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CHIRAL SYMMETRY AND CHIRAL SYMMETRY BREAKING*

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Chapter 1.

INTRODUCTION

These lectures concern the dynamics of fermions in strong interaction with gauge fields. Systems of fermions coupled by gauge forces have, as we shall see, a very rich structure of global symmetries, which I will call chiral symmetries. These lectures will focus on the realization of chiral symmetries and the causes and consequences of their spontaneous breaking.

From one viewpoint, the study of fermionic symmetries is a classical topic in high-energy physics: Some of the earliest applications of spontaneously broken symmetry in particle physics, including the classic papers of Gell-Mann and Levy [1] and Nambu and Jona-Lasinio [2], dealt with the chiral symmetries of the strong interactions, and much of the progress of theoretical particle physics in the 1960's occurred through exploration of the phenomenological consequences of this spontaneous chiral symmetry breaking. That progress has been summarized in numerous reviews (e.g., [3],[4],[5]). Our understanding of the underlying mechanism of chiral symmetry breaking, however, has not advanced so rapidly; to a great extent, the elucidation of this mechanism is still an open problem in the theory of the strong interactions.

The past few years have seen a renewed interest in this problem for three reasons, reflecting the successes of gauge theories in describing the fundamental interactions. First, numerical treatments of strong interaction gauge theories, especially in their lattice formulation, have been approaching a quantitative calculation of the hadron spectrum. There is, then, a need for physical ideas about quark dynamics of a power to match these numerical calculations. Secondly, the gauge-theoretic descriptions of the weak interactions have focused attention on the problem of explaining the quark and lepton spectrum. From the perspective of the gauge theories, the quark and lepton masses are simply parameters of chiral symmetry breaking in the interactions which determine the structure of these particles. Dynamical theories of the fermion mass matrix thus require an understanding of chiral symmetry in systems different from the usual strong interactions; such theories often require that chiral symmetry is realized in an unfamiliar way. Finally, the viewpoint provided by gauge theories has led to some striking qualitative conclusions about chiral symmetry which might form the basis of a more detailed theory. With these reasons in mind, I will try, in these lectures, to summarize our present understanding of the basis of chiral symmetry and its realization. My goal will be to bring this theory together at an intuitive level, to indicate what I think are its basic elements. I hope that in the next few years we can turn this qualitative theory into a quantitative one.

The plan of these lectures is as follows: I will begin with a brief introduction to the basic formalism and concepts of chiral symmetry breaking. Then, in sections 3 through 5, I will present some explicit calculations of chiral symmetry breaking in gauge theories, treating first parity-invariant and then chiral models. These calculations are meant to be illustrative rather than accurate; they make use of unjustified mathematical approximations which serve (I hope) to make the physics more clear. In sections 6 and 7, I will discuss some formal constraints on chiral symmetry breaking which will illuminate and extend the results of our more explicit analysis. Finally, sections 8 and 9 will present a brief review of the phenomenological theory of chiral symmetry breaking and will discuss some applications of this theory to problems in weak-interaction physics.

These lectures will be intuitive and rather ideosynchratic in tone. I should therefore apologize to the reader in advance for what I will omit: My presentation will be deliberately ahistorical; because of this, I will not review even major papers whose results do not bear directly on the issues I will discuss. In particular, I will not discuss the interaction between the study of chiral symmetry breaking and renormalization theory (e.g., [6]) which led to the realization that only asymptotically free theories allow chiral symmetry breaking at reasonable momentum scales [7,8]. (This point has been reviewed, for the light it sheds on asymptotic freedom, in [9].) Similarly, I will not concentrate on summarizing the most recent results in this field. At various points in my presentation I have deliberately oversimplified the arguments of papers I review in order to avoid technical digressions; the reader should be alert for warnings that this is going on. I hope that the streamlining which results from these three types of omissions brings the basic ideas more clearly into view. I have tried to make these lectures accessible to beginners in the field, although I should warn the reader that I address the problems of this field with the specific concerns of a particle physicist. I should also note that I have deliberately omitted an important and rapidly developing aspect of my subject, the realization of chiral symmetry in lattice gauge theories, since this topic is reviewed thoroughly in John Kogut's lectures in this volume.

Here, then, is the theory of chiral symmetries, in its present fascinating but incomplete state. I hope the reader of these lectures will be moved to develop this theory, and to ponder its lingering problems.

Chapter 2.

BASIC NOTIONS

2.1 WHAT ARE CHIRAL SYMMETRIES?

I will begin this set of lectures by presenting some basic formalism and describing, in the simplest terms, the physics I wish to explore. First of all, I should introduce the language I will use to describe the elements of this physics.

Chiral symmetries are normally introduced as formal symmetries of the massless Dirac Lagrangian. This Lagrangian, including a coupling to an (Abelian) gauge field, takes the form:

$$L = \bar{\psi} i\gamma^{\mu}(\partial_{\mu} - igA_{\mu})\psi \quad ; \quad \bar{\psi} = \psi^{\dagger}\gamma^{0} \quad . \tag{2.1}$$

There is an obvious symmetry

$$\psi \to e^{i\alpha}\psi \quad , \quad \bar{\psi} \to \bar{\psi} e^{-i\alpha} \quad . \tag{2.2}$$

which corresponds to fermion-number conservation. But the massless theory has another symmetry, using γ^5:

$$\psi \to \exp[i\alpha\gamma^5]\psi \quad , \quad \bar{\psi} \to \bar{\psi}\exp[i\alpha\gamma^5] \quad . \tag{2.3}$$

The exponentials cancel because $\{\gamma^5, \gamma^{\mu}\} = 0$.

To understand these symmetries physically, it is helpful to choose the following representation of the Dirac matrices:

$$\gamma^0 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix} \quad ,$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \quad , \quad \alpha^i = \gamma^0\gamma^i = \begin{pmatrix} -\sigma^i & \\ & \sigma^i \end{pmatrix} \quad . \tag{2.4}$$

In this representation, the Dirac Hamiltonian

$$H = \int \psi^\dagger [\vec{\alpha}\cdot(\vec{p}+g\vec{A}) - gA^0]\psi \qquad (2.5)$$

can be split apart. If we write

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

then

$$H = \int (\psi^\dagger{}_R[\vec{\sigma}\cdot(\vec{p}+g\vec{A}) - gA^0]\psi_R + \psi^\dagger{}_L[(-\vec{\sigma})\cdot(\vec{p}+g\vec{A}) - gA^0]\psi_L) \quad . \qquad (2.6)$$

Note that the positive-energy states of ψ_R have positive helicity and that the positive-energy states of ψ_L have negative helicity: ψ_R and ψ_L describe, respectively, right and left-handed massless fermions. The fermion numbers of ψ_R and ψ_L are (formally) separately conserved; this is the origin of the extra symmetry.

The two pieces of (2.6) are not actually of a different form. We can write ψ_R as a second type of ψ_L by applying charge conjugation, defining

$$\psi_{L2} = \psi^\dagger{}_R\sigma_2 \quad , \quad \psi^\dagger{}_{L2} = \sigma_2\psi_R \quad . \qquad (2.7)$$

Then, using the identity $\sigma_2\sigma_i\sigma_2 = -\sigma^T{}_i$ and an integration by parts, we can rewrite the first term of (2.6):

$$\int \psi^\dagger{}_R[\vec{\sigma}\cdot(\vec{p}+g\vec{A}_0) - gA^0]\psi_R = \int \psi^\dagger{}_{L2}[(-\vec{\sigma})\cdot(\vec{p}-g\vec{A}) + gA^0]\psi_{L2} \qquad (2.8)$$

as a ψ_L Hamiltonian with the opposite sign of the charge.

This construction is readily generalized to non-Abelian gauge theories. In the non-Abelian case, the Lagrangian is built from the covariant derivative

$$D_\mu = \partial_\mu - igA^a{}_\mu t^a{}_r \qquad (2.9)$$

where the index a runs over the generators of the gauge group and the matrices $t^a{}_r$ represent these generators in the representation r to which the fermions are assigned. Representation matrices for complex conjugate representations are related by

$$t^a{}_{\bar r} = -(t^a{}_r)^* = t^{aT}{}_r \quad . \tag{2.10}$$

This notation allows us to recast the Hamiltonian for a ψ_R as that of a ψ_L in the representation r:

$$\int \psi^\dagger{}_R[\vec\sigma\cdot(\vec p+g\vec A\cdot t) - gA^0\cdot t]\psi_R$$

$$= \int \psi^\dagger{}_{L2}[(-\vec\sigma)\cdot(\vec p+g\vec A\cdot(-t^T)) - gA^0\cdot(-t^T)]\psi_{L2} \quad . \tag{2.11}$$

In this notation, the most general Hamiltonian coupling massless fermions to gauge fields may be written compactly in the following form:

$$H = \sum_{\text{reps. } r} \sum_{i=1}^{n_r} \int \psi^\dagger{}_{Lri}[(-\vec\sigma)\cdot(\vec p+g\vec A\cdot t_r)-gA^0\cdot t_r]\psi_{Lri} \quad . \tag{2.12}$$

Once H has been cast into this form, it is easy to read off the global symmetries of this system: For each representation r, this Hamiltonian is (formally) invariant to the general unitary transformation:

$$\psi_{Lri} \rightarrow U_{ij}\psi_{Lrj} \quad . \tag{2.13}$$

Actually, one of these formal symmetries is illusory. A certain quantum correction to this theory, the Adler-Bell-Jackiw anomaly [10,11], spoils the conservation of the overall charge current

$$J_\mu = \sum_{ri} \bar\psi_{Lri}\, \gamma_\mu\, \psi_{Lri} \quad . \tag{2.14}$$

(In terms of ψ_L and ψ_R, this is the axial-vector current.) I will argue in detail, in section 4, that one should simply consider this symmetry as broken explicitly. The full global symmetry of the theory is, therefore,

$$G = [\prod_r U(n_r)]/U(1) \quad . \tag{2.15}$$

G is the group of <u>chiral symmetries</u> of such a theory.

As an example of this notation, consider the case of the strong interactions, which are described by a set of two almost massless Dirac fermions (quarks) coupled in the triplet representation to an SU(3) gauge group. These fermions may be written as left-handed fermions (henceforth, L-fermions), two in the 3 and two in the $\bar{3}$ representation of color SU(3). In the limit of zero quark masses, the chiral symmetry of this theory is

$$G = SU(2) \times SU(2) \times U(1) \quad . \qquad (2.16)$$

There is, as I have noted in the introduction, considerable evidence that the full group G is a symmetry of the strong interactions; however, hadrons do not form multiplets classified by G, but only by SU(2) × U(1) (isospin × baryon number). A part of G must, then, be spontaneously broken. It is our major goal in these lectures to understand why G should be spontaneously broken, and in what pattern. To begin, let me present a relatively simple intuitive argument which contains the right physics and leads to the right conclusion. This argument is due, in its original form, to Nambu and Jona-Lasinio [2], who, in turn, borrowed it from the theory of superconductivity. The gauge coupling of color SU(3) is asymptotically free and becomes strong at large distances. Let me assume it becomes arbitrarily strong. Let me think about the change in the structure of the vacuum state of this theory as I raise g from zero. Imagine that we can integrate over the quantum fluctuations of the gauge field; then H takes the form:

$$H = H_d + H_{o-d} \qquad (2.17)$$

where H_d is diagonal in the number of quark-antiquark pairs and H_{o-d} changes the number of such pairs. This decomposition is indicated in fig. 1. H_{o-d} is of order g^2 and is a small perturbation when g is small. In this regime it makes sense to approximate H by H_d;

diagonalizing H_d yields a ground state close to free-field vacuum. Now, slowly increase g. If the fermions have zero mass and experience attractive interactions, H_d decreases as g increases. H_{o-d}, of course, increases. At some value of g it becomes appropriate to treat H_{o-d} as our zeroth-order problem, and H_d as a perturbation. But H_{o-d} changes the number of pairs, so its ground state has an indefinite number of fermion pairs. We would still expect the ground state to be invariant to Lorentz transformations; hence these pairs must have vacuum quantum numbers: zero total momentum and angular momentum. The only pairs one can form from 3 and $\bar{3}$ L-fermions and their (right-handed) antiparticles which satisfy this condition are those of the form of fig. 2 and the corresponding pair of antifermions. The pair shown in fig. 2 carries a net charge under the transformations:

$$\psi_{L3i} \rightarrow e^{i\alpha}\psi_{L3i} \quad , \quad \psi_{L\bar{3}i} \rightarrow e^{i\alpha}\psi_{L\bar{3}i} \quad ,$$
$$\psi_{L3i} \rightarrow U_{ij}\psi_{L3j} \quad , \quad \psi_{L\bar{3}i} \rightarrow V_{ij}\psi_{L\bar{3}j} \quad . \qquad (2.18)$$

(The indices i,j = 1,2 are isospin labels.) The presences of an indefinite number of such pairs in the vacuum breaks these symmetries. More formally, we have found that the ground state $|\Omega\rangle$ of H has the property that an operator which destroys a fermion pair has a nonzero vacuum expectation value.

Let us assume that $|\Omega\rangle$ gives pair annihilation operators the rather simple expectation value:

$$\langle\Omega|\psi_{L3i}\psi_{L\bar{3}j}|\Omega\rangle = \Delta\delta_{ij} \qquad (2.19)$$

(where $\Delta \neq 0$), corresponding to equal condensation of pairs of each isospin. This expression is preserved by the transformations

$$\psi_{L3i} \to e^{i\alpha}\psi_{L3i} \quad , \quad \psi_{L\bar{3}i} \to e^{-i\alpha}\psi_{L\bar{3}i}$$

and

$$\psi_{L3i} \to U_{ij}\psi_{L3j} \quad , \quad \psi_{L\bar{3}i} \to \psi_{L\bar{3}j}U^{-1}{}_{ji} \quad . \tag{2.20}$$

This is an SU(2) × U(1) group of unbroken symmetries which corresponds precisely to isospin × baryon number. The remaining three symmetry directions of (2.16) must be spontaneously broken symmetries. Goldstone's theorem requires that each must generate a massless Goldstone boson. The three π mesons have the right quantum numbers to be identified with these three bosons. (This may be checked by rewriting $\psi_{L\bar{3}i}$ as a ψ_R; then the symmetries (2.20) correspond to vector currents and the broken symmetries to axial currents.)

This argument summarizes the basic points of physics that I wish to discuss in these lectures. It remains only to carry out this analysis more completely and precisely. We will make at least a little progress toward that goal.

2.2 THE EFFECTIVE POTENTIAL FORMALISM

Our first priority is to learn how to do a more quantitative computation of chiral symmetry breaking. Basically, we need to know how to test whether the energy of the vacuum is lowered if some fermion bilinear acquires a nonzero vacuum expectation value. If the quantity acquiring a vacuum expectation value is a scalar field ϕ, one has at one's disposal an object called the effective potential [12,13]. This object is equal to the energy of the vacuum under the constraint that the vacuum expectation value of ϕ has some definite value ϕ_0; it can also be computed straightforwardly in perturbation theory [14]. One need only compute this effective potential and minimize it with respect to ϕ_0 to determine the vacuum value of ϕ. For chiral symmetry breaking (henceforth, χSB) there is a similar construction due to Cornwall, Jackiw and Tomboulis [15].* Their construction is set up as follows:

If chiral symmetry is even formally exact, it will set to zero such vacuum expectation values as (2.19). For definiteness, consider expectation values of the (Dirac fermion) operator

$$\bar{\psi}_R \psi_L = \epsilon^{\alpha\beta} \psi_{L2\alpha} \psi_{L\beta} \tag{2.21}$$

where $\alpha, \beta = 1,2$ are spinor indices. To produce a vacuum expectation value of this operator, we must, in principle, turn on some external field (analogous to a magnetic field orienting a potentially ferromagnetic system), construct the ordered vacuum in the presence of this field, and then see if the order in this vacuum survives when we turn the field off. I will try to carry out that procedure explicitly. For the sake of familiarity and ease in finding minus signs, I will work with ψ_R, ψ_L for the moment and induce $\langle \bar{\psi}\psi \rangle \neq 0$.

Begin by writing a theory of massless fermions (in Euclidean space-time)

$$Z = \int \psi_A \exp[-\int (\bar{\psi} \not{D} \psi + 1/4\, F^2)] \quad . \tag{2.22}$$

The second term in the exponent is the gauge field Lagrangian; I will suppress this term from here on. Add to (2.22) a source K which induces χSB:

$$Z[K(x,y)] = \exp(-W[K]) = \int \psi_A \exp[-\int (\bar{\psi} \not{D} \psi - \bar{\psi}(x)\, K(x,y)\, \psi(y))] \ . \tag{2.23}$$

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*An early version of this formalism appears in [16]. A very clear discussion of it may be found in [17].

It is useful to take K to be nonlocal; this allows us to adjust K(x,y) so that

$$\langle\psi(x)\bar{\psi}(y)\rangle_K = \int \frac{d^4k}{(2\pi)^4} \exp[-ik\cdot(x-y)] \frac{1}{-i\not{k} + \Sigma(k^2)} \tag{2.24}$$

for any given generalized mass term $\Sigma(k^2)$. We must now determine for which function $\Sigma(k^2)$ this condition will be stable if we turn off K. Let us define $S(x,y) = \langle\psi(x)\bar{\psi}(y)\rangle$; then

$$S(x,y) = \frac{\delta}{\delta K(y,x)} (-\log Z[K]) = \frac{\delta}{\delta K(y,x)} W[K] \tag{2.25}$$

(assuming $\langle\psi\rangle = \langle\bar{\psi}\rangle = 0$).

It would be useful to exchange Σ or S for K as our basic variable. This can be done by making a Legendre transformation (replacing the Helmholtz by the Gibbs free energy). Define

$$\Gamma = W[K] - \int dxdy\, S(x,y)\, K(y,x) \tag{2.26}$$

(2.26) implies that

$$\frac{\delta\Gamma}{\delta S} = \int \frac{\delta K}{\delta S}\cdot\left[\frac{\delta W}{\delta K} - S\right] - K$$

or simply

$$\frac{\delta\Gamma}{\delta S(x,y)} = K(y,x) \tag{2.27}$$

S(x,y) can be stable if $\delta\Gamma/\delta S(x,y) = 0$. $\Gamma[S]$ is called the <u>effective action.</u>

Let us now compute Γ. Since the fermion propagator will eventually be given by the specified function S, it is convenient to evaluate (2.23) by taking S as the zeroth-order propagator or S^{-1} as the zeroth-order action. Rewrite (2.23) as:

$$\exp(-W[K]) = \int\psi \exp[-\int(\bar{\psi}S^{-1}\psi + \bar{\psi}(\not{D}-S^{-1})\psi + K\psi\bar{\psi})]$$

$$= (\det S^{-1})\cdot\exp[(\text{all vacuum diagrams})]$$

$$= \exp[\log\det(S^{-1}) + \mathrm{Tr}(\not{D}-S^{-1})S - \mathrm{Tr}\, KS + (\text{diagrams})] \tag{2.28}$$

where, in the last line, "diagrams" includes only those with at least one gluon vertex. Then

$$W = -\mathrm{Tr}\,\log(S^{-1}) - \mathrm{Tr}(\not{D}-S^{-1})S + \mathrm{Tr}\, KS - (\text{diagrams})$$

$$\Gamma = -\mathrm{Tr}\,\log S^{-1} + \mathrm{Tr}(S^{-1}-\not{D})S - (\text{diagrams}) \quad . \tag{2.29}$$

The diagrams in (2.29) may be divided into two classes as shown in fig. 3: Those whose only vertices are gluon vertices (fig. 3a) and those which involve the additional 2-fermion interaction $[\partial - S^{-1} - K]$ (fig. 3b). The diagrams of the second type involve the external source K; this source is determined by the condition that the exact propagator of the theory should be S(x,y). One should therefore adjust K to cancel any corrections to the propagator. This adjustment, however, makes the two diagrams shown in fig. 3 cancel precisely. One can check that all diagrams of the second type cancel against diagrams of the first type, leaving uncancelled only those diagrams which have no propagator corrections. These remaining diagrams may be characterized as "2-particle-irreducible" (2PI), i.e., as diagrams which cannot be split apart by removing 2 lines. Our final expression for $\Gamma[S]$ is (2.29) evaluated with only 2-particle-irreducible diagrams. A few of these diagrams are shown in fig. 4. This final expression for Γ does not require the explicit value of the source K; it is simply a function of S. Thus, one can readily compute $\delta\Gamma/\delta S$.

