}.$$

We have assumed that the charged vector boson couples to the fermions with a $1+\gamma^5$ term. The strategy of the cancellation is already clear in eq. (4.17) and we only use the properties of $\varepsilon^\pm$ to simplify the expression.

$$\frac{A}{(2\pi)^4} = \frac{4i}{\Delta_L} g^2\left[e_1 \frac{p_1\cdot\varepsilon}{p_1\cdot k} - e_2 \frac{p_2\cdot\varepsilon}{p_2\cdot k} + e_5 \frac{\left(p_1+p_2\right)\cdot\varepsilon}{\left(p_1+p_2\right)\cdot k}\right]J^{++}$$

$$+ \frac{4i}{\Delta_R} g^2\left[e_3 \frac{p_3\cdot\varepsilon}{p_3\cdot k} - e_4 \frac{p_4\cdot\varepsilon}{p_4\cdot k} - e_5 \frac{\left(p_3+p_4\right)\cdot\varepsilon}{\left(p_3+p_4\right)\cdot k}\right]J^{++}$$

$$+ \frac{2i}{\Delta_L} g^2\left(\frac{e_1}{p_1\cdot k} + \frac{e_2}{p_2\cdot k}\right)\left[\gamma^\mu\right]_R^+ \left[\not\varepsilon \not k \gamma^\mu\right]_L^+ + \frac{2i}{\Delta_R} g^2\left(\frac{e_3}{p_3\cdot k} + \frac{e_4}{p_4\cdot k}\right)\left[\not\varepsilon \not k \gamma^\mu\right]_R^+ \left[\gamma^\mu\right]_L^+$$

$$- \frac{4i}{\Delta_L} g^2 \left[\frac{e_2}{p_2\cdot k} - \frac{e_5}{\left(p_1+p_2\right)\cdot k}\right]\left\{[\not k]_R^+ [\not\varepsilon]_L^+ - [\not\varepsilon]_R^+ [\not k]_L^+\right\}$$

$$- \frac{4i}{\Delta_R} g^2 \left[\frac{e_4}{p_4\cdot k} + \frac{e_5}{\left(p_3+p_4\right)\cdot k}\right]\left\{[\not\varepsilon]_R^+ [\not k]_L^+ - [\not k]_R^+ [\not\varepsilon]_L^+\right\} \quad .$$

We find

$$p_{1,2}\cdot\varepsilon_L^{\pm} = \mp \frac{i}{\sqrt{2}\, N_L}\left(p_1\cdot p_2\right)\left(p_{1,2}\cdot k\right)$$

$$\left[\not\varepsilon_L^\lambda \not k \gamma^\mu\right]_L^+ = (1+\lambda)\ \frac{i\sqrt{2}}{N_L}\ p_1\cdot k \left\{p_2^\mu[\not k]_L^+ - p_2\cdot k\left[\gamma^\mu\right]_L^+\right\}$$

$$\left[\not\varepsilon_L^\lambda\right]_L^+ = \frac{i\lambda}{\sqrt{2}\, N_L}\ p_1\cdot p_2\ [\not k]_L^+$$

$$\left[\not\varepsilon_L^\lambda\right]_R^+ = \frac{i}{\sqrt{2}\, N_L}\ \left\{(1+\lambda)\ k\cdot p_3\left[\not p_2\right]_R^+ + (1+\lambda)\ p_2\cdot p_4[\not k]_R^+ - (1-\lambda)\ p_1\cdot p_3[\not k]_R^+ \right.$$
$$\left. - (1-\lambda)\ k\cdot p_4\left[\not p_1\right]_R^+ + \lambda p_1\cdot p_2[\not k]_R^+\right\} \qquad (\lambda = \pm 1) \quad .$$