To get some idea of how this formalism works, we might compute the variation $\delta\Gamma/\delta S$ from an approximate form for Γ obtained by including only the simple diagram of fig.5a in the evaluation of (2.29):

$$\frac{\delta}{\delta S}\Gamma \approx \frac{\delta}{\delta S}\left(+\,\mathrm{Tr}\,\log(S) + \mathrm{Tr}(S^{-1}-\not{\partial})S - (\text{fig. 5a})\right)$$

$$= S^{-1} - S^{-1}SS^{-1} + (S^{-1}-\not{\partial}) + (\text{fig. 5b}) \quad . \qquad (2.30)$$

The sign change in the last term comes from the factor (-1) for a closed fermion loop. Thus, in this approximation, $\delta\Gamma/\delta S = 0$ implies

$$S^{-1} = \not{\partial} - (\text{fig. 5b}) \quad . \qquad (2.31)$$

This is precisely the Hartree-Fock approximation to the equations of motion, obtained after integrating out the gauge field. To see this, note that if $\Delta_{\mu\nu}(x-y)$ is the gauge field propagator, the equation of motion for $\psi(x)$ may be written as:

$$\gamma^\mu(\partial_\mu - igA_\mu)\psi(x) = 0$$

$$\longrightarrow \quad \gamma^\mu(\partial_\mu + g^2 \int d^4y\, \Delta_{\mu\nu}(x-y)\, \bar{\psi}(y)\gamma^\nu\psi(y))\psi(x) = 0$$

$$\underset{\text{Hartree-Fock}}{\longrightarrow} \quad \not{\partial}\psi(x) + g^2 \int d^4y\, \gamma^\mu\, \Delta_{\mu\nu}(x-y)\, \langle\psi(x)\bar{\psi}(y)\rangle\, \gamma^\nu\psi(x) \quad . \qquad (2.32)$$

which is identical to (2.31). In the theory of superconductivity, eq. (2.31) is called the gap equation.

Let me end our discussion of the effective action by mentioning two problems with this formalism:

First, the formalism is not gauge invariant, because it relies on nonlocal source terms. One can fix up the gauge invariance of the source term by writing this term using line integrals of A_μ: $\int K(x,y)\, \bar{\psi}(x)\, P(\exp[ig\int A_\mu d\ell^\mu])\psi(y)$, where P denotes path-ordering. All the formalism goes through unchanged. But now K must regulate the value of $\langle\bar{\psi}(x)\, P\, \exp[ig\int A_\mu d\ell^\mu]\psi(y)\rangle$, and so does not cancel from Γ so easily. I will avoid this problem by choosing a gauge; Landau gauge ($\partial_\mu A^\mu = 0$) is particularly convenient [18]. If the fermion bilinear which obtains a vacuum expectation value also breaks global gauge invariance, then one has a worse problem; the formalism is inconsistent unless one also allows a mass for the gauge boson field. But this situation can be treated by letting Γ depend also on the gauge boson propagator $\Delta(x-y)$, and insisting also that $\delta\Gamma/\delta\Delta = 0$. This produces coupled equations for S and Δ. For example, in the approximation of keeping only fig. 5a in Γ, these equations are:

$$S^{-1} = \not{\partial} - (\text{fig. } 5b)$$

$$\Delta^{-1} = -(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu) - (\text{fig. } 5c) \qquad . \tag{2.33}$$

Second, and more seriously, the effective action is not bounded below. To see this, we can try the following exercise: Start with a free-field action $L = \bar{\psi}(\not{\partial}+m)\psi$ and to solve for the best exact propagator S using the effective action formalism. Take $S(p) = [i\not{p}+\Sigma(p)]^{-1}$, for simplicity. Then the evaluation of (2.29) gives:

$$\Gamma[\Sigma(p)] = -\text{Tr}\log(-i\not{p}+\Sigma) + \text{Tr}(\Sigma-m)\frac{1}{(-i\not{p}+\Sigma)}$$

$$= \int \frac{d^4p}{(2\pi)^4}\left[-2\log(p^2+\Sigma^2(p)) + 4\,\frac{(\Sigma(p)-m)\Sigma(p)}{p^2+\Sigma^2(p)}\right]$$

$$\frac{\delta\Gamma}{\delta\Sigma(p)} = -\frac{4\Sigma}{p^2+\Sigma^2} + \frac{4(\Sigma-m+\Sigma)}{p^2+\Sigma^2} - 8\,\frac{(\Sigma-m)\Sigma^2}{(p^2+\Sigma^2)^2} = \frac{(\Sigma-m)(p^2-\Sigma^2)}{(p^2+\Sigma^2)^2} \tag{2.34}$$

The correct value $\Sigma(p) = m$ is a solution to the condition $\delta\Gamma/\delta\Sigma = 0$. However, there is another solution at $\Sigma(p) = p$. Further, the shape of $\Gamma[\Sigma]$ is such that, after attaining a local minimum at $\Sigma(p) = m$, it rises until $\Sigma = p$ and then falls off, plummeting to $(-\infty)$ as $\Sigma \to \infty$. The stationarity condition $\delta\Gamma/\delta S = 0$ does produce the correct solution for Σ, but only if interpreted with care.

What can one do about these problems? In these lectures, I will simply apologize and ignore them. Banks and Raby [17] suggest a restriction to sources K(x,y) which are nonlocal only in space, not in time. This solves the problem with boundedness-below, but it sacrifices covariance and also does not allow one to get the specific solutions I will discuss in the next section. In any event, this formalism is good enough to let us do computations with real physics in them; let us, then, accept it and proceed.

I should, however, make note of an alternative, more formal but more precise, criterion for χSB. Consider the Dirac propagator, for fermions with small mass m, in a fixed background gauge field:

$$\langle\psi(x)\bar{\psi}(y)\rangle = (\not{D}+m)^{-1}(x,y) \tag{2.35}$$

By multiplying and dividing by (D-m) and using

$$\Delta_A = -(\not{D}+m)(\not{D}-m) = (-D^2+m + 1/4\ g\sigma^{\mu\nu}F_{\mu\nu}) \tag{2.36}$$

one can write

$$\langle\bar{\psi}\psi(x)\rangle = -m\cdot\Delta_A^{-1}(x,x) \quad . \tag{2.37}$$

We can convert this result to a result in an interacting gauge theory by inserting (2.37) under the functional integral over A_μ. χSB in the zero mass theory is equivalent to the behavior:

$$\langle\Delta_A^{-1}(x)\rangle_A \propto m^{-1} \text{ as } m \to 0 \quad . \tag{2.38}$$

Banks and Casher [19] have used this criterion to give a heuristic argument for χSB in lattice gauge theories; perhaps it can be put to more general use.

Chapter 3.

EXPLICIT COMPUTATIONS OF χSB — GLUONS

3.1 A SIMPLE STABILITY ANALYSIS

In the previous section, I introduced the formalism of Cornwall, Jackiw and Tomboulis [15] and described a simple approximation to this formalism in which we can compute whether χSB is induced by one-gluon exchange. In this lecture, I would like to analyze that approximation in some detail, to try to obtain a more concrete picture of the mechanism of χSB.

Recall that we had found the following condition for the value fermion propagator in the absence of external sources:

$$\frac{\delta}{\delta S}\Gamma[S] = 0 \tag{3.1}$$

where

$$\Gamma \underline{=} -\mathrm{Tr}\,\log S^{-1} + \mathrm{Tr}(S^{-1}-\not{p})S - (\text{2PI diagrams}) \quad . \tag{3.2}$$

In this lecture, I will make the truncation:

$$(\text{2PI diagrams}) = (\text{fig. 5a}) \quad . \tag{3.3}$$

Let us attempt to stationarize the truncated Γ and find the propagator which solves (3.1). As a first step, I will work within the following, very much simplified, framework: First, although renormalization effects in non-Abelian gauge theories cause the coupling constant to be a function of momentum scale, I will ignore this effect and consider g to be fixed and scale-invariant. Secondly, since I expect that, for g^2

sufficiently small, the vacuum is chirally symmetric, I will restrict my attention to studying the stability of the symmetric vacuum. Finally, I will use the Feynmann rules of QED, with one fermion flavor. Let me, then, insert into Γ the trial form

$$S = \frac{1}{-i\not{p} + \Sigma(p)} \tag{3.4}$$

expand Γ into quadratic order in Σ, and look for unstable modes. A similar computation was done by Banks and Raby in [17].

Using the trial form (3.4)

$$\begin{aligned} \mathrm{Tr}\log S^{-1} &= \int \frac{d^4p}{(2\pi)^4} \log\det(-i\not{p}+\Sigma(p^2)) \\ &= \int \frac{d^4p}{(2\pi)^4} \log(p^2+\Sigma^2) \\ &= (\mathrm{const}) + \int \frac{d^4p}{(2\pi)^4}\, 2\, \frac{\Sigma^2(p)}{p^2} + O(\Sigma^4) \end{aligned} \tag{3.5}$$

$$\begin{aligned} \mathrm{Tr}(S^{-1}-\not{\partial})S &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\Sigma \frac{1}{(-i\not{p}+\Sigma)}\right] = \int \frac{d^4p}{(2\pi)^4} \frac{4\Sigma^2}{p^2+\Sigma^2} \\ &\int \frac{d^4p}{(2\pi)^4}\, 4\, \frac{\Sigma^2(p)}{p^2} + O(\Sigma^4) \end{aligned} \tag{3.6}$$

so that

$$\Gamma = \int \frac{d^4p}{(2\pi)^4} \left[\frac{2\Sigma^2}{p^2} + \cdots\right] - (\mathrm{diagrams}) \quad . \tag{3.7}$$

Note that these essentially kinematical terms stabilize the chirally symmetric state $\Sigma = 0$; the interactions must counteract this effect. To see how, we must expand fig. 5a in powers of Σ; this expansion is sketched in fig. 6, in which the double lines represent full propagators and the heavy dots insertions of Σ. The first term on the right-hand side is independent of Σ. The second term vanishes if we work in Landau gauge ($\partial_\mu A^\mu=0$); let us make that choice from here on. This leaves the third diagram, whose value is:

$$- \frac{1}{2}(ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{\left(g^{\mu\nu} - \frac{(k-p)^\mu (k-p)^\nu}{(k-p)^2}\right)}{(k-p)^2} \mathrm{Tr}\left[\gamma_\mu \frac{\Sigma(p)}{p^2} \gamma_\mu \frac{\Sigma(k)}{k^2}\right]$$

$$= \frac{g^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi^4)} \frac{1}{(k-p)^2} \frac{\Sigma(p)}{p^2} \frac{\Sigma(k)}{k^2} \cdot 12$$

$$= \frac{3g^2}{16\pi^5} \int_0^\infty dk\ k\ \Sigma(k) \int_0^\infty dp\ p\ \Sigma(p) \int_0^\pi d\theta\ \sin^2\theta \frac{1}{[k^2 + p^2 - 2kp\cos\theta]} \quad . \tag{3.8}$$

To evaluate this, we need

$$\int_0^\pi d\theta\ \sin^2\theta \frac{1}{(k^2 + p^2 - 2kp\cos\theta)} = \frac{\pi}{2} \frac{1}{kp} \cdot \left(\min\left\{\frac{k}{p}, \frac{p}{k}\right\}\right) \tag{3.9}$$

then

$$(\text{fig. } 6) = (\text{const}) + \frac{3g^2}{32\pi^4} \int_0^\infty dk\ \Sigma(k) \int_0^\infty dp\ \Sigma(p) \exp[-|\log k/p|] \quad . \tag{3.10}$$

The full term in Γ quadratic in Σ is therefore:

$$\Gamma_2 = \frac{1}{4\pi^2} \int_0^\infty dp\ p\ \Sigma^2(p) - \frac{3g^2}{32\pi^4} \int_0^\infty dk\ \Sigma(k) \int_0^\infty dp\ \Sigma(p) \exp[-|\log k/p|] \quad . \tag{3.11}$$

To diagonalize this quadratic form, we may exploit the scale-invariance of our restricted problem. To do this, set

$$p = p_0 \exp(\eta) \quad , \quad k = p_0 \exp(\xi) \quad , \quad \Sigma(p) = 1/p\ \sigma(\eta) \quad . \tag{3.12}$$

In (3.12), p_0 is a fixed reference momentum. With this change of variables (3.11) becomes

$$\Gamma_2 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\eta d\xi \; \sigma(\eta) \left(\delta(\eta-\xi) - \frac{3g^2}{8\pi^2} \exp[-|\eta-\xi|] \right) \sigma(\xi) \quad . \tag{3.13}$$

If we Fourier transform in the variables η, ξ:

$$\sigma(q) = \int d\eta \; \exp[iq\eta] \; \sigma(\eta) \tag{3.14}$$

then Γ_2 takes the form:

$$\Gamma_2 = \frac{1}{4\pi^2} \int \frac{dq}{3\pi} \sigma(q)\sigma(-q) \left[1 - \left(\frac{3g^2}{4\pi^2} \right) \frac{1}{1 + q^2} \right] \quad . \tag{3.15}$$

In this truncation of the effective action, the chirally symmetric vacuum is unstable for

$$\frac{g^2}{4\pi} > \left(\frac{\pi}{3} \right) \tag{3.16}$$

Apparently, the gauge coupling must become quite strong to induce chiral symmetry breaking.

How does this computation change if we consider a non-Abelian gauge theory? For the moment, think only about theories of Dirac fermions. I will, for definiteness, consider a theory with n flavors of Dirac fermions belonging to some representation r of the gauge group. Let us denote the dimensionality of r by d_r and the representation matrices of the gauge generators by $t^a{}_r$. The quadratic Casimir operator, the analogue of the rotational invariant L^2 for a general Lie group, is defined by

$$\sum_a t^a{}_r t^a{}_r = C_2(r) \cdot \underline{1} \tag{3.17}$$

where $\underline{1}$ denotes the unit matrix. For future use, I will make a few more definitions: Denote by G the representation in which generators of the gauge group lie (the adjoint representation); then d_G is the number of generators. Define $C(r)$ by

$$\mathrm{Tr}\, t^a{}_r t^b{}_r = \delta^{ab} C(r) \quad ; \quad C(r) = d_r/d_G \, C_2(r) \quad . \tag{3.18}$$

For the group SU(N), $d_G = N^2-1$, $C_2(G) = N$, and, for the fundamental representation N (or N),

$$C_2(N) = \frac{N^2 1}{2N} \quad , \quad C(N) = \frac{1}{2} \tag{3.19}$$

The computation we have just done is readily generalized to this context. The kinematical terms given by (3.5) and (3.6) must be multiplied by the total number of Dirac fermions $(n \cdot d_r)$. Since the gauge boson vertex now contains a factor of $t^a{}_r$, the contribution (3.10) is multiplied by

$$n\, \mathrm{Tr}\, t^a{}_r t^a{}_r = n\, d_r\, C_2(r) \quad . \tag{3.20}$$

Then the criterion for an instability becomes:

$$\frac{g^2 C_2(r)}{4\pi} > \frac{\pi}{3} \quad . \tag{3.21}$$

For quarks, in the 3 of SU(3)

$$\frac{g^2}{4\pi} > \frac{\pi}{4} \quad . \tag{3.22}$$

For the N of SU(N), as $N \to \infty$, (3.20) involves the combination $(g^2 N)$, as it should.

3.2 EFFECTS OF COUPLING CONSTANT RENORMALIZATION

The analysis we have just done ignores all effects of coupling constant renormalization. In a non-Abelian gauge theory, however, the coupling constant renormalization is an essential element of the theory, since it insures that phenomena which happen only in the regime of strong coupling do indeed occur at some scale. To think about this effect, we will consider a special limit in which it is easy to study.

Coupling constant renormalization in non-Abelian gauge theories is given by the equation [9]

$$\frac{d}{d \log p} g(p) = \beta(g) \quad ; \quad \beta(g) = -b_0 \frac{g^3}{(4\pi)^2} + \cdots \quad . \tag{3.23}$$

Using this approximation for $\beta(g)$,

$$g^2(p) = \frac{g_0{}^2}{\left[1 + b_0 \frac{g_0{}^2}{(4\pi)^2} \log(p^2/p_0{}^2)\right]} \tag{3.24}$$

where $g_0 = g(p_0)$. If $b_0 > 0$, g^2 increases slowly as p^2 decreases. In non-Abelian gauge theories of the type we were considering

$$b_0 = \left[\frac{11}{3} C_2(G) - \frac{4}{3} n\, C(r)\right] \tag{3.25}$$

where the various group theory factors were defined at the end of section 3.1.

Let us study χSB in the following limit: $C_2(r) \to \infty$, so that chiral symmetry breaking takes place when g^2 is small, but $n \to 0$ at the same time so that b_0 remains fixed. In this limit χSB takes place at a scale where g^2 is evolving slowly. (I was moved to consider the limit by the Monte Carlo calculations of Kogut et. al.[20], who work in this limit.)

To consider this situation, we should insert a slow evolution of the coupling constant into our equation for Σ. For this analysis, it will be easiest to start from eq. (2.31) obtained by variation of the effective action. To analyze this equation properly, we should insert the most general structure:

$$S = \frac{1}{[Z(p^2)\cdot(-i\not{p}) + \Sigma(p^2)]} \tag{3.26}$$

but I will ignore $Z(p^2)$ in the analysis to follow. (That is an acceptable approximation for the results I will actually discuss, because (in Landau gauge) $Z(p^2) \to 1$ when $p \gg \Sigma$.)

Let us identify the mass-like terms (terms without γ matrices) on the left- and right-hand sides of (2.31), and integrate out angular variables. This yields the equation:

$$p\Sigma(p) = \frac{3g^2C_2(r)}{8\pi^2}\int_0^\infty \frac{dk}{k}\, k\Sigma(k)\, \exp[-|\log p/k|] \cdot \left(\frac{k^2}{k^2 + \Sigma^2(k)}\right) \tag{3.27}$$

Except for the last factor, this is just the integral equation associated with the approximation (3.11) for Γ. We may introduce coupling constant renormalization into (3.27) by considering g^2 to be a function of momenta, to be evaluated at the largest of (p,k,Σ).

By examining eq. (3.27), we can make some statements about the limiting behavior of $\Sigma(p)$.

1. As $p \to 0$, $\Sigma(p) \to \Sigma(0)$, a constant, where

$$\Sigma(0) = \frac{3C_2(R)}{8\pi^2}\int_0^\infty \frac{dk}{k}\left(\frac{k^2\Sigma(k)}{k^2 + \Sigma^2(k)}\right) \cdot g^2\ (k \text{ or } \Sigma) \tag{3.28}$$

2. As $p \to \infty$, $\Sigma(p) \propto 1/p^2$ if the integral

$$\int^\infty \frac{dk}{k}\left(\frac{k^2}{k^2 + \Sigma^2(k)}\right) k^2\Sigma(k) \cdot g^2(k) \tag{3.29}$$

converges. For this to happen, $(k\Sigma(k))$, which was constant in our earlier treatment, must be persuaded to tend to zero as $k \to \infty$. The fact that $g^2(k) \to 0$ will surely help this to occur. (More properly, the form for Σ which asymptotically satisfies (3.27) self-consistently is

$$\Sigma(p) \propto \frac{(\log k)^{\alpha}}{k^2} \quad , \quad \alpha = \frac{3C_2(r)}{b_0} \quad . \tag{3.30}$$

This result was derived carefully by Lane in [21].)

Let us now study eq. (3.27) in the case of a slowly evolving coupling constant. At scales p where $g^2(p)$ just satisfies the condition (3.21) for an instability to χSB, $\sigma(k) = k\Sigma(k)$ will appear dominantly in modes near $q = 0$ (in terms of q in (3.14)), so $(k\Sigma(k))$ will also be slowly varying. I will assume, and check a posteriori, that $\sigma(k)$ varies on a logarithmic scale but does not vary as slowly as $g^2(k)$. But eq. (3.27) tells us that $\sigma(k)$ can vary slowly only if $k \gg \Sigma$, so the analysis to follow will apply only in this regime.

Introduce logarithmic momentum variables as indicated in eq. (3.12), and take p_0 to be defined as the momentum such that

$$\frac{3C_2(r)\; g^2(p_0)}{4\pi^2} = 1 \quad . \tag{3.31}$$

Then (3.27) becomes:

$$\sigma(\eta) = \int_{-\infty}^{\infty} d\xi \, \exp(-|\eta-\xi|) \left[\frac{3C_2(r)\; g^2(k \text{ or } p)}{8\pi^2}\right] \sigma(\xi) \tag{3.32}$$

the exponential factor varies more rapidly than any other factor in the integrand; it is therefore appropriate to expand $\sigma(\xi)$ about $\sigma(\eta)$. I ignore the corresponding variation of g^2. In this approximation:

$$\sigma(\eta) = \int d\xi \, \exp(-|\eta-\xi|) \left[\frac{3C_2(r)\ g^2}{8\pi^2}\right]$$

$$\cdot[\sigma(\eta) + (\xi-\eta)\sigma'(\eta) + 1/2(\xi-\eta)^2\sigma''(\eta) + \cdots]$$

$$\sigma(\eta)\left[1 - \frac{3g^2\ C_2}{4\pi^2}\right] = \left(\frac{3g^2\ C_2}{4\pi^2}\right)\frac{d^2}{d\eta^2}\,\sigma + \cdots \tag{3.33}$$

If $\sigma(\eta)$ varies slowly, we can ignore the terms not written explicitly. Equations (3.24) and (3.31) imply that

$$\left(\frac{3C_2(r)}{4\pi^2}\right) g^2(p=p_0e^{\eta}) = \frac{1}{\left[1 + \left(\frac{b_0}{6\pi C_2(r)}\right)\eta\right]} \quad . \tag{3.34}$$

Recall that, in our approximation, $b_0 \ll C_2(r)$. Inserting (3.34) into (3.33) we find the equation:

$$-\frac{d^2}{d\eta^2}\,\sigma(\eta) + \left[\frac{b_0}{6\pi C_2(r)}\cdot\eta\right]\sigma(\eta) = 0 \quad . \tag{3.35}$$

This is just the Schrödinger equation with a shallow linear potential. $\sigma = p\Sigma(p)$ is the wave function:

$$\sigma(\eta) = \mathrm{Ai}\left(\left(\frac{b_0}{6\pi C_2(r)}\right)^{1/3}\eta\right) \tag{3.36}$$

where Ai(x) is the Airy function. If $b_0 \ll C_2(r)$, this function is indeed more rapidly varying than $g^2(p)$. Thus for $p > \Sigma(p)$,

$$\Sigma(p) = \frac{C}{p}\cdot \mathrm{Ai}\left(\left(\frac{b_0}{6\pi C_2(r)}\right)^{1/3}\log(p/p_0)\right)$$

$$\xrightarrow[p\to\infty]{} \frac{D}{p}\,\frac{1}{(\log p/p_0)^{1/4}}\exp\left[-\frac{2}{3}\left(\frac{b_0}{6\pi C_2(r)}\right)^{1/2}(\log p/p_0)^{3/2}\right] \quad . \tag{3.37}$$

where C and D are constants. This function should describe well the behavior of $\Sigma(p)$ just above $p = p_0$. For very large p, however, (3.37) falls to zero faster than any power of p; then our approximation (3.33) is no longer valid and the behavior of $\Sigma(p)$ reverts to that of (3.30). A qualitative picture of $\Sigma(p)$ which assembles all the information we have obtained is given in fig. 7.

3.3 MORE GENERAL FERMION REPRESENTATIONS

Before finishing this discussion, I should indicate its generalization to situations allowing more complicated forms of the fermion condensate. It is not hard to treat the most general situation. Label the L-fermions as $\psi_{\alpha i}$, where $\alpha = 1,2$ is a spin index and i schematically indexes colors and flavors. We can imagine constructing the effective action Γ corresponding to the most general scalar source term

$$\int K^{ij}(y,x)[(\epsilon^{\alpha\beta}\psi_{\alpha i}(x)\psi_{\beta j}(y)) + h.c.] \tag{3.38}$$

For the case of Dirac fermions, we can write $\psi_{L2} = \psi^{\dagger}{}_{R}\sigma_2$, then

$$\epsilon^{\alpha\beta}\psi_{L2\alpha}\psi_{L\beta} + h.c. = \bar{\psi}\psi \tag{3.39}$$

but (3.38) allows more possible patterns of condensation. The only restriction on the form of K^{ij} is that it must be symmetric under interchange of i and j, as the result of fermion anticommutation and the antisymmetry of $\epsilon^{\alpha\beta}$. The formalism we developed in section 2.2 goes through for the source term (3.38) with only the modification that we should use a more general class of propagators:

$$\bar{\psi}S^{-1}\psi \rightarrow \bar{\psi}_i \not{\partial}\psi_i + 1/2[\Sigma^{ij}\epsilon^{\alpha\beta}\psi_{\alpha i}\psi_{\beta j} + h.c.] \quad . \tag{3.40}$$

Like K, Σ must be symmetric in its indices.

To obtain an idea of the physics of this situation, let us carry this system through the stability analysis of section 3.1. The discussion given there is easily generalized, leading to the result

$$\Gamma_2 = \frac{1}{4\pi^2}\int\frac{dq}{2\pi}\left[\sigma^{*ij}(q)\sigma^{ij}(q) + \frac{3g^2}{4\pi^2}\frac{1}{(1+q^2)}(\sigma^{*ij}(q)(t^a)_{im}(t^a)_{jn}\sigma^{mn}(q))\right] \tag{3.41}$$

where $\sigma^{ij}(q)$ is the Fourier transform of $(k\Sigma^{ij}(k))$. The routing of indices in the second term corresponds to the structure of fig. 8. To complete the diagonization of Γ_2, we need to do a little more group theory. A pair of indices (i,j) of Σ corresponds to a pair of L-fermions in the representations r_i, r_j of the gauge group. Let us decompose the product $r_i \times r_j$ into irreducible representations R. Then σ^{ij} may be written:

$$\sigma^{ij} = \sum_R \sigma_{(R)}{}^k \cdot C_k{}^{ij} \tag{3.42}$$

where k indexes the representation R and $C_k{}^{ij}$ is a Clebsch-Gordon coefficient. With this notation, we may compute:

$$(t^a t^a)_{im}\sigma^{mj} = C_2(r_i)\sigma^{ij}$$

$$(t^a t^a)_{jn}\sigma^{in} = C_2(r_j)\sigma^{ij}$$

$$[(t^a)_{ip}\delta_{jq} + \delta_{ip}(t^a)_{jq}][(t^a)_{pm}\delta_{qn} + \delta_{pm}(t^a)_{qn}]\sigma^{mn}$$

$$= (\text{total } t^a)^2\sigma^{ij} = \sum_R C_2(R)\sigma_{(R)}{}^k \cdot C_k{}^{ij} \quad . \tag{3.43}$$

Then, the group theory factors in (3.41) reduce to:

$$(t^a)\cdot(t^a) = 1/2\ [C_2(R) - C_2(r_i) - C_2(r_j)] \quad . \tag{3.44}$$

The eigenvalues of the quadratic form are therefore proportional to

$$\left\{1 - \frac{3g^2}{4\pi^2}\frac{1}{(1+q^2)}[C_2(r_i) + C_2(r_j) - C_2(R)]\right\} \quad . \tag{3.45}$$

An instability will appear, for sufficiently strong coupling, for any representation R in $r_i \times r_j$ such that

$$(C_2(r_i) + C_2(r_j) - C_2(R)) > 0 \quad . \tag{3.46}$$

We can check this result against that found in section 3.1. The discussion there corresponds to the special case of L-fermions in complex conjugate representations r and $\bar{r}$ condensing in a mode which corresponds to a singlet under the gauge group ($R = 1$, $C_2(R) = 0$). In this case, (3.46) properly reproduces (3.21).

It is not hard to see that there is always a representation in $r_i \times r_j$ satisfying (3.46): The quantity appearing there is essentially the eigenvalue of $(t^a)\cdot(t^a)$ in (3.44). But

$$\mathrm{Tr}(t^a)\cdot(t^a) = (t^a)_{ii}(t^a)_{jj} = 0 \tag{3.47}$$

so this matrix must have a negative eigenvalue. (Occasionally, though, the negative eigenvector is forbidden from appearing in Σ^{ij} by the symmetry of this object.) Usually, there are several possibilities. In such a case, the analysis of section 3.2 can be used to suggest which mode of condensation is preferred. The critical coupling is inversely proportional to the factor in (3.46). Choosing the minimal value of $C_2(R)$ maximizes this factor and thus minimizes the critical coupling. The analysis of section 3.2 suggests that, in an asymptotically free gauge theory, this choice maximizes the value of the momentum scale p_0 at which the fermion condensate appears and, thus, maximizes the energy of stabilization of the condensate. Our analysis suggests, then, that the condensate which will appear is the one which maximizes (3.46). Equivalently, the condensate appears in the channel which is most attractive with respect to 1-gluon exchange. This Maximally Attractive Channel (MAC) criterion was first formulated some time ago by Cornwall [18]; it has recently been revived and popularized by Raby, Dimopoulos, and Susskind [22].

For the case of Dirac fermions, the MAC criterion implies that the preferred condensate will be a color singlet, as we had assumed implicitly in section 3.1, and as would be required to produce the pattern of chiral symmetry breaking in QCD described at the end of section 2.1. More complex systems of fermions, however, may condense into nonsinglet representations. I will show some examples of this in section 5. I should, however, make one comment on the applicability of our formalism to this situation. I had commented at the end of section 2.2 that if Σ breaks the gauge symmetry, it will induce gauge boson masses $M^2{}_{ab}$; one must solve self-consistently for Σ and M^2. This change in the formalism, however, does not at all affect the conclusions of the stability analysis we have performed in this section: Any term in Γ which contains a factor Σ also contains a factor Σ^*. Hence, the most general form possible for Γ_2 is:

$$\Gamma_2 = \Sigma^* \cdot (G) \cdot \Sigma + M^2 \cdot (H) \cdot M^2 \qquad (3.48)$$

where G and H are operators. In this section, we found the circumstances under which G has a negative eigenvalue. Even though H is positive definite, the presence of a negative eigenvalue in G signals an instability in the coupled fermion-gauge boson system.

I should finish this section by noting a possible application of the dependence of the scale of chiral symmetry breaking on the group-theoretic factor (3.46). By considering models involving fermions in several representations, one can easily imagine scenarios in which some fermions condense and gain mass at a scale Λ_1, but others are left massless there and gain mass only at a lower scale Λ_2. If our criterion that condensation takes place when g^2 is small and evolving slowly at this value, it is possible that $\Lambda_2 \ll \Lambda_1$, so that the theory has a hierarchy of scales. Raby, Dimopoulos and Susskind [22] and Marciano [23] have offered this mechanism to explain why quark mass differences are so large, or why the W boson mass is so much larger than the proton mass. Unfortunately, our estimate (3.21) corresponds to a large critical coupling except in the artificial limit used in section 3.2. Proponents of this idea must, then, hope that higher-order corrections reduce that estimate substantially. There is some numerical evidence that this may be the case [20].

Chapter 4.

EXPLICIT CALCULATIONS OF χSB — INSTANTONS

4.1 A PEDESTRIAN INTRODUCTION TO INSTANTONOLOGY

In section 2.1, I remarked that (when we have written all fermions as L-fermions) the symmetry of overall U(1) phase rotation: $\psi \to e^{i\alpha}\psi$, whose corresponding current is that of overall fermion number:

$$J^\mu = \sum_i \bar{\psi}_{Li}\gamma^\mu\psi_{Li} \tag{4.1}$$

is destroyed by quantum effects. I should now explain that remark more carefully. That explanation will lead us to another mechanism of χSB. My discussion of this effect will be brief and concentrate on essentials; I will give references to more complete reviews of this subject.

Let me first observe the effect of the diagrams involving J^μ shown in fig. 9. These diagrams may appear as subdiagrams in any vertex function involving J^μ. The diagrams of fig. 9 are superficially linearly divergent; the divergent part is actually $\int d^4k/k^4 \cdot k^\mu = 0$, but short distance effects can influence the residual finite part. Specifically, a kinematically allowed term proportional to

$$\epsilon^{\mu\nu\lambda\alpha}(k-p)_\alpha \tag{4.2}$$

in terms of the notation of fig. 10, appears when one computes these diagrams. This term does not respect $\partial_\lambda J^\lambda = 0$. It may be seen to survive the cleverest of (gauge-invariant) regularization procedures [10,11]; one always finds:

$$\partial_\mu \text{ (fig. 10)} = -\frac{g^2}{4\pi^2} \mathrm{Tr}(t^a t^b) \epsilon^{\mu\nu\alpha\beta} F^a{}_{\mu\alpha} F^b{}_{\alpha\beta} \tag{4.3}$$

where the trace is over all representations, flavors, and colors. Equation (4.3) is known as the Adler-Bell-Jackiw axial vector anomaly. The result is actually correct to all orders in perturbation theory [24]. (A detailed review of the derivation and consequences of (4.3) may be found in [25].) For our purposes, it will be useful to recast eq. (4.3) in the following way. Let us write

$$F\tilde{F} = 1/2\ \epsilon^{\mu\nu\alpha\beta} F^a{}_{\mu\nu} F^a{}_{\alpha\beta} \tag{4.4}$$

Then (4.3) takes the form:

$$\partial_\mu J^\mu = \frac{g^2}{32\pi^2} (F\tilde{F}) \sum_r n_r \cdot 2C(r) \tag{4.5}$$

where n_r is the number of L-fermions in the representation r of the gauge group, and C(r) is the invariant defined in (3.18).

The anomaly equation tells us that $\partial_\mu J^\mu \neq 0$. Examined formally, this breaking of the conservation of J^μ seems not especially efficacious: Since one can write (4.4) as

$$F\tilde{F} = \partial_\mu [2\epsilon^{\mu\nu\alpha\beta} \{ A_{a\nu} \partial_\alpha A^a{}_\beta + g/3\ f^{abc} A^a{}_\nu A^b{}_\alpha A^c{}_\beta \}] \tag{4.6}$$

(The f^{abc} are the structure constants of the gauge group, which appear in F for non-Abelian groups), this operator is just a surface term. However, this surface term nevertheless has a strong influence on physics throughout space-time, due to a series of miracles. These miracles were discovered by Belavin, Polyakov, Schwartz, and Tyupkin [26] and 't Hooft [27]; their effects are reviewed in detail in lectures of Coleman [28].

If $F_{\mu\nu}(x)$ is to vanish as $|x| \to \infty$, A_μ must tend to a gauge transform of $A_\mu = 0$. Thus, as $|x| \to \infty$,

$$A^a_\mu(x)t^a \to i/g\ U(x)\partial_\mu U^\dagger(x) \tag{4.7}$$

where U(x) is an element of the gauge group. The function U(x) is a mapping of the (three-dimensional) sphere at $|x| = \infty$ into the gauge group. For any simple Lie group, such mappings fall into homotopy classes, characterized by an integer (called the Pontryagin index). One can show that

$$n = \int d^4x \frac{g^2}{32\pi^2} F\tilde{F} \quad . \tag{4.8}$$

In the class of fields with n = 1 or -1, one can readily find a localized solution of the classical equation of motion in Euclidean space, an instanton. To solve these equations, we must minimize the classical action S, subject to (4.8). To do this, note that

$$0 \le \int(F-\tilde{F})^2 = \int(F^2+\tilde{F}^2-2F\tilde{F}) = 2[\int F^2 - \int F\tilde{F}] \quad . \tag{4.9}$$

Thus, S satisfies

$$S = \int \frac{1}{4} F^2 \ge \frac{8\pi^2}{g^2} \cdot n \tag{4.10}$$

and the bound is saturated for configurations where $F = \tilde{F}$. Belavin et. al., solved the equation $F = \tilde{F}$ for the case n = 1; they found a solution $A^a_\mu(x)$ for the gauge group SU(2) which gives

$$gF^a_{\mu\nu}(x) = \frac{4\eta_{a\mu\nu}\rho^2}{(x-x_0)^2 + \rho^2} \tag{4.11}$$

where $\eta_{a\mu\nu}$ is a numerical tensor. Since classical Yang-Mills theory is translation- and scale-invariant, the solution has an arbitrary center x_0 and size ρ. This field configuration may be considered an n = 1 solution to $F = \tilde{F}$ for an arbitrary gauge group by considering SU(2) as a subgroup of that group; this construction gives the most general $n = 1$ solution for any group. (I should note that the general solution to $F = \tilde{F}$ has since been found for any n [29].) Despite the fact that the solution (4.11) appears in (4.5) only as a surface term, its influence on physics is clearly localized to a region about the point x_0. In a configuration with Pontryagin index n,

$$\int d^4x \; \partial_\mu J^\mu = \Delta Q = n \cdot \left[\sum_r n_r \cdot 2C(r) \right] \quad . \tag{4.12}$$

In a space-time containing such a gauge field configuration, fermions must disappear. It is worth exploring in detail how this occurs.