Equivalent results hold for the R-type terms. Thus

$$\left[\gamma^\mu\right]_R^+ \left[\not\varepsilon_L^\lambda \not k \gamma^\mu\right]_L^+ = (1+\lambda)\ \frac{i\sqrt{2}}{N_L}\ p_1\cdot k\left\{p_2\cdot\left(p_1+p_4\right)J^{++} - \left[\not p_2\right]_R^+ \left[\not p_4\right]_L^+\right\}$$

$$\left[\not\varepsilon^\lambda \not k \gamma^\mu\right]_R^+ \left[\gamma^\mu\right]_L^+ = (1+\lambda)\ \frac{i\sqrt{2}}{N_R}\ p_3\cdot k\left\{p_4\cdot\left(p_2+p_3\right)J^{++} - \left[\not p_2\right]_R^+ \left[\not p_4\right]_L^+\right\}$$

$$[\not k]_R^+ \left[\not\epsilon_L^\lambda\right]_L^+ - \left[\not\epsilon_L^\lambda\right]_R^+ [\not k]_L^+ =$$

$$\frac{i}{\sqrt{2}\, N_L}(1+\lambda)\Big[\big(p_3\cdot k\big)\big(p_2\cdot p_3\big) - \big(p_2\cdot p_4\big)\big(p_1\cdot p_4\big) - \big(p_2\cdot p_4\big)\big(p_2\cdot p_3\big)\Big] J^{++}$$

$$+\frac{i}{\sqrt{2}\, N_L}(1-\lambda)\Big[\big(p_1\cdot p_3\big)\big(p_1\cdot p_4\big) + \big(p_1\cdot p_3\big)\big(p_2\cdot p_3\big) - \big(p_4\cdot k\big)\big(p_1\cdot p_4\big)\Big] J^{++}$$

$$+\frac{i}{\sqrt{2}\, N_L}(1+\lambda)\big(p_3\cdot k - p_2\cdot p_4\big)\left[\not p_2\right]_R^+ \left[\not p_4\right]_L^+ + \frac{i}{\sqrt{2}\, N_L}(1-\lambda)\; p_1\cdot p_3\left[\not p_2\right]_R^+ \left[\not p_4\right]_L^+$$

$$-\frac{i}{\sqrt{2}\, N_L}(1+\lambda)\; p_2\cdot p_4\left[\not p_1\right]_R^+ \left[\not p_3\right]_L^+ + \frac{i}{\sqrt{2}\, N_L}(1-\lambda)\big(p_1\cdot p_3 - p_4\cdot k\big)\left[\not p_1\right]_R^+ \left[\not p_3\right]_L^+ .$$

Finally,

$$\frac{A(\lambda)}{(2\pi)^4} = \frac{2\sqrt{2}}{N_L\Delta_L}\, g^2\big(p_1\cdot p_2\big)\left[e_1 + e_2 - e_5\,\frac{k\cdot\big(p_2 - p_1\big)}{k\cdot\big(p_2 + p_1\big)}\right] J^{++}$$

$$+\frac{2\sqrt{2}}{N_R\Delta_R}\, e^{i\varphi_\lambda}\, g^2\big(p_3\cdot p_4\big)\left[e_3 + e_4 + e_5\,\frac{k\cdot\big(p_4 - p_3\big)}{k\cdot\big(p_4 + p_3\big)}\right] J^{++}$$

$$-\frac{2\sqrt{2}}{N_L\Delta_L}\, g^2(1+\lambda)\left(\frac{e_1}{p_1\cdot k} + \frac{e_2}{p_2\cdot k}\right) p_1\cdot k\left\{p_2\cdot\big(p_1 + p_4\big)\; J^{++} - K_{24}^{++}\right\}$$

$$-\frac{2\sqrt{2}}{N_R\Delta_R}\, e^{i\varphi_\lambda}\, g^2(1+\lambda)\left(\frac{e_3}{p_3\cdot k} + \frac{e_4}{p_4\cdot k}\right) p_3\cdot k\left\{p_4\cdot\big(p_2 + p_3\big)\; J^{++} - K_{24}^{++}\right\}$$

$$+\frac{2\sqrt{2}}{N_L\Delta_L}\, g^2\left[\frac{e_2}{p_2\cdot k} - \frac{e_5}{\big(p_1 + p_2\big)\cdot k}\right]\left[a_L^\lambda\, J^{++} + b_L^\lambda\, K_{24}^{++} + c_L^\lambda\, K_{13}^{++}\right]$$