One can show that the Dirac equation for massless fermions in the representation r in the instanton field has 2C(r) zero eigenvalues. This corresponds to one zero eigenvalue for each chiral fermion in the representation r = N of SU(N). If the gauge group is SU(2), we can write the associated eigenmode (which I will call a zero mode) in the following way:

$$\psi_{0\alpha\beta}(x) = \frac{1}{(x^2+\rho^2)^{3/2}} \epsilon_{\alpha\beta} \quad . \tag{4.13}$$

In (4.13), $\alpha = 1,2$ is the gauge index, $\beta = 1,2$ is the spinor index, and ϵ is the invariant contraction. Equation (4.13) satisfies

$$0 = \not{D}_L \psi_0 = [(\partial_0 - iA_0 \cdot t) - \vec{\sigma} \cdot (\vec{\partial} - i\vec{A} \cdot t)]\psi_0 \tag{4.14}$$

In Euclidean space $\not{D}_L$ is not (anti)-Hermitian, so the equation $[\not{D}_L]^+ \bar{\psi} = 0$ need have no such zero mode. Indeed, it does not. $\bar{\psi}$, but not ψ, has a zero mode in the anti-instanton field.

To see the consequences of these zero modes, expand ψ and $\bar{\psi}$ in eigenstates of $\not{D}$. In the instanton field

$$\psi(x) = \xi_0\psi_0(x) + \sum_i \xi_i\psi_i(x) \qquad \bar{\psi}(x) = \sum_i \bar{\psi}_i(x)\bar{\xi}_i \quad . \tag{4.15}$$

The eigenfunctions $\psi_i(x)$, $\bar{\psi}_i(x)$ are c-number functions. Since $\psi(x)$, $\bar{\psi}(x)$ are Grassman (anticommuting) fields, the parameters ξ, $\bar{\xi}$ must be Grassman variables. We can represent

$$\int_{\psi\bar{\psi}} = \int d\xi_0 \int \prod d\xi_i \int \prod d\bar{\xi}_i \tag{4.16}$$

The rule for integrating over Grassman parameters if the following [30]: The most general function of ξ is $f(\xi) = (a+b\xi)$, since $\xi^2 = 0$. Its integral is

$$\int d\xi \; f(\xi) = b \tag{4.17}$$

This allows us to evaluate

$$\begin{aligned} \int_{\psi\bar{\psi}} \exp[-\int \bar{\psi}\not{D}_L\psi &= \int_{\xi,\bar{\xi}} \exp[-\sum_i \lambda_i\bar{\xi}_i\xi_i] \\ &= \int_{\xi\bar{\xi}} \prod_i (1-\lambda_i\bar{\xi}_i\xi) \end{aligned} \tag{4.18}$$

If there were no zero modes, this would become:

$$(4.18) = \prod_i \lambda_i = \det(\not{D}_L) \tag{4.19}$$

the standard result of an integral over fermions. In the instanton field, however, there is an extra factor:

$$\int d\xi_0 \cdot (\text{independent of } \xi_0) = 0 \tag{4.20}$$

so (4.18) = 0. However,

$$\begin{aligned} \int_{\psi\bar{\psi}} \exp[-\int \bar{\psi}\not{D}_L\psi] \; \psi(x) &= \int_{\xi\bar{\xi}} \exp[-\sum_i \lambda_i\bar{\xi}_i\xi_i] \; (\xi_0\psi_0 + \sum_i \xi_i\psi_i) \\ &= \mathrm{Det}'(\not{D}_L)\cdot\psi_0(x) \end{aligned} \tag{4.21}$$

where Det′(M) is the product of nonzero eigenvalues of M.

Apparently, the instanton creates an extra L-fermion. If the theory contains one N and one $\bar{N}$ of L-fermions (one Dirac fermion), the instanton creates one N and one $\bar{N}$. 't Hooft showed that, at distances much larger than the size ρ of the instanton, the instanton has the same effect as the operator:

$$\Sigma^* \epsilon^{\alpha\beta} \psi^{\dagger}_{\alpha(N)}(x_0) \psi^{\dagger}_{\beta(\bar{N})}(x_0) = \Sigma^* \bar{\psi}_L\psi_R(x_0) \tag{4.22}$$

The anti-instanton's influence is that of $\Sigma\cdot\bar{\psi}_R\psi_L(x_0)$. Thus, the instantons effectively generate a mass term and completely destroy the chiral symmetry of this model. Quite generally, the mechanism I have just displayed destroys the overall U(1) chiral symmetry and the conservation of the total number of L-fermions. It may be seen to preserve all other chiral symmetries, since these have no Adler-Bell-Jackiw anomalies.

The interplay between instantons and massless fermions, however, suggests that instantons can provide a mechanism for the spontaneous breaking of other chiral symmetries. I will explain this mechanism by showing how instantons can produce mass spontaneously in an SU(N) gauge theory of $n\cdot(N+\bar{N})$ L-fermions. My calculation will be very crude, just enough to see the essential physics. For a more complete analysis (including the numberical factors), the reader should consult the original papers [31,32,33].

4.2 AN INSTABILITY TO χSB

I wish to argue that the chirally asymmetric vacuum state is stabilized by the presence of instantons. To do this, let me modify the formula (2.29) of Cornwall, Jackiw and Tomboulis by replacing the term (diagrams) by (instantons). More properly, I wish to include the effects of instantons on the evaluation of Z[K] in the derivation of (2.29). The integral over A may be separated into contributions from each topological sector n. The n = 1 fields may be expanded about the minimum action configurations in this sector, the instantons. Analyzing Z[K] in this way, we have:

$$Z[K] = \sum_n \exp[-\textstyle\int K\Sigma]\, Z_n(\Sigma)$$

$$= \exp[-\textstyle\int K\Sigma]\, Z_0(\Sigma)\cdot\left(1 + \sum_{n\neq 0} A_n(\Sigma)\right) \quad . \tag{4.23}$$

In (4.23), $Z_0(\Sigma)$ is the value given by perturbation theory. We will call $A_1(\Sigma)$ the 1-instanton amplitude; $A_{-1}(\Sigma) = (A_1(\Sigma))^*$. A standard approximation to (4.23) is the dilute gas approximation [32]:

$$\left(1 + \sum_{n\neq 0} A_n(\Sigma)\right) \approx \exp[(A_1(\Sigma) + A_{-1}(\Sigma))] \quad . \tag{4.24}$$

In this approximation

$$\Gamma(\Sigma) = (\text{kinematic terms}) - (A_1(\Sigma) + A_{-1}(\Sigma)) \tag{4.25}$$

plus the effects of 2PI diagrams, which I will ignore for the remainder of this section.

The value of the 1-instanton amplitude, to leading order in perturbation theory, is:

$$A_1(\Sigma) = \frac{\int_{A(n=1)}\int_{\psi\bar{\psi}} \exp\left[-\int\left(\frac{1}{4} F^2+\bar{\psi}\not{D}\psi+\bar{\psi}\Sigma\psi\right)\right]}{\int_{A(n=0)} \cdots}$$

$$= \int d^4x \int d\rho \; A(g^2) \exp[-8\pi^2/g^2(\rho)] \prod_{i=1}^{n} \left(\frac{\det(\not{D}+\Sigma)}{\det(\not{\partial}+\Sigma)}\right) \quad . \qquad (4.26)$$

The integral in the numerator should be expanded about the instanton field; D in the second line should evaluated in the instanton field. Because of the zero mode, the factor in parentheses vanishes when $\Sigma = 0$. We can get an idea of the shape of this term by evaluating it for $\Sigma = m$, a constant. Extracting from the ratio of determinants a factor which depends on the scale μ of coupling-constant renormalization, we have

$$\left(\frac{\det \not{D}+m}{\det \not{\partial}+m}\right) = I(m) \exp\left[-\frac{2}{3} C(r) \log \rho^2\mu^2\right] \quad . \qquad (4.27)$$

I(m) has the limiting behavior:

$$I(m) \rightarrow B\cdot m \qquad (m \rightarrow 0)$$

$$\rightarrow \exp[2/3 \; C(r) \log m^2\rho^2] \qquad (m \rightarrow \infty) \qquad (4.28)$$

B is a numerical constant computed by 't Hooft [34] and others [35]. The behavior as $m \rightarrow \infty$ may be understood by remembering that a heavy fermion decouples at scales lower than m, so that the coupling constant renormalization contains only the factor $\log(\mu^2/m^2)$ and is independent of ρ. The general form of I(m) is indicated in fig. 11. I(m), and the whole amplitude $A_1(\Sigma)$, is real and positive in this approximation.

Now that we understand the form of $A_1(\Sigma)$, we can construct the effective action Γ in the approximation (4.25). For simplicity, let us further approximate the determinant for instantons of size ρ:

$$\left.\left(\frac{\det \not{p}+\Sigma}{\det \not{\partial}+\Sigma}\right)\right|_{\rho} \approx I(\Sigma(p = 1/\rho))\cdot\exp\left[-\frac{2}{3}\, C(r)\,\log\,(\rho^2\mu^2)\right] \quad . \tag{4.29}$$

Then the interaction terms in (4.25) are negative and depend on $\Sigma(p)$ as $(\Sigma(p))^n$ when $\Sigma(p)$ is small. If $n > 2$, the origin of the configuration space is always stable. However, as ρ becomes large (or p becomes small), $g^2(\rho)$ increases and so the coefficient of $(\Sigma(p))^n$ becomes large. Eventually, the effective action, as a function of $\Sigma(p)$, acquires the form shown in fig. 12. The strong growth of $A_1(\Sigma)$ generates an instability toward χSB.

Two central elements of this argument are more easily understood in a more general context. Let me now review them briefly, using the notation of L-fermions in which the fermion action takes the form

$$\int\left[\bar{\psi}\not{\partial}_L\psi + \frac{1}{2}\,(\epsilon^{\alpha\beta}\psi_\alpha\psi_\beta\Sigma + \text{h.c.})\right] \tag{4.30}$$

Σ multiplies a term which destroys two L-fermions; thus, if the conservation of J^μ in (4.1) were respected by quantum effects, Γ would depend only on the combination $(\Sigma^*\Sigma)$. The instanton, however, creates a number of fermions equal to the number of zero modes. These fermions can be destroyed only by the mass term; hence the instanton amplitude must take the form:

$$A_1(\Sigma) = (\Sigma)^k\ f(\Sigma^*\Sigma) \tag{4.31}$$

where k is half the number of zero modes

$$2k = \sum_r n_r\cdot 2C(r) \tag{4.32}$$

Equation (4.31) has two consequences: First, the vanishing of $A_1(\Sigma)$ at $\Sigma = 0$ is a general feature of models of massless fermions, and the power law with which $A_1(\Sigma)$ vanishes is given simply by (4.32). Second, the phase of $A_1(\Sigma)$ may be adjusted by adjusting the phase of Σ. Thus it was actually irrelevant that (4.26) was real and positive; $A_1(\Sigma)$ can always be made real and positive (making $\Gamma(\Sigma)$ maximally negative) by varying the phase of Σ.

These considerations make clear that the instanton mechanism of χSB applies to a wide class of gauge models, including chiral gauge theories. The generalization to chiral models is, however, a bit complex. We will discuss it in stages in the next section.

Chapter 5.

χSB IN CHIRAL GAUGE THEORIES

5.1 PATTERNS OF χSB — GENERAL OBSERVATIONS

We have spent the last two sections investigating mechanisms of χSB. Though we have not found approximation schemes which allow quantitative calculation, we have found qualitative physical pictures of χSB which seem very appealing. One might wish to re-examine the calculations I have done and attempt to improve their accuracy, but I will not pursue that problem here. Instead I will move to a deeper level of analysis and apply the method of the previous sections to examine a more detailed, yet still qualitative, question. That question is the following: Of all patterns of χSB available to a given gauge theory model, which is actually chosen by the dynamics? I will first briefly recapitulate what we have learned about this question in models with massless Dirac fermions. Then I will examine some examples of chiral gauge theories, theories in which the L-fermions belong to a complex representation of the gauge group and so cannot be grouped into Dirac fermions. In such theories, our simple methods of analysis will yield patterns of mass generation rather more intricate than those of the cases we have examined so far. This discussion will furnish us with some phenomena which are curious in their own right and also relevant to the discussion from more general principles which I will present in section 6.

Let me first say a few words about the pattern of chiral symmetry breaking in theories with n Dirac fermions belonging to a complex representation r of the gauge group. We would write the content of this theory as n $(r+\bar{r})$ L-fermions; these fermions are described by fields

$$\psi_{(r)\alpha i} \qquad \psi_{(\bar{r})\alpha i} \tag{5.1}$$

where $\alpha = 1,2$ is the spinor index and $i = 1, \ldots, n$. As we discussed in the first lecture, this theory has the anomaly-free chiral symmetries:

$$G = U(1) \times SU(n) \times SU(n) \quad . \tag{5.2}$$

According to the arguments of the last two lectures, this theory has spontaneous breaking of symmetry corresponding to the acquisition of a mass term

$$m\bar{\psi}\psi = m(\epsilon^{\alpha\beta}\,\psi_{(r)\alpha}{}^{i}\,\psi_{(\bar{r})\beta i} + \text{h.c.}) \tag{5.3}$$

This term respects the full group of gauge symmetries. Further, if all fermions acquire equal mass, this term is invariant to the subgroup $H = U(1) \times SU(n)$ of G:

$$\psi_{(r)\alpha}{}^{i} \rightarrow U^{i}{}_{j}\,\psi_{(r)}{}^{j} \quad ; \quad \psi_{(\bar{r})\beta i} \rightarrow \psi_{(\bar{r})\beta j}(U^{-1})^{j}{}_{i} \tag{5.4}$$

This pattern of symmetry breaking is at least one of the ones preferred by the analyses of sections 3 and 4. It corresponds to the earliest instability with respect to one gluon-exchange; it also may be seen to give the largest coefficient of the one-instanton amplitude. This is also the pattern of symmetry breaking observed in the strong interactions:

$$[\text{color } SU(3)] \times U(1) \times SU(2) \times SU(2) \rightarrow [\text{color } SU(3)] \times U(1) \times SU(2) \tag{5.5}$$

so it is sensible also from the viewpoint of phenomenology. Note that this pattern corresponds to the maximal global symmetry which allows all fermions to acquire mass. I will guess that this principle — that fermions retain the maximum global symmetry consistent with dynamical mass generation — is one of general significance. Eventually, I will use it as a guide.

On the other hand, the pattern of χSB in chiral gauge theories cannot be so simple as the one we have just discussed. To see this, one need only note that chiral gauge theories are precisely those in which fermion mass generation cannot occur without also breaking the gauge symmetry: A fermion mass term is generally of the form

$$\epsilon^{\alpha\beta}\psi_{\alpha A}\psi_{\beta B}\Sigma^{AB} + \text{h.c.} \tag{5.6}$$

If the fermions belong to the (reducible) representation R of the gauge group; this object is in the representation R × R. If $R \neq \bar{R}$ (the feature which defines chiral models), some fermion will be forbidden by global gauge invariance from acquiring mass. In models of this type, χSB necessarily also induces spontaneous breaking of the gauge symmetry; we must, then, work out interlocking patterns of chiral and gauge breaking. These patterns are most easily understood by consideration of specific examples, to which we will turn in a moment.

5.2 ANOMALY CONSTRAINTS ON CHIRAL REPRESENTATIONS

Before beginning a discussion of χSB, however, I should point out that there is a strong restriction on the consistency of chiral gauge theories. This restriction, due to Gross and Jackiw [36], is another consequence of the Adler-Bell-Jackiw anomaly discussed in section 4.1. Let us slightly generalize the argument given there by computing the graphs of fig. 9 using for external current one of the gauge currents $J^c{}_\mu = \bar{\psi}_i\gamma^\mu t^c\psi_i$. This calculation could, potentially, yield a term with the structure of the anomaly; such a term would destroy the conservation of J^c_μ and ruin the Ward identities of the gauge theory. We must insist that the coefficient of the structure (4.2) should vanish. But this coefficient contains only factors of π and g^2, and the group theoretic factor

$$\mathrm{Tr}(\{t^a,t^b\}t^c) \tag{5.7}$$

which replaces the factor $\mathrm{Tr}(t^at^b)$ of eq. (4.3). A chiral gauge theory is, then, only consistent if it satisfies the group-theoretic constraint:

$$\sum_r n_r \, \mathrm{Tr}(\{t^a{}_r,t^b{}_r\}t^c{}_r) = 0 \tag{5.8}$$

where n_r is the number of L-fermions in the representation r of the gauge group.

It is worth spending some effort to simplify the condition (5.8) [37]. The trace indicated in (5.8) is a totally symmetric group-invariant tensor with three indices in the adjoint representation; it may therefore be written in terms of a standard set of such invariant tensors. The SU(N) groups (N > 2) have only one such invariant, the symbol d^{abc} which appears in the formula for representation matrices of the fundamental representation:

$$\{t^a,t^b\} = \frac{1}{N}\,\delta^{ab}\cdot 1 + 2d^{abc}t^c \tag{5.9}$$

Thus, for any representation r,

$$\mathrm{Tr}(\{t^a{}_r,t^b{}_r\}t^c{}_r) = A(r)d^{abc} \tag{5.10}$$

where A(r) is an overall constant, called the anomaly coefficient. Equation (5.9) implies that for r = N of SU(N), A(N) = 1; this is essentially a normalization convention. From the relation $t^a{}_{\bar{r}} = -(t^a{}_r)^T$, one can readily determine that

$$A(\bar{r}) = -A(r) \tag{5.11}$$

Thus, pairs of L-fermions in complex conjugate representations give cancelling contributions to (5.8); we may say that a Dirac fermion gives zero anomaly. The condition (5.8) is a stringent one only in theories where the fermions are intrinsically chiral.*

For future reference, I will quote the anomaly coefficients for two-index tensor representations of SU(N). In general, the n-index totally antisymmetric and totally symmetric tensors form irreducible representations of SU(N); I will henceforth denote their representations as [n] and $\{n\}$, respectively. $[1] = \{1\} = N$ of SU(N). Using this notation,

$$A([2]) = (N-4) \quad , \quad A[\{2\}] = (N+4) \quad . \tag{5.12}$$

(The anomaly coefficient for a general representation of SU(N) may be found in [39].) Notice that theories with one [2] and (N-4) $\bar{N}$ representations are anomaly-free and thus consistent (barring further unknown difficulties). For an SU(5) gauge group, this corresponds to one L-fermion in each of the representations

$$10 + \bar{5} \quad , \tag{5.13}$$

the content of the Georgi-Glashow grand unified theory [40]. This class of theories will provide us a useful set of examples for analysis.

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*An additional restriction, which applies to SU(2) and Sp(2n) gauge theories, has recently been discovered by Witten [38].

5.3 χSB IN THE GEORGI-GLASHOW MODEL

Let us consider first the Georgi-Glashow SU(5) model, whose fermion content is given by (5.13). Let us investigate what our simple gluon and instanton mechanisms predict for the pattern of symmetry breaking in this model for strong gauge coupling. To begin, we should catalog the possible mass terms which could be included: Denote the two fermion fields as:

$$\bar{5}: \quad \psi_{\alpha a} \qquad\qquad 10: \quad \psi_{\alpha}{}^{ab} \tag{5.14}$$

where a,b = 1,...,5. Then the possible fermion bilinears are

$$\begin{gathered} \epsilon^{\alpha\beta}\psi_{\alpha a}\psi_{\beta b}\Sigma^{ab} \\ \epsilon^{\alpha\beta}\psi_{\alpha a}\psi_{\beta}{}^{bc}\Sigma^{a}{}_{bc} \\ \epsilon^{\alpha\beta}\psi_{\alpha}{}^{ab}\psi_{\beta}{}^{cd}\Sigma_{abcd} \end{gathered} \tag{5.15}$$

Since Σ^{ab} must be symmetric in its indices, the first term of (5.15) involves a fermion bilinear in the $\{\bar{2}\} = \overline{15}$ representation of SU(5). The second term contains a bilinear in the reducible representation

$$\bar{5} \times 10 = 5 + 45 \tag{5.16}$$

The third term contains a bilinear in the representation

$$(10 \times 10)_{symm} = \bar{5} + 50 \tag{5.17}$$

(Note that $\bar{5}$ = [4].) Fermion pairs may thus condense in any of five distinct channels. We must determine which channels are favored by the dynamics.

Let us first apply our simple instanton model; we must compute $A_1(\Sigma)$ at least well enough to examine its behavior for small Σ and see which of the possible forms of Σ leads to an instability.

We can construct $A_1(\Sigma)$ for small Σ by following the prescription given at the end of section 4.2. We should first count zero modes to identify the instanton amplitude in powers of Σ, using sufficiently many $\Sigma\psi\psi$ terms to destroy all of these extra fermions.

To count zero modes, recall that the instanton lives in an SU(2) subgroup of SU(5). We may choose coordinates so that this SU(2) group acts on only the first two components of a 5-vector.* The 5 of SU(5) thus transforms under this SU(2) as a 2 and three singlets. Similarly,

$$\bar{5} \to 2 + 3\cdot 1$$

$$10 \to 3\cdot 2 + 4\cdot 1 \qquad (5.18)$$

(The second line is the antisymmetric product of two 5's.) The instanton apparently produces one $\bar{5}$ and three 10 fermions; to absorb these fermions, we need one factor of the $(\bar{5}\times 10)$ mass term and one factor of the (10×10) mass term from eq. (5.15). Thus, for small Σ, $A_1(\Sigma)$ has the form:

$$A_1(\Sigma) \propto \Sigma(\bar{5}\times 10)\cdot\Sigma(10\times 10) \qquad (5.19)$$

However, when we construct $A_1(\Sigma)$, we must sum over all instanton solutions. This sum includes an integral in the group SU(5). This implies that A must be SU(5)-invariant, that is, that it must depend only on SU(5)-singlet combinations of the Σ's. Equation (5.19) can be a singlet only if we choose a specific mode of condensation from each of (5.16) and (5.17):

$$A_1(\Sigma) \propto \Sigma(\bar{5}\times 10 \to 5)\cdot\Sigma(10\times 10 \to \bar{5}) \qquad (5.20)$$

By the logic of section 4.2, there is an instability in the effective action which allows these two Σ terms to acquire nonzero values. This corresponds to inducing the mass terms

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*There are inequivalent embeddings of SU(2) in SU(5) for which this choice of coordinates is not possible. However, these embeddings produce solutions to $F = \tilde{F}$ with $n > 1$ [41].

$$\Sigma_1 \epsilon^{\alpha\beta}\psi_{\alpha a}\psi_\beta{}^{a5}$$

$$\Sigma_2 \epsilon^{\alpha\beta}\psi_\alpha{}^{ab}\psi_\beta{}^{cd}\epsilon_{abcd5} \tag{5.21}$$

The first of these terms allows $\psi_{\alpha i}$ and $\psi_\alpha{}^{i5}$ ($i=1,\ldots,4$) to pair to a massive Dirac fermion; the second allows $\psi_\alpha{}^{ij}$ ($i,j=1,\ldots,4$) to become massive. Both terms break the gauge symmetry to SU(4); the massive fermions transform as a $(4 + \bar{4})$ and a 6, respectively, of SU(4). One fermion, $\psi_{\alpha 5}$, is still left massless.

Now that we have found the pattern of symmetry-breaking induced by the instantons, we should ask what pattern of condensates the gluons favor. This analysis is much simpler, since we can apply the results of section 3.3. The result is precisely the same as that of the instanton analysis: Each of the condensations

$$\bar{5}\times 10 \rightarrow 5 \qquad\qquad 10\times 10 \rightarrow \bar{5} \tag{5.22}$$

corresponds to the minimal $C_2(r_i)$ possible for that pair of fermions; each of these channels is maximally attractive [22,42]. Even though the gluon and instanton mechanisms of χSB contain completely different physics, they lead to the same qualitative result. That is the first surprise in the physics of this model. But there will be one more.

Now that we have identified the pattern of condensation in this model, we should try to identify any residual global symmetries. The original model had only one anomaly-free global symmetry, the U(1) rotation whose charge Q is given by:

$$Q\psi_a = 3\cdot\psi_a \qquad\qquad Q\psi^{ab} = (-1)\cdot\psi^{ab} \tag{5.23}$$

Note that instanton physics conserves this charge: The complement of fermions created by the instanton has total charge zero. (More formally, this Q satisfies

$$0 = \mathrm{Tr}[Qt^a t^b] = \delta^{ab}\sum_r n_r\, Q(r)C(r) \tag{5.24}$$

so that fig. 9, computed with this charge, vanishes.)

The symmetry generated by Q is spontaneously broken by the mass generation we have discussed. However, this does not mean there is no residual unbroken global symmetry. Consider the U(1) charge

$$\tilde{Q} = 1/5\ (2Q + Q_{(5)}) \tag{5.25}$$

where $Q_{(5)}$ is the SU(5) generator

$$Q_{(5)} = \left(\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ \hline & & & & -4 \end{array}\right) \qquad \text{(acting on 5)} \tag{5.26}$$

which is also broken by the condensates (5.21). Let us compute the action of $\tilde{Q}$ on the components of the $\bar{5}$ and 10 fermions. For $i, j = 1, \ldots, 4$,

$$\tilde{Q}\psi_a = \frac{1}{5}\left(\begin{array}{c} (6-1)\psi_i \\ \hline (6+4)\psi_5 \end{array}\right) = \left(\begin{array}{c} 1\cdot\psi_i \\ \hline 2\cdot\psi_5 \end{array}\right)$$

$$\tilde{Q}\psi^{ab} = \left(\begin{array}{c|c} 0\cdot\psi^{ij} & (-1)\cdot\psi^{i5} \\ \hline & 0 \end{array}\right) \tag{5.27}$$

The condensates (5.21) have total charge $\tilde{Q} = 0$. The fermion ψ_5, however, has charge $\tilde{Q} = 2$; since there are no $\tilde{Q} = -2$ fermions, it may not acquire mass without further symmetry breaking. But this fermion is an SU(4) singlet; it has no strong gauge interactions at low momentum. Thus, despite all the tumult of the SU(5) strong interactions, this fermion remains absolutely massless.

5.4 χSB IN A MORE GENERAL CLASS OF MODELS

Let us now extend the analysis of the previous section to the whole class of chiral gauge models based on antisymmetric tensor representations, the models with gauge group SU(N) and fermion content

$$[2] + (N-4)\cdot\bar{N} \tag{5.28}$$

We will see that all of the major conclusions of section 4.3 generalize to this whole class of models.

Let us first analyze the condensates from the gluon viewpoint. The maximally attractive channels are:

$$\begin{aligned} &[2] \times \bar{N} \rightarrow N \\ &[2] \times [2] \rightarrow [4] \end{aligned} \tag{5.29}$$

The second of these condensates breaks the gauge group SU(N) to a subgroup in which [4] is an invariant; the largest such subgroup is SU(4). (N-4) orthogonal condensates of the first form will also break SU(N) to SU(4).

If we represent the [2] and $\bar{N}$ fermion fields by ψ^{ab} and $\psi_{a,i}$, respectively (a,b = 1,...,N; i = 1,...,(N-4)), we can write the fermion mass terms corresponding to the condensates (5.29) as follows:

$$\begin{aligned} &\Sigma_1 \epsilon^{\alpha\beta} \psi_{\alpha a,i} \psi_\beta{}^{a(i+4)} \\ &\Sigma_2 \epsilon^{\alpha\beta} \psi_\alpha{}^{ab} \psi_\beta{}^{cd} \epsilon_{abcd5\ldots N} \end{aligned} \tag{5.30}$$

Equation (5.30) gives mass to (N-4) fermions in the representation $(4 + \bar{4})$ and to one fermion in the 6 of the residual gauge group SU(4). Notice that, in (5.30), I have assumed that the $(4 + \bar{4})$ fermions acquire equal masses; this is an application of the principle mentioned in section 5.1, that the theory should retain the maximal global symmetry consistent with fermion mass generation. Despite the fact that this symmetry does not rest on a firmer basis, I will make use of it below.

We might now try to ascertain whether the instantons also favor the pattern of condensates shown in (5.1). A first step is to construct $A_1(\Sigma)$ for small Σ. One can count zero modes in all of these examples by the procedure of (5.18): Each $\bar{N}$ has one zero mode (for a total of (N-4)), and the [2] has (N-2) zero modes. It is clear that one can absorb the corresponding fermions using the mass terms in (5.30); the required terms yield the following form for $A_1(\Sigma)$:

$$A_1(\Sigma) \propto (\Sigma([2] \times \bar{N} \to N))^{(N-4)} \cdot (\Sigma([2] \times [2] \to [4]))^1 \qquad (5.31)$$

Further, (5.31) contains an SU(N) invariant: Combining the factors on the right-hand side totally antisymmetrically yields an object in the [N] of SU(N), which is equivalent to the singlet. The instantons therefore do induce the pattern of condensates (5.30). (I have not yet argued that they favor this pattern over others; I will sketch that argument a bit later.)

What global symmetries do the condensates (5.30) leave unbroken? The original anomaly-free chiral symmetry of the theory is $G = U(1) \times SU(N-4)$. The SU(N-4) is the group of flavor transformations of the $\bar{N}$'s; the U(1) symmetry is that generated by the following charge Q:

$$Q\psi_{a,i} = (N-2)\cdot\psi_{a,i} \qquad Q\psi^{ab} = -(N-4)\cdot\psi^{ab} \qquad (5.32)$$

which satisfies (5.24). Equation (5.30) spontaneously breaks all of these global symmetries. However, many symmetries within the gauge group SU(N) have also been spontaneously broken. As in the example of section 5.3, we can identify combinations of chiral and global gauge charges which generate symmetries undisturbed by (5.30). Let $Q^a{}_{(N)}$ be a generator of the SU(N-4) subgroup of the gauge group which commutes with the residual gauge symmetry SU(4); this is the group of unitary transformations which act on the components 5 through N of an N-vector. Let Q^a be the corresponding generator of the flavor SU(N-4). Then

$$\tilde{Q}^a = (Q^a + Q^a{}_{(N)}) \tag{5.33}$$

is respected by both terms of (5.30). Similarly, let $Q_{(N)}$ be the broken SU(N) generator

$$Q_{(N)} = \left(\begin{array}{cccc|ccc} N-4 & & & & & & \\ & N-4 & & & & & \\ & & N-4 & & & & \\ & & & N-4 & & & \\ \hline & & & & -4 & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & -4 \end{array}\right) \qquad \text{(acting on N)} \tag{5.34}$$

and define the combination

$$\tilde{Q} = 1/N\ (2Q + Q_{(N)}) \tag{5.35}$$

We can compute the $\tilde{Q}$ charges of the various fermions as we did in eq. (5.27). For c,d = 1, ... 4 and i,j = 1, ... (N-4):

$$\tilde{Q}\psi_{a,i} = \frac{1}{N}\left(\frac{[2(N-2) - (N-4)]\psi_{c,i}}{[2(N-2) + 4]\psi_{j,i}}\right) = \left(\frac{1\cdot\psi_{c,i}}{2\cdot\psi_{j,i}}\right)$$

$$\tilde{Q}\psi^{ab} = \left(\begin{array}{c|c} 0\cdot\psi^{cd} & (-1)\cdot\psi^{c(i+4)} \\ \hline & (-2)\cdot\psi^{(i+4)(j+4)} \end{array}\right) \tag{5.36}$$

The mass terms of (5.30) have zero total charge; thus, they are left invariant by the symmetry generated by $\tilde{Q}$. We have now identified a complete U(1) × SU(N-4) group of residual global symmetries. In a certain sense, no chiral symmetry has been broken.

Equation (5.36) makes clear that the fermions $\psi_{i,j}$ and $\psi^{(i+4)(j+4)}$ have the right quantum numbers to pair up and acquire mass. Such masses are presumably induced by radiative corrections to this leading-order analysis. However, ψ^{ab} is antisymmetric, so the components of $\psi_{i,j}$ symmetric under intercharge of i and j have no natural partners. These fermions, all singlets under the residual group, remain massless.

We may denote the complete pattern of symmetry-breaking by:

$$[SU(N)] \times U(1) \times SU(4) \rightarrow [SU(4)] \times U(1) \times SU(N-4) \tag{5.37}$$

The gauged symmetry is indicated in brackets. The fermions transform under the unbroken symmetries as

$$\begin{aligned} [2] &\rightarrow (6,\ 0,\ 1) + (4,\ -1,\ \overline{N-4}) + (1,\ -2,\ [\bar{2}]) \\ (N-4) \times \bar{N} &\rightarrow (\bar{4},\ +1,\ N-4) + (1,\ +2,\ [2]) + (1,\ +2,\ \{2\}) \end{aligned} \tag{5.38}$$

The last multiplet, in the $\{2\}$ of SU(N-4), is protected from acquiring mass by its nontrivial quantum numbers under the unbroken chiral symmetry.

It is remarkable that the gluons and instantons give the same pattern of symmetry breaking, especially since, for the gluons, one must look channel by channel, while, for the instantons, one must look at the global pattern of χSB and induce masses for all fermions seen by a given instanton. I might shed a little light on the mystery by noting that both schemes require the pattern of symmetry breaking to respect the following properties:

(1) The spontaneous breaking of the original gauge group G_s may stop at a subgroup H_s only if all fermions transforming nontrivially under H_s can acquire mass.

(2) The condensation of a pair of fermions in the representation N of of SU(N) may occur only in the channel $N \times N \rightarrow [2]$, not in $N \times N \rightarrow \{2\}$.

The gluon scheme implies (1) because, as we saw in section 3.3, as long as there are strong gauge forces coupling massless fermions, there is always an attractive channel to further χSB. The instanton scheme implies (1) because the instanton amplitude vanishes if even one fermion in the instanton field is massless. The gluon scheme implies (2) because $\{2\}$ is not an attractive channel. The instanton scheme implies (2) because each N has only one zero mode, whereas a symmetric mass term destroys two fermions. These rules turn out to be quite restrictive; it is straightforward to enumerate all of the patterns of chiral symmetry breaking which satisfy them. In the model considered in this section, one can check that, among these patterns, the simple one I have selected leads to the largest coefficient of the 1-instanton amplitude.

Perhaps this coincidence of conclusions is evidence that the pattern of symmetry-breaking I have suggested for this class of models is the correct one. In any event, let me emphasize that the question we have addressed in this section — the qualitative issue of which pattern of chiral symmetry breaking a given gauge model chooses — is one for which we have no definitive solution. It is certainly a problem worthy of further attention.

Chapter 6.

CAN STRONGLY-COUPLED GAUGE THEORIES LEAVE MASSLESS FERMIONS?

6.1 THE 'T HOOFT ANOMALY CONDITION

A surprising aspect of the patterns of χSB I have described in the previous section is the appearance of massless fermions — fermions protected from acquiring mass by unbroken chiral symmetries. Massless bosons appear in a wide variety of physical systems, as a consequence of Goldstone's theorem, but, in Nature, massless fermions seem rarely, if ever, to arise as a consequence of strong interactions. Nevertheless, one might be tempted to suggest for these gauge models an analogue of Goldstone's theorem which I might state as follows: If spontaneously broken chiral symmetries imply the existence of massless (Goldstone) bosons, then unbroken chiral symmetries imply the existence of massless fermions. Remarkably enough, a more precise version of this statement is actually true. It was first realized, and proved, by 't Hooft [43].*

't Hooft's theorem has had a wide-ranging importance in the study of chiral symmetry, first of all, since it gives a very strong constraint on the existence and pattern of χSB, and, secondly, since it is one of the few really solid results in this subject. I will, therefore, devote this section to a discussion of this theorem and its consequences.

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*Some aspects of 't Hooft's argument were presented also in work of Ansel'm [44] and Zee [45].

Let me first present 't Hooft's beautiful and very simple argument: Imagine that I have a theory with chiral symmetry group G which has strong-coupling dynamics (and perhaps confinement) at a momentum scale Λ. At momenta much greater than Λ, one can see the elementary fermions. At the scale Λ, these fermions form bound states, most of which have mass of order Λ but some of which might be massless. At momenta much less than Λ, the massive states are irrelevant (in the technical sense) to the dynamics, only the massless states contribute. I will assume that G is not at all spontaneously broken; however, the conclusions of this analysis will also apply to any unbroken subgroup of G if G is partially broken.

Now imagine adding to the theory gauge bosons coupled, very weakly, to the currents of the chiral symmetry G. In general, the conservation of the gauge currents will be spoiled by anomalies, as I discussed in section 5.2. If this is so, invent some new fermions χ which couple only to the new gauge bosons, and add them to the theory to cancel the anomalies. Now look at the therory at momenta well below Λ. At such momentum scales, the theory effectively contains only the G gauge bosons, the fermions χ, and the massless bound states of the strongly interacting fermions. But nothing has happened to spoil the local gauge invariance with respect to G. Hence, the massless bound states of the strongly interacting fermions must have just the right quantum numbers to cancel the G anomaly of the χ fermions. This implies that the total G anomaly of these bound states must be equal to the total G anomaly of the original elementary fermions. If A(r) is the anomaly coefficient of the representation r, defined in (5.10) for any simple subgroup of G:

$$\sum_{\substack{\text{original fermions} \\ \text{representations } r}} A(r) = \sum_{\substack{\text{massless fermion bound} \\ \text{state representations } r_B}} A(r_B) \tag{6.1}$$

This relation is known as the 't Hooft anomaly condition. This condition is displayed schematically in fig. 13. If the left-hand side of (6.1) is nonzero, either G must be spontaneously broken, or massless fermions must appear as a result of the strong interaction dynamics.

To aid in understanding this result, let me sketch a more formal proof given by Frishman, Schwimmer, Banks, and Yankielowicz [46]. Another argument, which fills in even more details (including a demonstration that bound states of higher spin cannot contribute to the right-hand side of 't Hooft's equation) has been given recently by Coleman and Grossman [47]. Frishman et. al., analyze dispersively the vertex function of three G symmetry currents, shown in fig. 14. In describing their analysis, I will work directly in the limit $k^2 = p^2 = 0$ (in terms of the kinematics of fig. 14) and in the limit of zero fermion masses, simply assuming that these limits are reached smoothly. I will also suppress the group indices a,b,c.