$$-\frac{2\sqrt{2}}{N_R\Delta_R}\, g^2\, e^{i\varphi_\lambda}\left[\frac{e_4}{p_4\cdot k} + \frac{e_5}{\big(p_3 + p_4\big)\cdot k}\right]\left[a_R^\lambda\, J^{++} + b_R^\lambda\, K_{24}^{++} + c_R^\lambda\, K_{13}^{++}\right]$$

$$K_{ij}^{++} = \left[\not p_i\right]_R^+ \; \left[\not p_j\right]_L^+ .$$

$$a_L^\lambda = (1+\lambda)\Big[\big(p_3\cdot k\big)\big(p_2\cdot p_3\big) - \big(p_1\cdot p_4\big)\big(p_2\cdot p_4\big) - \big(p_2\cdot p_3\big)\big(p_2\cdot p_4\big)\Big]$$

$$+ (1-\lambda)\Big[\big(p_1\cdot p_3\big)\big(p_1\cdot p_4\big) + \big(p_1\cdot p_3\big)\big(p_2\cdot p_3\big) - \big(p_4\cdot k\big)\big(p_1\cdot p_4\big)\Big] .$$

$$b_L^\lambda = (1+\lambda)\left(p_3\cdot k - p_2\cdot p_4\right) + (1-\lambda)\ p_1\cdot p_3 \ .$$

$$c_L^\lambda = (1+\lambda)\ p_2\cdot p_4 + (1-\lambda)\left(p_1\cdot p_3 - p_4\cdot k\right) \ .$$

The R coefficients are obtained by the exchange $p_1 \leftrightarrow p_3$, $p_2 \leftrightarrow p_4$.

As an example of the general formulas we give the radiation representation for the process $e^-\mu^- \to e^-\mu^-\gamma$ in massless QED. Now

p_2 momentum of incoming e^-

p_4 momentum of incoming μ^-

$-p_1$ momentum of outgoing e^-

$-p_3$ momentum of outgoing μ^- .

Moreover, $e_a = e_b = -e$ and $g_+ = g_- = -(i/2)e$. The amplitude is decomposed into a sum of terms $A^{\lambda_\gamma}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ where λ_γ is the photon polarization while λ_1, λ_2, λ_3 and λ_4 denote respectively the incoming e^-,μ^- and outgoing e^-,μ^- polarizations. Thus

$$A^+(+,+,+,+) = (2\pi)^4\ i\sqrt{2}\ e^3\left(C_L + C_R^+\right) K^{++}\left(p_2,p_4\right)$$

$$A^+(-,+,-,+) = (2\pi)^4\ i\sqrt{2}\ e^3\left(C_L + C_R^+\right) K^{+-}\left(p_1,p_4\right)$$

$$A^+(+,-,+,-) = (2\pi)^4\ i\sqrt{2}\ e^3\left(C_L + C_R^+\right) K^{-+}\left(p_2,p_3\right)$$

$$A^+(-,-,-,-) = (2\pi)^4\ i\sqrt{2}\ e^3\left(C_L + C_R^+\right) K^{--}\left(p_1,p_3\right) \ .$$

Summing over lepton polarizations

$$A^+ = (2\pi)^4\, i\sqrt{2}\, e^3\left(C_L + C_R^+\right)\left[K^{++}\left(p_2,p_4\right) + K^{+-}\left(p_1,p_4\right) + K^{-+}\left(p_2,p_3\right) + K^{--}\left(p_1,p_3\right)\right].$$

Similarly

$$A^- = (2\pi)^4\, i\sqrt{2}\, e^3 \left(C_L + C_R^-\right)\left[K^{++}(1,3) + K^{+-}(2,3) + K^{-+}(1,4) + K^{--}(2,4)\right]$$

where

$$K^{ij}\left(p_\ell, p_m\right) = -p_\ell \cdot p_m\, \bar{u}_{3i}\, \gamma^\mu\, u_{4i}\, \bar{u}_{1j}\, \gamma^\mu\, u_{2j} + \bar{u}_{3i}\, \not{p}_\ell\, u_{4i}\, \bar{u}_{1j}\, \not{p}_m\, u_{2j} \quad (i,j = +,-),$$

$$C_L = \frac{1}{2p_3 \cdot p_4\, N_L}\ , \qquad C_R^\pm = \frac{e^{i\varphi_\pm}}{2p_1 \cdot p_2\, N_R}\ .$$

The presence of the common factor $C_L + C_R^\pm$ guarantees the vanishing of the amplitude on the null zone.