In analyzing fig. 14, we may take two of the three currents to be conserved; the third is spoiled by the anomaly. Let the currents J_ν, J_λ be conserved; then this amplitude is symmetric under the interchange of (ν,k) and (λ,p). The most general structure for an (odd parity) amplitude with this symmetry is:

$$\begin{aligned}\Gamma_{\mu\nu\lambda} = \; & A_1(q^2)\cdot\epsilon_{\mu\nu\lambda\alpha}(k-p)^\alpha \\ & + A_2(q^2)\cdot(q_\mu\epsilon_{\nu\lambda\alpha\beta}k^\alpha p^\beta) \\ & + A_3(q^2)\cdot(k_\nu\epsilon_{\mu\lambda\alpha\beta}k^\alpha p^\beta - p_\lambda\epsilon_{\mu\nu\alpha\beta}k^\alpha p^\beta) \\ & + A_4(q^2)\cdot(k_\lambda\epsilon_{\mu\nu\alpha\beta}k^\alpha p^\beta - p_\nu\epsilon_{\mu\lambda\alpha\beta}k^\alpha p^\beta)\end{aligned} \tag{6.2}$$

The value of $A_1(q^2)$ at $q^2 = 0$ is determined by the anomaly. The Adler-Bardeen theorem [24] fixes this value to be:

$$A_1(q^2) = \frac{1}{4\pi^2} \sum_r A(r) \tag{6.3}$$

This term arises from short-distance dynamics and is therefore purely real. Current conservation at the vertex ν ($k^\nu \Gamma_{\mu\nu\lambda} = 0$) implies

$$-A_1(q^2) + \frac{q^2}{2} A_4(q^2) = 0 \tag{6.4}$$

We may recast this equation, using a dispersion relation for $A_4(q^2)$, as

$$A_1(q^2) = \frac{q^2}{2\pi} \int ds \frac{\text{disc } A_4(s)}{s - q^2} \tag{6.5}$$

However, $A_1(q^2)$ is real, so

$$\text{disc } A_1(q^2) = \frac{q^2}{2\pi} \text{ disc } A_4(q^2) = 0 \tag{6.6}$$

Equations (6.3), (6.5), and (6.6) are incompatible unless

$$\text{disc } A_4(q^2) = C\, \delta(q^2) \quad ; \quad C = \frac{1}{2\pi} \sum_r A(r) \tag{6.7}$$

Such a discontinuity of a vertex function signals the presence of physical intermediate states of zero mass. However, two different types of states can give rise to the behavior (6.7). A single massless boson created by the J_μ produces a discontinuity of this form, but a current can create a single massless boson only if its associated symmetry is spontaneously broken. Alternatively, a pair of fermions created by the current produces a discontinuity which tends, in the limit of zero fermion mass, to the form $\delta(q^2)$. The coefficient of this delta function is given by computing the triangle diagrams which yield the anomaly for fermions with the quantum numbers of the physical massless fermions of the theory. Since (6.3) arose from computing the same triangle diagrams with the elementary fermions, (6.7) implies eq. (6.1) (or fig. 13) directly in this case. Both of these alternatives may be avoided if the limit $k^2 \to 0$ or $p^2 \to 0$ is singular, but only if the singularity corresponds to the pole of a massless boson created by J_ν or J_λ. In either case G must be spontaneously broken, as we have noted above. We can conclude that either G is spontaneously broken, or the 't Hooft anomaly condition, eq. (6.1), holds.

6.2 'T HOOFT'S ANOMALY CONDITION IN QCD

Equation (6.1), the 't Hooft anomaly condition, is an extremely powerful constraint on the pattern of χSB. It implies that chiral symmetries which protect fermions from acquiring mass must be spontaneously broken unless it is possible to form physical fermions with the right quantum numbers to match the anomalies of the original fermions. Since the quantities to be matched are cubic in charges, this condition is not at all straightforward to satisfy, as the validity of Fermat's Last Theorem for $n = 3$ might remind us. I will spend the remainder of section 6 describing several applications of the 't Hooft anomaly condition, first, to theories of Dirac fermions, and then to chiral theories.

't Hooft, in his original paper [43], used his condition to argue that chiral symmetry must be broken in the usual strong interactions, as described by QCD. I would like to explain the logic of that argument by presenting some specific examples. (For the details of the general analysis, the reader should consult the paper of 't Hooft. Some refinements of this argument have been given in [46,48,49].) I will first consider the case of an SU(3) color gauge theory with two massless quarks belonging to the 3 of color. In this case

$$G = U(1) \times SU(2)_L \times SU(2)_R \tag{6.8}$$

Let us assume that color is confined; then any physical fermions contributing to (6.1) must be color-singlet spin one-half bound states of quarks.

It is easy to form color-singlet combinations of quark fields; to project these singlet states into spin one-half is also straightforward if one uses some elementary properties of the Lorentz group [50]: Left- and right-handed fermions transform according to distinct, complex conjugate, two-dimensional representations of the Lorentz group. (We used this fact implicitly in the development of section (2.1).) These representations may be considered as the spin one-half representations of two different angular momenta; all other finite-dimensional representations of the Lorentz group can be built up from these by addition of angular momenta. Let us denote objects which transform as L- and R-fermions by spinor indices α, β and η,λ, respectively. Then an L-fermion composite state is formed by contracting all η,λ indices and all but one α index with the invariant tensors $\epsilon^{\eta\lambda}$, $\epsilon^{\alpha\beta}$. In trying to enumerate all possible such composite states, one should remember that the complex conjugate of any R-fermion composite is an L-fermion composite.

Using the notation described in the previous paragraph, we can write the left- and right-handed quark fields as

$$\psi_{\alpha a}{}^{i} \quad , \quad \psi_{\eta a i} \tag{6.9}$$

where $\alpha,\eta = 1,2$, $a = 1,2,3$ is the color index, and $i = 1,2$ is the flavor index. The U(1) factor in (6.8) is quark number; it assigns to both of these fields the charge (+1). It will be useful to refer to the quarks in the notation of (6.9) for the purpose of constructing composite states. However, it will be easiest to compute anomalies by recording the quantum numbers of L-fermions, imagining that R-fermions have been charge-conjugated. With this convention, the (L-fermion) quantum numbers under G of the fields in (6.9) are:

$$((+1),\ 2,\ 1) \quad \text{and} \quad ((-1),\ 1,\ 2) \quad . \tag{6.10}$$

By combining triplets of the quarks (6.9), we can form color-singlet spin one-half states with the quantum numbers of

$$\epsilon_{abc}\psi_{\alpha a}{}^{i}\psi_{\eta bj}\psi_{\lambda ck}\epsilon^{\eta\lambda}$$

and

$$\epsilon_{abc}\psi_{\eta ai}\psi_{\alpha b}{}^{j}\psi_{\beta c}{}^{k}\epsilon^{\alpha\beta} \tag{6.11}$$

Both of these objects are antisymmetric with respect to interchange of j and k, so that j and k must pair to an SU(2) singlet. These objects then transform under G as:

$$((+3),\ 2,\ 1) \quad \text{and} \quad ((-3),\ 1,\ 2) \quad , \tag{6.12}$$

respectively, in the notation of (6.10). If chiral symmetry were broken spontaneously, to U(1) × (isospin SU(2)), the corresponding composite fermions would have precisely conjugate quantum numbers and could pair to form an isospin doublet of massive Dirac fermions. These fermions would, in fact, be the familiar proton and neutron. But if G remains unbroken, these states are left massless, protected by chiral invariance. Does this situation satisfy 't Hooft?

Because G is not a simple group, we must check (6.1) for all combinations of components of G. Some examples of anomalies which must be matched are shown in fig. 15. Many of these anomalies, though, vanish trivially. Since the 2 of SU(2) is its own complex conjugate, the anomaly (a) of fig. 15 vanishes by (5.11) for both the elementary and the physical fermions. Since (b) involves only one $SU(2)_L$ current, its group theory weight contains the factor

$$Tr(t^a) = 0 \tag{6.13}$$

and therefore vanishes. Other anomalies involving one $SU(2)_L$ or $SU(2)_R$ current vanish similarly. Because the U(1) charges of both elementary and physical fermions are paired, (c) must also be zero. In fact, the only nontrivial constraint comes from the anomalies of type (d). (The corresponding anomaly with $SU(2)_R$ currents is equal and opposite, by parity.) (d) is proportional to the factor:

$$Tr[Qt^a t^b] = \sum_r n_r Q(r) C(r) \tag{6.14}$$

Let us tabulate the contributions to (6.14) implied by (6.10) and (6.12) (remembering that the elementary fermions have three color states):

elementary fermions	$n_r \cdot Q(r) \cdot C(r)$:	
((1), 2, 1):	$3 \cdot 1 \cdot (1/2)$	sum = 3/2
((-1), 1, 2):	$3 \cdot (-1) \cdot 0$	
composite fermions	**$n_r \cdot Q(r) \cdot C(r)$:**	
((3), 2, 1):	$1 \cdot 3 \cdot (1/2)$	sum = 3/2
((-3), 1, 2):	$1 \cdot (-3) \cdot 0$	

(6.15)

So it is apparently formally consistent to have a theory with two massless flavors in which G is unbroken and realized with a massless proton and neutron.

What if there are three massless quark flavors? One might naively imagine that simply changing $i = 1,2$ to $i = 1,2,3$ in the above analysis would give a consistent solution. Let us check. The chiral symmetry G is now $U(1) \times SU(2)_L \times SU(3)_R$; the elementary fermions now transform as

$$((+1),\ 3,\ 1) \quad \text{and} \quad ((-1),\ 1,\ \bar{3}) \tag{6.16}$$

and the composite states (6.11) as

$$((+3),\ 3,\ \bar{3}) \quad \text{and} \quad ((-3),\ 3,\ \bar{3}) \tag{6.17}$$

We have used the fact that, in SU(3), the 3 and $\bar{3}$ are inequivalent and $[2] = \bar{3}$. Since SU(3) representations are not necessarily real, we must check the matching of the anomalies (a) and well as (d) of fig. 15; (a) is proportional to

$$\sum_r n_r A(r) \tag{6.18}$$

Let us, then, tabulate:

elementary fermions:	$n_r \cdot A_r$:		$n_r \cdot Q(r) \cdot C(r)$:	
$(+1,\ 3,\ 1)$	$3 \cdot 1$	3	$3 \cdot 1 \cdot (1/2)$	3/2
$(-1,\ 1,\ \bar{3})$	$3 \cdot 0$		$3 \cdot (-1) \cdot 0$	
composite fermions:				
$(+3,\ 3,\ \bar{3})$	$3 \cdot 1$	6	$3 \cdot 3 \cdot (1/2)$	0
$(-3,\ 3,\ \bar{3})$	$3 \cdot 1$		$3 \cdot (-3) \cdot (1/2)$	

(6.19)

(For the composite fermions, the $SU(3)_R$ multiplicity contributes $n_r = 3$.) The anomaly matching is a disaster. Trial and error indicates that adding more complicated color-singlet bound states only makes matters worse.

To obtain a more general result, 't Hooft introduces a further assumption: One should consider only solutions to the anomaly conditions for n flavors with the property that, if the mass of any one flavor is taken to be nonzero, every composite state containing this fermion may acquire mass by pairing with another composite with the same quantum numbers under the subgroup of chiral symmetries $(U(1) \times U(1) \times SU(n-1)_L \times SU(n-1)_R)$ which preserves this mass term. This condition has been labelled the "Appelquist-Carazone decoupling" or the "persistent mass" condition. It is straightforward to show that there are no solutions to (6.1) satisfying this condition for any color group SU(N) [43,46,48].

One might be tempted to conclude from this argument that chiral symmetry must be broken in any SU(N) gauge theory with n Dirac flavors, if $n > 2$. However, the question is really far from settled. If the mass of one flavor is taken to infinity, it is clear that all composite states containing this fermion must become infinity heavy, but if one flavor has a nonzero but small mass, the strong interactions might well arrange that composites containing this fermion have zero mass [51]. Dimopoulos and Preskill have given some examples in which it is particularly plausible that massless composite fermions should contain massive constituents [52]. This method of escape from 't Hooft's analysis requires, however, a complicated phase structure: The mass spectrum of the theory must change discontinuously as a function of the fermion masses. I should also note that relatively few solutions to (6.1) are known even for particular choices of the number of colors N and the number of flavors n. Weinberg [53] has found a solution for $N = 5$, $n = 3$; Albright [54] has constructed a rather lengthy catalogue. The most striking feature of these particular solutions, though, is their complexity and ugliness; one must, in general, accept a large multiplet of massless fermions in order to maintain chiral symmetry.

Coleman and Witten [55] have made further use of the 't Hooft anomaly condition by combining it with the limit $N \to \infty$ to obtain information on the pattern of χSB. There are indications from strong-interaction phenomenology that QCD with $N = 3$ is already rather close to this limit [56]. Equation (6.1) has no solution smooth in N as $N \to \infty$; thus, Coleman and Witten conclude, chiral symmetry must be broken in this limit. But one can then use another property of this limit, that graphs with internal fermion loops are suppressed by powers of N^{-1}. A general color singlet mass term may be diagonalized as follows:

$$\bar{\psi}_{Ri}\Sigma_{ij}\psi_{Lj} = \sum_i (\bar{\psi}'_{Ri} m_i \psi'_{Li}) \tag{6.20}$$

by making independent unitary transformations on ψ_{Ri} and ψ_{Lj}. The leading contribution of Σ to $\Gamma(\Sigma)$ in powers of N^{-1} will involve graphs with precisely one fermion loop (with arbitrary gluon dressing); $\Gamma(\Sigma)$ will therefore have the form:

$$\Gamma(\Sigma) = N\left(\sum_i F(m_i) + O(N^{-1})\right) \tag{6.21}$$

Since chiral symmetry is broken, F must have its minimum at some $m_0 \neq 0$. Then each flavor acquires this same mass m_0. Thus, in the $N \to \infty$ limit, the SU(n) flavor symmetry is not broken; the pattern of χSB is

$$U(1) \times SU(n) \times SU(n) \longrightarrow U(1) \times SU(n) \quad , \tag{6.22}$$

as we see in the familiar strong interactions.

6.3 'T HOOFT'S ANOMALY CONDITION IN CHIRAL MODELS

In chiral gauge theories, 't Hooft's anomaly condition may be realized in a more interesting way. As an example of what might happen, let us consider again the strong interactions of the SU(5) model of Georgi and Glashow. In section 5.3, we analyzed this model rather thoroughly in terms of possible modes of chiral symmetry breaking. I would now like to approach this model from the opposite viewpoint: I will assume that both the gauge and chiral symmetries of the model remain unbroken and ask whether one can find composite fermions, bound by confining SU(5) forces, which can satisfy 't Hooft's anomaly condition.

Let me recall that the SU(5) model contains fermions in the 10 and $\bar{5}$ representations of SU(5) (eq. (5.14)) and only one U(1) global symmetry, whose charge Q satisfies (5.23). The anomaly of the elementary fermions with respect to three U(1) currents is proportional to

$$\mathrm{Tr}\, Q^3 = \sum_r n_r (Q(r))^3 = 5\cdot(3)^3 + 10\cdot(-1)^3 = 125 \qquad (6.23)$$

But consider the following SU(5)-singlet composite state:

$$\epsilon^{\alpha\beta\underline{\gamma}}\psi_{\alpha a}\psi_{\beta}{}^{ab}\psi_{\gamma b} \quad ; \qquad (6.24)$$

this state has charge Q = 5 and thus contributes an anomaly

$$\mathrm{Tr}\, Q^3 = (5)^3 = 125 \qquad (6.25)$$

(Note that the charge $\tilde{Q}$ defined in (5.25) has the value $\tilde{Q} = 2$ in this state.) This example was discovered by Dimopoulos, Raby, and Susskind [57]. They also gave this example a beautiful physical interpretation, which I will discuss in a moment. Let me first present their demonstration that this construction works for all the theories with fermion content ([2] + (N-4)$\cdot\bar{N}$) which we considered last time.

The fermions of this class of theories are those which appear in (5.32); the global symmetries of these theories are the SU(N-4) transformations of the $\bar{N}$'s and the U(1) symmetry generated by Q in (5.32). I claim that the following color singlet composite satisfies the t' Hooft anomaly conditions with respect to these symmetries:

$$\epsilon^{\alpha\beta}\psi_{\alpha}{}^{ab}(\psi_{\beta a,i}\psi_{\gamma b,j} + \psi_{\gamma a,i}\psi_{\beta b,j}) \tag{6.26}$$

(6.26) generalizes (6.24), and preserves its property of being symmetric in the spins of the two $\bar{N}$'s. This state has Q = N and belongs to the representation $\{2\}$ of SU(N-4). (Under $\tilde{Q}$ in (5.35), it has charge $\tilde{Q}$ = 2.) The anomaly of three SU(N-4) currents is proportional to

$$\begin{aligned} \sum_r n_r A(r) &= N\cdot A(N-4) \quad \text{for the elementary fermions} \\ &= 1\cdot A(\{2\}) \quad \text{for the composite (6.26)} \end{aligned} \tag{6.27}$$

These two expressions are equal by virtue of (5.12). The anomaly matching must also be checked for three U(1) currents and for one U(1) and two SU(N-4) currents. This last anomaly is proportional to

$$\mathrm{Tr}\; Qt^a t^b = \delta^{ab} \sum_r n_r Q(r)\, C(r) \tag{6.28}$$

To check this condition, one needs the values of C(r) for SU(M) representations

$$C(M) = \frac{1}{2} \quad , \quad C([2]) = \frac{M-2}{2} \quad \left(C(\{2\}) = \frac{M+2}{2} \right) \tag{6.29}$$

With this information, it is a simple exercise (strongly recommended to the reader) to see that (6.26) does indeed balance the anomalies of the elementary fermions.

6.4 COMPLEMENTARITY

It is surprising that the anomaly conditions may be solved so simply for the models we considered in the previous section. But this solution contains a further surprise which, perhaps, has not escaped the reader's notice: The massless composite fermion which we constructed has precisely the same quantum numbers – $\tilde{Q} = 2$, $\{2\}$ of SU(N-4) – as the multiplet of massless fermions which emerged from our analysis of symmetry-breaking patterns for these models in section 5.4. Dimopoulos, Raby, and Susskind [57] recognized that this is not an accident; they showed that these solutions follow from an intriguing physical picture which I wish, in this section, to explain.

I will work toward the picture of Dimopoulos, Raby, and Susskind in two stages. First, I will show that if, in a theory of massless fermions, one breaks the gauge symmetry in such a way that a subset of the original fermions remain massless, those massless fermions always obey the 't Hooft anomaly conditions with respect to the unbroken chiral symmetries [58]. This remark implies that the multiplets of massless fermions we found in section 5.4 indeed satisfy 't Hooft's conditions. Second, I will argue that we can convert the massless elementary fermions we found in section 5.4 into massless composite fermions.

The proof of the first claim is very easy. Recall that in our examples of section 5, the final global symmetry generators $\tilde{Q}^a$ were obtained by forming linear combinations of gauge-invariant global symmetry charges Q^a and global gauge charges $Q_{(N)}{}^a$, as in eqs. (5.33), (5.35). The charges $\tilde{Q}^a$ were constructed to leave the dynamically generated mass terms invariant. In this general context, let us compute the anomaly of the massless fermions with respect to three $\tilde{Q}^a$ currents. The calculation is indicated schematically in fig. 16. Since the dynamical mass terms respect the charges $\tilde{Q}^a$, the fermions which acquire mass must appear in pairs of complex-conjugate representations of the symmetry group generated by the $\tilde{Q}^a$. Equation (5.11) implies that each pair of representations gives two equal and opposite contributions to the anomaly of three $\tilde{Q}^a$ currents. Hence the sum over massless fermions may be replaced by a sum over all of the original fermions. Now we can expand

$$\tilde{Q}^a = Q^a + Q^a{}_{(N)} \quad ; \tag{6.30}$$

the result of this expansion is indicated in the second line of fig. 16. The second term on this line is proportional to the trace of a gauge group generator $Q^a{}_{(N)}$ and therefore vanishes. The third term vanishes if all of the global symmetries Q^a are anomaly-free with respect to the strong gauge interactions. The fourth term is the anomaly of three gauge currents, which must vanish by the considerations of section 5.2. What remains is the first term, which is precisely the left-hand side of (6.1). Thus, 't Hooft's condition is automatically satisfied.

I will now argue that the multiplets of massless composite fermions described in section 6.3 and the multiplets of massless fermions from section 5.4, which satisfy (6.1) by virtue of this argument, are closely related. To see that relation, we should think about the competition between chiral symmetry breaking and confinement in this class of gauge models. In the analysis of section 5.4, we ignored the effects of confinement completely. We found that our simple picture of the dynamics gave a vacuum expectation value to a fermion bilinear

$$\Phi^b{}_i = \epsilon^{\alpha\beta}\psi_{\alpha a,i}\psi_\beta{}^{ab} \tag{6.31}$$

Such a vacuum expectation value would spontaneously break the gauge symmetry. Let us now consider the opposite situation, by assuming that confinement is the dominant effect. Then the vacuum must be locally a color singlet. $\Phi^b{}_i$ cannot acquire a vacuum expectation value; it merely creates a light fermion pair. In any local region of space-time, the vacuum contains pairs of pairs with cancelling color quantum numbers. Pairs of the structure $(\Phi^b{}_i\Phi^{\dagger i}{}_b)$ would break no global or local symmetries.

In such a confining theory, a single elementary fermion $\psi_{\alpha a,i}$ could not be an asymptotic state. However, one could convert it to an asymptotic state by contracting it on its color index with a light pair $\Phi^b{}_j$. Since $\Phi^b{}_j$ is a singlet under $\tilde{Q}$, this contraction does not change the $\tilde{Q}$ quantum numbers of the original fermion. The contracted state, however, is a color singlet; we may evaluate the quantum numbers of this state by replacing $\tilde{Q}^a$ by Q^a. In general, we may convert all of the elementary fermions to color singlets by contracting them with light pairs; the resulting composite fermions will have the same quantum numbers with respect to the Q^a as the elementary fermions would have with respect to the $\tilde{Q}^a$ in an analysis based on χSB through condensation of the pairs Φ. Pairs of these composite fermions with conjugate Q^a quantum numbers may acquire mass; the remaining composite fermions will, by the argument of fig. 16, satisfy the 't Hooft anomaly conditions.

We have now seen that the same conclusions about the unbroken global symmetries and the content of massless fermions in a chiral gauge theory may arise from two different pictures of the dynamics — one in which the gauge symmetry is broken by pair condensation and the massless fermions are elementary, another in which the gauge symmetry remains exact and confining and the massless fermions are color-singlet composites. Fradkin and Shenker [59] have given examples of lattice gauge theories with elementary Higgs fields in which one can move continuously from a region of the parameters of the theory in which the gauge symmetry is confining to one in which the gauge symmetry is spontaneously broken, without crossing through a phase transition point or changing the qualitative behavior of gauge-invariant observables. The analysis we have just given implies that this is possible also in cases where the Higgs field is a fermion bilinear. Dimopoulos, Raby, and Susskind refer to this property as "complementarity."

One should remember, though, that a chiral gauge theory of massless fermions is a unique theory, with no adjustable parameter except for a mass scale. Thus, even if we do not know whether the theory prefers one or the other of the pictures just described, the theory presumably knows, and follows its preference. This preference should reflect the answer to the general question of whether confinement or chiral symmetry breaking is a stronger effect in gauge theories with fermions. Perhaps the considerations of this section will suggest a way to answer this question.

Before moving on to my next topic, I should note parenthetically that the solutions to the 't Hooft conditions found by Dimopoulos, Raby, and Susskind have been generalized by Banks, Yankielowicz, Schwimmer, and Bars [60,61] through a beautiful construction involving graded Lie algebras. Those readers who have not yet been exhausted by group theory should certainly consult these papers.

Chapter 7.

χSB IN SUPERSYMMETRIC THEORIES

7.1 INTRODUCTION AND ORIENTATION

In this section, I would like to indicate how the physics we have discussed so far generalizes to gauge theories with supersymmetry — a symmetry which interchanges bosonic and fermionic states. Supersymmetric field theories have become a topic of intense interest in the past few years; reviews of their general structure have been given in [62,63]. However, most of what we know about these theories applies only to the region of weak coupling. The behavior of supersymmetric gauge theories in the strong-coupling regime is not yet understood even qualitatively. The problem of how these theories behave is an important one — important, I feel, not only to those primarily interested in the study of supersymmetry but also to those more generally interested in the problem of the realization of chiral symmetries.

At first sight, supersymmetric gauge theories do not look at all unusual; they are, to all appearances, ordinary gauge theories of fermions supplemented by the addition of a few innocuous elementary bosons. On the other hand, the supersymmetry of these theories provides extremely powerful constraints on their dynamics. I would like, then, to summarize what is known about the strong-coupling behavior of these theories and to attempt to reconcile that with the intuitive picture of χSB we have been developing.

The central element in this discussion will be a remarkable result proved recently by Witten [64]: In supersymmetric gauge theories containing fermions in a real representation of the gauge group, supersymmetry cannot be spontaneously broken. This result is not at all straightforward, and, when combined with the Ward identities of supersymmetry, it yields some unusual consequences. Witten proved this result by developing a novel intuitive description of supersymmetry. In the remainder of section 7.1, I will introduce supersymmetry through Witten's description and explain the general logic of his proof. In section 7.2, I will briefly review the structure of supersymmetric gauge theories and explain why Witten's result is counter-intuitive. In section 7.3, I will show that, nevertheless, it is true. Section 7.4 will then present a set of possibilities for the behavior of supersymmetric gauge theories; I invite the reader to puzzle out which choice is correct.

We should first ask what, more precisely, is supersymmetry. A theory is supersymmetric if it has a conserved charge Q_α which converts bosons to fermions and vice versa. If Q is taken to transform as an L-fermion, $Q^\dagger$ transforms as an R-fermion, and the quantity

$$(Q_\alpha Q^\dagger{}_\eta + Q^\dagger{}_\eta Q_\alpha) \tag{7.1}$$

then transforms as a spin-1 object, with no scalar piece. Nevertheless (7.1) is a conserved charge, which, further, can vanish only if Q itself vanishes. This would be highly unusual unless (7.1) is proportional to $P^\mu = (H,\vec{P})$, the energy-momentum 4-vector. In fact, any other choice may be seen to forbid nontrivial scattering [65]. Thus, we can form a Hermitian linear combination of Q_α and $Q^\dagger{}_\eta$, which I will also call Q, which satisfies

$$Q^2 = H \tag{7.2}$$

on the subspace $\vec{P} = 0$. Remarkably, many interesting Hamiltonians may be represented in this form. Equation (7.2) implies that H is non-negative. Further, the fact that H can be represented as the square of Q implies that the eigenstates of H appear in boson-fermion pairs:

$$Q|b\rangle = \lambda|f\rangle \qquad Q|f\rangle = \lambda|b\rangle \qquad \text{with} \qquad \lambda = \sqrt{E} \tag{7.3}$$

This pairing of eigenstates holds even in a finite volume where the spectrum of H is discrete, as long as the boundary conditions respect the conservation of Q. (Periodic boundary conditions have this property.) The only exceptions to this pairing are the states annihilated by Q; these may be arbitrary in number. In general, then, the spectrum of H, in a finite volume, has the form shown in fig. 17.

We say that supersymmetry is unbroken if the ground state $|\Omega\rangle$ of H is annihilated by Q:

$$Q|\Omega\rangle = 0 \tag{7.4}$$

Since H is non-negative, any state which satisfies (7.4) will be the ground state; thus supersymmetry is unbroken if there exists any state which satisfies (7.4). Further, if, in any finite volume, there exists a state $|\Omega\rangle$ annihilated by Q, there will also be such a state in the infinite volume limit. Thus, it suffices to examine the spectrum of H in a finite volume to show that supersymmetry is not spontaneously broken; this is one respect in which supersymmetry differs from an ordinary global symmetry.

If H depends on a parameter g, we might think about the spectrum of H as a function of g. As g is changed continuously, the energy levels move continuously. But if H(g) is supersymmetric for any g, the energy levels with energy $E > 0$ must be paired. Thus, zero eigenstates can move away from zero only in pairs. This means that all Hamiltonians $H(g)$ which can be reached from one another by continuous deformation have the same value of the following quantity, called Witten's index:

$$Tr(-1)^F = (\text{number of bosonic zero-energy states}) - (\text{number of fermionic zero-energy states}) \tag{7.5}$$

But if $Tr(-1)^F$ 6 0, a zero-energy state exists and supersymmetry is unbroken. The strategy of Witten's proof is to start from an interesting Hamiltonian $H(g)$ and adjust g (which may be, for example, the coupling constant or a particle mass) to a value which makes it easy to prove that $Tr(-1)^F$ 6 0.

I should illustrate this strategy as it applies to a particularly simple example of a supersymmetric theory — supersymmetric quantum mechanics [66]. Let us define

$$Q = \frac{1}{\sqrt{2}} (p\sigma^1 + W(q)\sigma^2) \tag{7.6}$$

where q,p are coordinate and momentum variables and σ^1, σ^2 are the Pauli matrices. Then

$$Q^2 = H = \left[\frac{1}{2} p^2 + \frac{1}{2} (W(q))^2 + \frac{1}{2}\left(\frac{dW}{dq}\right)\sigma^3\right] \tag{7.7}$$

This Hamiltonian describes a quantum-mechanical particle with spin interacting with an external potential $V(q) = 1/2\, W^2(q)$ and an external magnetic field. A typical form for $W(q)$, and the corresponding potential $V(q)$, is shown in fig. 18. By analogy to the higher-dimensional case, I will call spin-up states fermonic and spin-down states bosonic.

Near a zero of $W(q)$

$$W(q) \approx \omega(q-q_0) \tag{7.8}$$

and the potential takes the form of a harmonic oscillator. If the nonlinear terms in W(q) are small, we can find the spectrum of states near q_0 by diagonalizing the harmonic oscillator Hamiltonian:

$$H = 1/2\ p^2 + 1/2\ \omega^2\ (q-q_0)^2 + 1/2\ \omega\sigma^3 \tag{7.9}$$

The eigenvalues of H in (7.9) have the form

$$E_n = n|\omega| \tag{7.10}$$

where n begins with 0 or 1, depending on the spin. The spectra for $\omega > 0$ and $\omega < 0$ are shown in fig. 19. If we treat each zero of W in the harmonic approximation, there is one zero-energy state associated with each such zero; it is bosonic or fermionic according to the sign of ω. Thus the difference in the number of bosonic and fermionic zero-energy states is predicted to be:

$$\begin{aligned} \mathrm{Tr}(-1)^F &= (\text{number of zeros of } W \text{ with } \omega > 0) \\ &\quad - (\text{number of zeros of } W \text{ with } \omega < 0) \\ &= \begin{cases} 1 \text{ if } W > 0 \text{ as } q \to \infty,\ W < 0 \text{ as } q \to -\infty \\ 0 \text{ if } W > 0 \text{ or } W < 0 \text{ for both } q \to \pm\infty \\ -1 \text{ if } W < 0 \text{ as } q \to \infty,\ W > 0 \text{ as } q \to -\infty \end{cases} \quad . \end{aligned} \tag{7.11}$$

In this simple example, we can also solve the equation $Q\psi = 0$ directly. Since

$$Q = \frac{-i}{\sqrt{2}} \left(\begin{array}{c|c} & \dfrac{d}{dq} + W(q) \\ \hline \dfrac{d}{dq} - W(q) & \end{array} \right) \tag{7.12}$$

$Q\psi = 0$ if and only if

$$\Psi(q) = \exp\left[-\int_0^q dx\ W(x)\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{(bosonic state)}$$

or

$$\Psi(q) = \exp\left[+\int_0^q dx\ W(x)\right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{(fermionic state)} \tag{7.13}$$

In general, only one of these states will be a normalizable wave function; the condition that the bosonic or fermion state of (7.13) is normalizable is precisely the condition given in (7.11) for $\mathrm{Tr}(-1)^F = 1$ or -1. The conclusion of (7.11) is therefore exact, providing a first check on Witten's formalism. This example also provides at least an idea of what perturbations of H constitute continuous deformations to which Witten's analysis apply. If $W(q) = Aq^3$, we can change A or add a q^2 term without changing $\mathrm{Tr}(-1)^F$. However, if we change the asymptotic behavior of W by adding a term Bq^4, we bring in a new zero from ∞; this must be a singular perturbation.

7.2 SUPERSYMMETRIC THEORIES IN FOUR DIMENSIONS

Now that we have some idea of the general structure of supersymmetric theories, it is time that we surveyed in more detail the structure of supersymmetric theories in four dimensions [62,63]. To establish notation, let me write the Dirac matrices in the form:

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \tag{7.14}$$

as we did in (2.4). Then we can write the algebra of Q_α and $(Q_\alpha)^\dagger = \bar{Q}_\eta$ as follows:

$$\left\{Q_\alpha, \bar{Q}_\eta\right\} = 2(\bar{\sigma}^\mu)_{\eta\alpha} P_\mu$$

$$\left\{Q_\alpha, Q_\beta\right\} = \left\{\bar{Q}_\eta, \bar{Q}_\lambda\right\} = 0 \tag{7.15}$$

Then

$$Q = \frac{1}{\sqrt{2}} (Q_\alpha + \bar{Q}_\eta) \qquad \text{for} \qquad \alpha = \eta = 1 \text{ or } 2 \tag{7.16}$$

satisfies

$$Q^2 = H \tag{7.17}$$

on the subspace $\vec{P} = 0$.

It is possible to represent the algebra (7.15) by the following action on a multiplet of fields (A, ψ_α, F), where A and F are complex scalar fields, and ψ_α is an L-fermion:

$$\begin{aligned} [Q_\alpha, A] &= \psi_\alpha & [\bar{Q}_\eta, A] &= 0 \\ \{Q_\alpha, \psi_\beta\} &= \epsilon_{\alpha\beta} F & \{\bar{Q}_\eta, \psi_\beta\} &= +2i(\bar{\sigma}^\mu)_{\eta\beta} \partial_\mu A \\ [Q_\alpha, F] &= 0 & [\bar{Q}_\eta, F] &= -2i(\sigma^\mu)_{\beta\eta} \partial_\mu \psi_\beta \end{aligned} \tag{7.18}$$

Equation (7.18) involves anticommutators between pairs of fermionic operators. You may check that (7.18) implies that

$$\{Q_\alpha, \bar{Q}_\eta\} = 2i(\bar{\sigma}^\mu)_{\eta\alpha} \partial_\mu \tag{7.19}$$

acting on A, ψ, F. I will refer to a set of fields such as (A,ψ,F) mixed by the supersymmetry algebra as a supermultiplet.

For a theory with several supermultiplets, the following Lagrangian is invariant under the transformations (7.18):

$$L = \bar{\psi}_i i \not{\partial}_L \psi_i + \partial^\mu A^\dagger{}_i \partial_\mu A_i + F^\dagger{}_i F_i - \left[\left(\frac{1}{2} \epsilon^{\alpha\beta} \psi_{\alpha i} \psi_{\beta j} \frac{\partial^2 V(A)}{\partial A_i \partial A_j} + F_i \frac{\partial V(A)}{\partial A_i}\right) + \text{h.c.}\right] \tag{7.20}$$

(The indices i,j are to be summed over all the supermultiplets of the theory.) V(A) is an arbitrary function of the A_i; it must be cubic or lower if the theory is to be renormalizable. F_i is just a Lagrange multiplier; it may be integrated away. This theory has one complex boson for each chiral fermion and constraints between boson and fermion vertices, but it is otherwise fairly arbitrary in form.

If the system (7.20) possesses a global symmetry, it may be coupled supersymmetrically to the gauge bosons A^a_μ of this symmetry group. The gauge field belongs to a supermultiplet $(A^a_\mu, \lambda^a_\alpha, D^a)$, where λ is an L-fermion and D is a real scalar field; all three fields belong to the adjoint representation of the gauge group. To couple this multiplet to (7.20), change ∂_μ to $D_\mu = (\partial_\mu - igA_\mu \cdot t)$ in that Lagrangian and add the set of couplings

$$L_I = g(\epsilon^{\alpha\beta}A^\dagger{}_i\lambda_\alpha \cdot t\psi_{\beta i} + h.c.) + g(A^\dagger{}_i t^a A_i)D^a \qquad (7.21)$$

and the kinetic terms for the gauge fields

$$L_G = -\frac{1}{4}(F^a{}_{\mu\nu})^2 + \bar{\lambda}i\not{\partial}_L\lambda + \frac{1}{2}(D^a)^2 \qquad (7.22)$$

(A, λ, D) have rather complicated transformation laws, but the supersymmetry transformations of gauge-invariant combinations of (A, ψ, F) are not changed by the gauging.

If $V(A) = 0$, the Lagrangian composed of the sum of (7.20), (7.21), and (7.22) has a set of global symmetries which straightforwardly generalize the symmetries (2.15). If this Lagrangian contains n_r matter supermultiplets (A, ψ, F) transforming under the gauge group according to a representation r, the Lagrangian formally has the full $U(n_r)$ global flavor symmetry. It also has, formally, a U(1) symmetry of phase transformations on the fermions:

$$\lambda^a \to \exp[i\beta]\lambda^a \qquad \psi_i \to \exp[i\beta]\psi_i \qquad (7.23)$$

often called R-invariance. The anomaly removes one U(1) symmetry from this set of symmetries; the global symmetry of the model is therefore

$$G = \prod_r U(n_r) \qquad (7.24)$$

If the matter multiplets form complex-conjugate pairs, it is possible to break some of these chiral symmetries softly by adding a mass term

$$V(A) = \frac{1}{2} \sum_{\substack{\text{pairs} \\ r,\bar{r}}} \sum_{j=1}^{n_r} m_r A_{rj} A_{\bar{r}j} \tag{7.25}$$

The potential in (7.25) gives equal masses m_r to the bosons and fermions belonging to r.