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Appendix

For $W^+ \to t\bar{b}\gamma$ we compute

$$\frac{d^2\Gamma(\lambda)}{dsdt} = \frac{G_F\alpha}{18\pi^2 M} h^2 H^\lambda$$

where λ denotes the photon polarization. It follows

$$H^+ = G_1^2 I_1 - G_1 G_5 I_{22}^+ - G_1 G_6 I_{23} + G_5^2 I_{52} + G_5 G_6 I_{63} + G_6^2 I_{53}$$

$$\begin{aligned} H^- = {} & \left(G_1 + 2G_2\right)\left[\left(G_1 + 2G_2\right) I_1 - 2G_3 I_{21} - G_5 I_{22}^- + G_6 I_{23} - G_4 I_3 + 2G_7 I_4\right] \\ & + 4G_3^2 I_{51} + G_5^2 I_{52} + G_6^2 I_{53} + 2G_3 G_5 I_{61} - 2G_3 G_6 I_{62} - G_5 G_6 I_{63} \\ & + 2G_3 G_4 I_{71} + G_5 G_4 I_{72} - G_6 G_4 I_{73} - 4G_3 G_7 I_{81} - 2G_5 G_7 I_{82} + 2G_6 G_7 I_{83} \\ & + G_4^2 I_9 - 2G_4 G_7 I_{10} + 4G_7^2 I_{11} \end{aligned}$$

where, neglecting m_b

$$G_1 = adF_1 + (ae + d)F_2 \;, \qquad G_2 = deF_3$$

$$G_3 = dF_3 \;, \quad G_4 = G_3 \;, \quad G_5 = F_4 \;, \quad G_6 = aF_4 \;, \quad G_7 = dF_4$$

$$a = \frac{1}{2}(1 - \tilde{s}) \;, \quad b = \frac{1}{2}\left(\tilde{t} - \tilde{M}^2\right) \;, \quad c = \frac{1}{2}\left(\tilde{u} - \tilde{M}^2 - 1\right)$$

$$d = \frac{1}{2}(1 - \tilde{t}) \;, \quad e = -\frac{1}{2}\tilde{u} \;, \quad f = \frac{1}{2}\left(\tilde{s} - \tilde{M}^2\right)$$

$$F_1 = \frac{1}{\tilde{u}\left(\tilde{M}^2 - \tilde{s}\right)} \;, \quad F_2 = \frac{1}{(1 - \tilde{t})\left(\tilde{s} - \tilde{M}^2\right)} \;, \quad F_3 = \frac{1}{\tilde{u}(\tilde{t} - 1)} \;, \quad F_4 = F_2 \;.$$

Moreover,

$$I_1 = \frac{16}{\tilde{M}^2} bc - 8a$$

$$I_{21} = -16d + 16\frac{c}{\tilde{M}^2}\left(dc - fa + eb\right)$$

$$I_{22}^{\lambda} = 16\left(dg_2 + eg_1\right) + 16\frac{g_3}{\tilde{M}^2}\left(dc - fa + eb\right) + 16\lambda\left(d^2 + 2dea\right)$$

$$I_{23} = 16\frac{f}{\tilde{M}^2}\left(dc - fa + eb\right)$$

$$I_3 = 48d\ , \quad I_4 = 16\left(d + \frac{fb}{\tilde{M}^2}\right)$$

$$I_{51} = 16\left(\frac{c^2}{\tilde{M}^2} - 1\right)de\ , \quad I_{52} = 16\left(\frac{g_3^2}{\tilde{M}^2} + 2n^2\right)de\ , \quad I_{53} = 16\frac{f^2}{\tilde{M}^2}de$$

$$I_{61} = 32\left(g_2 + \frac{cg_3}{\tilde{M}^2}\right)de\ , \quad I_{62} = 32\left(e + \frac{fc}{\tilde{M}^2}\right)de\ , \quad I_{63} = 32\frac{fg_3}{\tilde{M}^2}de$$

$$I_{71} = 16\left(\frac{c}{\tilde{M}^2}fd + de\right)\ , \quad I_{72} = 32\frac{g_3}{\tilde{M}^2}fd\ , \quad I_{73} = 32\frac{f^2}{\tilde{M}^2}d$$

$$I_{81} = 16\left(\frac{fc}{\tilde{M}^2} + e\right)d\ , \quad I_{82} = 16\frac{g_3 f}{\tilde{M}^2}d\ , \quad I_{83} = 16\frac{f^2}{\tilde{M}^2}d$$

$$I_9 = 16\left(de - 2\frac{d}{\tilde{M}^2}fc\right)$$

$$I_{10} = 32\,de + 16\frac{f}{\tilde{M}^2}\left(fa - be + cd\right)$$

$$I_{11} = 8\frac{f^2}{\tilde{M}^2}a\ ,$$

with

$$2a = 1 - \tilde{s}\ , \quad 2b = \tilde{t} - \tilde{M}^2\ , \quad 2c = \tilde{u} - \tilde{M}^2 - 1$$

$$2d = -\tilde{u}\ , \quad 2e = 1 - \tilde{t}\ , \quad 2f = \tilde{s} - \tilde{M}^2$$

$$g_1 = -ad\ , \quad g_2 = ae + d\ , \quad g_3 = eb - dc$$

$$n^2 = -2ade - d^2\ .$$

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Table 1

m_t = 25 GeV			m_t = 30 GeV		
$\frac{s}{M_W^2}$	$\cos\theta^L_{\gamma t}$	$\cos\theta^H_{\gamma t}$	$\frac{s}{M_W^2}$	$\cos\theta^L_{\gamma t}$	$\cos\theta^H_{\gamma t}$
0.3	0.30	-0.44	0.4	0.51	-0.50
0.4	0.01	-0.51	0.5	0.13	-0.59
0.5	-0.23	-0.59	0.6	-0.28	-0.71
0.6	-0.44	-0.69			
0.7	-0.67	-0.80			

Table 2

m_t = 40 GeV			m_t = 50 GeV		
E_γ(GeV)	$\frac{t^0}{M_W^2}$	$\cos\theta^0_{\gamma t}$	E_γ(GeV)	$\frac{t^0}{M_W^2}$	$\cos\theta^0_{\gamma t}$
16	0.517	-0.45	12	0.591	-0.15
20	0.583	-0.63	16	0.658	-0.91
24	0.650	-0.94			

Figure Captions

Fig. 1. The process $1+2 \to 3 + \ldots + n + \gamma$

Fig. 2. The null zone for $1+2 \to 3+\gamma$.
Conventions are: $s = -(p_1+p_2)^2$, $t = -(p_1+p_3)^2$.

Fig. 3. The W radiative decay.

Fig. 4. The γ phase space for $W(Q) \to F(p_-)+\bar{F}(p_+)+\gamma(k)$.
The dashed line is the null zone. The physical region is represented by the slashed area. Moreover $s = -(p_+ + p_-)^2$ and $t = -(p_- + k)^2$.

Fig. 5. The diagrams contributing to W radiative decay.

Fig. 6. The process $FF \to FF\gamma$.

Fig. 7. $d^2\Gamma/dsdt$ for $W^+ \to t\bar{b}\gamma$. $s = -(p_t+p_b)^2$, $t = -(p_t+p_\gamma)^2$.
Moreover $m_t = 40$ GeV, $M_W = 80$ GeV.

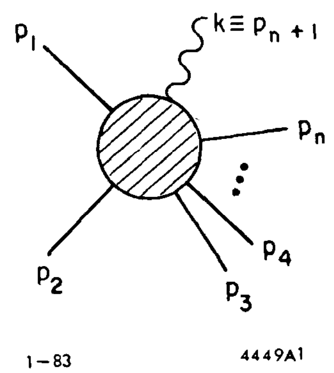

Fig. 1

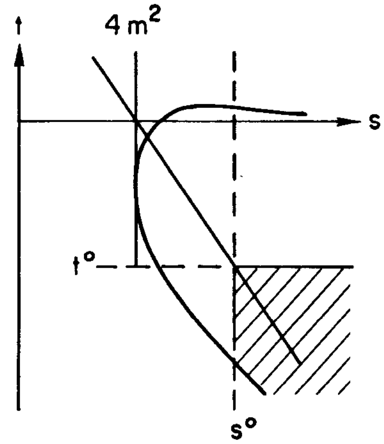

Fig. 2

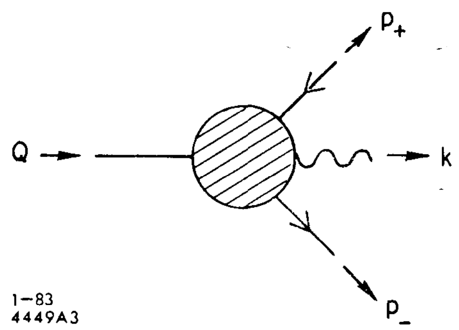

Fig. 3

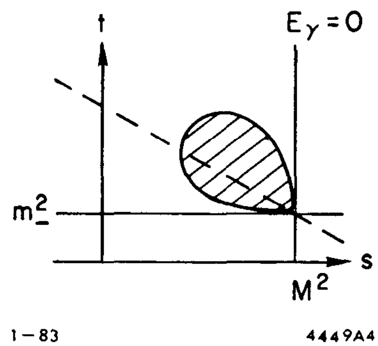

Fig. 4

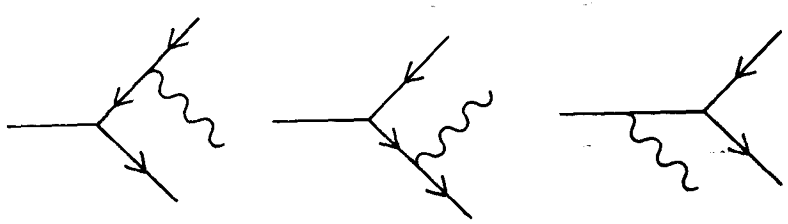

Fig. 5

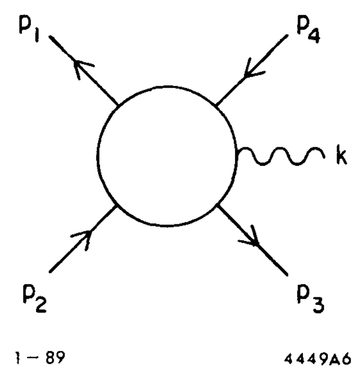

Fig. 6

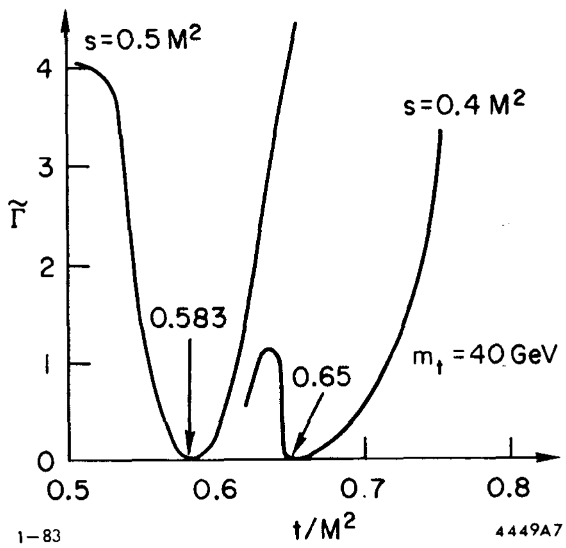
s=0.5 M²
s=0.4 M²
0.583
0.65
m_t =40 GeV
Γ̃
4
3
2
1
0
0.5
0.6
0.7
0.8
t/M²
1–83
4449A7

Fig. 7