In this latter class of theories, an application of supersymmetry leads to a rather unusual identity [67,68]. Consider the anticommutator

$$\left\{ Q_\alpha, A_r \cdot \psi_{\bar{r}\beta} \right\} = \psi_{r\alpha} \cdot \psi_{\bar{r}\beta} + \epsilon_{\alpha\beta} A_r \cdot F_{\bar{r}} \tag{7.26}$$

If supersymmetry is not spontaneously broken, $Q_\alpha|\Omega\rangle = 0$ and the vacuum expectation value of (7.26) takes the form:

$$0 = \langle\Omega|\psi_{r\alpha} \cdot \psi_{\bar{r}\beta}|\Omega\rangle + \epsilon_{\alpha\beta}\langle\Omega|A_r \cdot F_{\bar{r}}|\Omega\rangle \tag{7.27}$$

The first term on the left-hand side is equal to $1/2\ \epsilon_{\alpha\beta}\langle\bar{\psi}_L \psi_R\rangle$; thus

$$\frac{1}{2} \langle\Omega|\bar{\psi}_L \psi_R|\Omega\rangle = - \langle\Omega|A_r \cdot F_{\bar{r}}|\Omega\rangle \tag{7.28}$$

However, if $V(A) = 0$, $F_{\bar{r}} = 0$ by its equation of motion, and therefore $\langle\bar{\psi}\psi\rangle = 0$. If we try to perform this analysis more carefully by adding a small mass term (7.25) and then sending $m_r \to 0$, we find $F_{\bar{r}} = mA^\dagger{}_r$, so that

$$\frac{1}{2} \langle\bar{\psi}_L \psi_R\rangle = -m\langle|A_r|^2\rangle \tag{7.29}$$

One can check that, if the dynamical parts of boson and fermion masses $\Sigma(p)$ have the behavior (3.30), the left- and right-hand sides of (7.29) contain equal ultraviolet-divergent contributions proportional to m and are otherwise not ultraviolet-sensitive. We seem to find, then, that $\langle\bar{\psi}\psi\rangle \to 0$ as m is taken to zero, as a consequence of manifest supersymmetry.

The authors who discovered this Ward identity — Dimopoulos and Raby [67] and Dine, Fischler, and Srednicki [68] — used it to argue, not implausibly, that supersymmetry must be broken in this class of theories, since a nonzero vacuum expectation value for $\bar{\psi}\psi$ would follow from the (presumably) well-understood dynamics we have studied in the past few sections. Thus, it was quite surprising that Witten could prove that supersymmetry is not broken.

I should note, however, that it is not unreasonable that $\langle |A_r|^2 \rangle$ might be singular as m^{-1} as m is taken to zero. We have already seen that the propagator of a related equation has this property:* This singular behavior is just what we found in eq. (2.38) for the modified Klein-Gordon propagation (2.36). One can plausibly argue that the extra $(\sigma \cdot F)$ term in (2.36) and the extra coupling terms for A_r in (7.21) are similar, at least in that both have the effect of cancelling the additive mass renormalization normally present for scalar bosons.

I should, finally, remark that there is no corresponding conceptual barrier which forbids the pair condensation of the fermions λ:

$$\langle \epsilon^{\alpha\beta} \lambda^a{}_\alpha \lambda^a{}_\beta \rangle \neq 0 \tag{7.30}$$

Nilles [69] and Veneziano and Yankielowicz [70] have argued in detail that (7.30) can be embedded in a supersymmetric effective description of supersymmetric Yang-Mills theory.

*I am grateful to Giorgio Parisi for calling eq. (2.38) to my attention in this context.

7.3 WITTEN'S INDEX FOR SUPERSYMMETRIC GAUGE THEORIES

Now that we have surveyed a bit of the structure of supersymmetric gauge theories, we should return to Witten's analysis and compute the index $Tr(-1)^F$ for these theories. My discussion will present the basic lines of this computation, but I will not enter into its many subtleties. I strongly recommend to anyone tempted by this discussion to study the original paper [64].

Let us begin by considering the pure gauge theories (7.22), without matter fields (A,ψ,F). The easiest case is supersymmetric QED, the U(1) gauge theory; this is actually a free theory containing a photon and a neutral fermion λ. Let us quantize this theory in $A^0 = 0$ gauge, in a finite volume with periodic boundary conditions, and study the spectrum of gauge-invariant states.

What are the zero-energy eigenstates of H? Any state containing photons or fermions of finite momentum has energy $2\pi n/L$, where n is a positive integer and L is the size of the box. We may therefore concentrate on states containing only particles of zero momentum. Note that the zero-momentum component of $\vec{A}$ cannot be gauged away, since the quantity

$$\exp\left[ig\int_P d\ell\cdot A\right] \quad , \tag{7.31}$$

defined on a closed path P which wraps around the periodically connected volume, is gauge invariant. However, each component of $\vec{A}$ may be changed by $(2\pi/gL)$ without affecting (7.31). The action for the zero-momentum modes is given by

$$\int dt\int d^3x\ (7.22)\Big|_{\vec{p}=0} = \int dt\ L^3\left(\frac{1}{2}\left(\frac{d}{dt}\vec{A}_0\right)^2 + \lambda^\dagger_{0\alpha}\frac{d}{dt}\lambda_{0\alpha}\right) \tag{7.32}$$

The corresponding Hamiltonian is

$$H = \frac{1}{2L^3} (\vec{\Pi}_0)^2 \tag{7.33}$$

where $\vec{\Pi}_0$ is the conjugate momentum to $\vec{A}_0$. If we insist that the wave function $\psi(A_0)$ is periodic with periodicity $(2\pi/gL)$, there is a unique zero-energy state $|\Psi_0\rangle$ of the gauge fields; the next states have energy

$$\frac{1}{2}\left(\frac{g}{2\pi}\right)^2 \frac{1}{L} \tag{7.34}$$

To enumerate all the zero energy states of H, we must consider adding zero momentum fermions to this gauge-field state. There are, in all, four zero-energy states:

$$|\Psi_0\rangle \quad , \quad a^\dagger_\uparrow|\Psi_0\rangle \quad , \quad a^\dagger_\downarrow|\Psi_0\rangle \quad , \quad a^\dagger_\uparrow a^\dagger_\downarrow|\Psi_0\rangle \tag{7.35}$$

where the operators $a^\dagger$ create zero-momentum fermions λ. Apparently, $Tr(-1)^F = 0$. But all is not lost. The theory has a symmetry of charge conjugation

$$A_\mu \to -A_\mu \quad , \quad \lambda \to -\lambda \quad . \tag{7.36}$$

The supersymmetry charge Q is even under C, so finite energy states must be paired separately in the C = +1 and C = −1 sectors. Thus, a nonzero index in either sector indicates that supersymmetry is not broken. The catalogue (7.35) implies:

$$Tr_{(C=+1)}(-1)^F = +2 \qquad Tr_{(C=-1)}(-1)^F = -2 \tag{7.37}$$

so supersymmetry cannot be spontaneously broken.

Now add matter fields to this theory. Let us add L-fermions ψ in pairs (ψ_+,ψ_-) of opposite electric charge, so that they pair to Dirac fermions and can be given large masses. If the masses are added supersymmetrically, using a potential of the form (7.25), the corresponding bosons A_+, A_- receive the same large masses. These massive particles contribute no new zero-energy states. The dynamics are still invariant to charge conjugation if (7.36) is supplemented by:

$$\psi_+ \to \psi_- \qquad A_+ \to A_-$$

$$\psi_- \to \psi_+ \qquad A_- \to A_+ \tag{7.38}$$

so the counting of zero-energy states is unchanged from (7.37) and supersymmetry cannot be broken. One might now tune the matter field masses continuously to zero. The indices should still be unchanged from (7.37), so that this chirally-symmetric theory has no supersymmetry breaking.

This argument may be generalized to non-Abelian gauge theories. Let us consider first the pure supersymmetric gauge theory. We are permitted to tune the coupling constant g so that g is small at all length scales up to the size L of the box. The fermions λ are then weakly coupled to the gauge fields. In this situation, we may analyze the zero-energy states of the pure gauge system first, then consider adding the fermions. As in the Abelian case, we need only consider zero-momentum components of $\vec{A}$. Each component of $\vec{A}_0$ may be gauge-transformed to the form

$$A_0{}^{ia} t^a = A_0{}^{ic} t^c \tag{7.39}$$

where $\{t^c\}$ is a set of mutually commuting generators of the gauge group. (The index c takes the values c = 1,...,r, where r is the rank of the gauge group; r = (N-1) for SU(N).) The various vector components of $\vec{A}\cdot t$ must all be mutually commuting; otherwise we would find

$$F^{ij}\cdot t = -ig[A^i\cdot t, A^j\cdot t] \neq 0 \tag{7.40}$$

so that this $\vec{A}$ field would produce a nonzero magnetic field and thus a nonzero energy. The components A_0^{ic} are defined only modulo $(2\pi/gL)$; further, the transformations $A_0^{ic} \to -A_0^{ic}$ and permutations of the indices c are elements of the gauge group. We must construct a state invariant to all of these transformations. The Hamiltonian on the space of variables A_0^{ic} is:

$$H = \frac{1}{2L^3} (\Pi_0^{ic})^2 \tag{7.41}$$

where the Π_0^{ic} are the conjugate momenta. This has a unique ground state $|\Psi_0\rangle$ with the required symmetry and an energy gap of magnitude (7.34) to the first excited state.

We can create a zero-energy state with fermions from $|\Psi_0\rangle$ only by placing fermions λ into the zero-energy fermion modes in the background field $\vec{A}\cdot t$. For fields of the form (7.39), these modes have the form

$$\lambda \cdot t = \lambda_0^c t^c \tag{7.42}$$

There are 2r such modes. To form gauge-invariant states, however, we must populate these modes in such a way as to respect the subgroup of gauge transformations which act on the components λ^c. Certainly we must insist that a gauge-invariant state is invariant to

$$A_0^{ic} \to -A_0^{ic} \qquad \lambda_0^c \to -\lambda_0^c \tag{7.43}$$

and to permutations of the indices c. One way to construct such states is to define the gauge-invariant operator

$$U = \sum_{c=1}^{r} a^{\dagger c}{}_{\uparrow} a^{\dagger c}{}_{\downarrow} \tag{7.44}$$

where the $a^\dagger$ are fermion creation operators, as in (7.35). Then we can form the zero-energy states

$$|\Psi_0\rangle \ , \quad U|\Psi_0\rangle \ , \quad U^2|\Psi_0\rangle \ , \ \dots \ , \quad U^r|\Psi_0\rangle \tag{7.45}$$

This series of states terminates because $U^{r+1} = 0$. It is plausible that U is the only gauge-invariant operator which creates zero-energy fermions; Witten argues out this point in more detail. All of the states indicated in (7.45) are bosonic; hence

$$\begin{aligned} Tr(-1)^F &= r + 1 \\ &= N \text{ for } SU(N) \text{ gauge theories} \end{aligned} \quad (7.46)$$

As in the Abelian case, we can add massive fermions and bosons in conjugate representations $(r+\bar{r})$ without disturbing this counting. Then we can smoothly increase g and decrease m until $m \ll L^{-1} \ll \Lambda$, where Λ is the momentum scale at which g becomes strong. This process also preserves (7.46) and leaves us with the conclusion that supersymmetry is not spontaneously broken in this class of theories.

It is not known whether this argument generalizes to supersymmetric chiral gauge theories. Nilles [69] has presented examples of such theories in which condensates such as we discussed in section 5 can appear without endangering supersymmetry. But perhaps other theories of this type may allow dynamical supersymmetry breaking. I will say a bit more on this point at the end of the next section.

7.4 THE QUALITATIVE BEHAVIOR OF SUPERSYMMETRIC GAUGE THEORIES

Now that we have established one definite property of the zero mass limit of supersymmetric gauge theories, it is appropriate to combine this property with intuition and speculation to map out a coherent picture of the behavior of these theories. Unfortunately, I do not know enough about the behavior of the theories to be able to present you with a uniquely compelling picture. I therefore choose to indicate the range of possibilities still available by presenting two very different scenarios. The first is rather perverse, but has a taste of plausibility. The second is more conservative, but still has a number of unusual features.

The two scenarios are distinguished, first of all, by their assumptions about whether the state in which chiral symmetry is broken by condensates while the gauge symmetry remains unbroken is a supersymmetric state. Let us assume first that it is not, perhaps because $\langle |A_r|^2 \rangle$ in (7.29) is nonsingular in the limit $m \to 0$. This possibility is not excluded by Witten's theorem, as Witten himself is careful to point out [64]. The theorem does imply, however, that there must be a supersymmetric state somewhere in the space of states, perhaps at a point where some scalar field has a large vacuum expectation value. That vacuum expectation value might, in fact, move to infinity as $m \to 0$. But, wherever this state may be located, if it is annihilated by the Q_α it is necessarily the ground state. It is possible, then, that the bosons we added to make the theory supersymmetric acquire large vacuum expectation values and completely change the qualitative physics of the systems.

The plausibility of bosons acquiring large vacuum expectation values is emphasized by examination of the scalar field potential which follows from the Lagrangian (7.21), (7.22). If we eliminate the Lagrange multipliers D^a, we find

$$V(A) = \frac{g^2}{2}\left[\sum_r A_r t^a{}_r A_r\right]^2 \qquad (7.47)$$

This potential is obviously non-negative, but one can find a sizable space on which it is zero. (I will demonstrate this explicitly in a moment.) The form of the vacuum energy on this space is determined entirely by nonperturbative effects, and everything depends on what specifically one assumes about these effects. As an extreme example, Srednicki [71] has suggested that the picture of [67,68], in which chiral symmetry breaking causes spontaneous supersymmetry breaking, can be saved from Witten's theorem by assuming a potential of the form of fig. 20a. For any finite m, there is a zero of the vacuum energy at a finite value of the expectation value of some field ϕ, separated from the state where $\langle\phi\rangle = 0$ by a potential barrier. As $m \to 0$, the barrier becomes arbitrarily higher and the supersymmetric state, though it exists in principle, becomes inaccessible.

I prefer, for my first alternative, a scenario suggested more directly by the form of (7.47). Let me illustrate this scenario by considering an SU(N) gauge theory with one N and one $\bar{N}$ matter supermultiplet. The matter fields are paired, as Witten's argument required. Consider a state in which the two boson fields take the vacuum expectation values:

$$\langle A_N\rangle = C\begin{pmatrix}1\\0\\0\\\cdot\\\cdot\\\cdot\\0\end{pmatrix} \qquad \langle A_{\bar{N}}\rangle = C\begin{pmatrix}1\\0\\0\\\cdot\\\cdot\\\cdot\\0\end{pmatrix} \tag{7.48}$$

where C is some constant with the dimensions of mass. Since

$$t^a{}_{\bar{N}} = -(t^a{}_N)^T \tag{7.49}$$

the potential (7.47), evaluated with the field values (7.48), vanishes for any C. It is known that this property of the vanishing of the potential persists to all orders in perturbation theory in a supersymmetric theory if it is present at the classical level [72]. Thus, the vacuum energy as a function of C is given entirely by nonperturbative contributions. If C is large, I would expect these contributions to be small, of the form

$$V(C) \propto C^4 \exp[-(B/g^2(C))] \tag{7.50}$$

where A,B are numerical constants and $g^2(C)$ is the scale dependent gauge coupling. This suggests a potential of the form of fig. 20b, a slowly decreasing potential with its minimum at $C = \infty$ (for $m = 0$).

The vacuum expectation values (7.48) break the gauge symmetry SU(N) to SU(N-1). The gauge bosons corresponding to broken symmetry directions acquire mass and so do their fermionic partners; most of the light degrees of freedom in the matter multiplets are swallowed up in the process. Indeed, the only particles which do not acquire masses of order C are the SU(N-1) gauge bosons and fermions and a boson-fermion pair which is neutral under SU(N-1). The massive particles, and the neutral ones, decouple from low-energy physics, so the theory appears at energies much less than C to be a supersymmetric gauge theory without matter fields. This scenario generalizes to accommodate arbitrary numbers of $(N+\bar{N})$, or $(r+\bar{r})$, supermultiplets: Either by eliminating all light charged matter fields or by reducing the gauge group to a product of U(1) factors, one can remove the possibility of strongly-coupled matter supermultiplets by allowing some of their boson fields to acquire large vacuum expectation values.

The second scenario I will present involves the more conservative (and, I believe, more likely) assumption that chiral symmetry breaking can proceed without violating supersymmetry. In this case it is natural to assume a supersymmetric generalization of the symmetric breaking pattern described in section 5.2: A model containing n pairs of $r+\bar{r}$ matter multiplets, which according to (7.24) has the chiral symmetry $U(n) \times U(n)$, should spontaneously break this symmetry:

$$U(n) \times U(n) \rightarrow U(n) \tag{7.51}$$

This breaking pattern yields n^2 Goldstone bosons; supersymmetry requires that these be accompanied by n^2 massless fermions.

Closer examination, however, shows that supersymmetry constrains this system even more powerfully. To explain this, I must anticipate some results which will be presented in detail in section 9. I will argue there that the low-energy dynamics of the Goldstone bosons resulting from the spontaneous symmetry-breaking $G \rightarrow H$ is described by a nonlinear Lagrangian built from a field whose value is a point in the coset space G/H. Zumino has shown that such Lagrangians can be made supersymmetric only if the space of values of the field is a Kähler manifold, a complex manifold satisfying certain additional restrictions [73]. The symmetry-breaking pattern (7.51), however, yields

$$\frac{G}{H} = \frac{U(n) \times U(n)}{U(n)} \tag{7.52}$$

This is not a complex manifold; indeed, (7.52) does not generally have an even number of real dimensions, the minimal requirement for a complex parameterization. The only way to make the pattern (7.51) consistent with supersymmetry is, then, to enlarge the space of massless particles. This requirement has a natural physical interpretation: One can form from the matter particles two different types of light pseudoscalar mesons, with the quantum numbers of

$$A_{ri}A_{\bar{r}j} \quad \text{and} \quad \Psi_{ri}\Psi_{\bar{r}j} \quad . \tag{7.53}$$

These would be the Goldstone bosons of chiral symmetry breaking in a theory with only bosons or only fermions, respectively. A given broken chiral current creates one linear combination of these states; this is the true Goldstone boson. However, the other linear combination should also be light, and supersymmetry could well require it to be massless. In any event, one can show that (7.52) is a submanifold of

$$\frac{U(2n)}{U(n) \times U(n)} \tag{7.54}$$

which is a Kahler manifold*. Equation (7.54) has $2n^2$ coordinates, so it can accommodate all of the states (7.53); these $2n^2$ coordinates form, appropriately, two adjoint representations of $H = U(n)$. It is therefore likely that supersymmetric gauge theories with matter multiplets and chiral symmetry breaking according to (7.51) would have $2n^2$ massless bosons and their fermionic partners — twice as many pairs as a naive argument would suggest.

I cannot resist adding one more speculative ingredient to this picture. Recently, Ong [74] and Bagger and Witten [75] have shown that when supersymmetric nonlinear Lagrangians whose fields take values in a compact coset space are coupled to gauge fields corresponding to broken symmetry generators, these Lagrangians show spontaneous supersymmetry breaking at the classical level. This indicates to me that strongly-interacting supersymmetric gauge theories whose broken chiral symmetries

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*I am grateful to I. M. Singer for showing me how to do this.

are coupled to additional gauge bosons should also show supersymmetry breaking. If the additional gauge bosons could be coupled in such a way that the matter fields belonged to a real representation of the full gauge group, such supersymmetry breaking would violate Witten's theorem. However, I will argue in section 8.2 that in this case the additional gauge symmetry realigns itself (in a sense I will make precise there) to remain unbroken. The more general situation, in which the additional gauge symmetry is coupled chirally, seems then an attractive candidate for a system with dynamical supersymmetry breaking.

Chapter 8.

GOLDSTONE BOSONS AND VACUUM ALIGNMENT

8.1 COUNTING GOLDSTONE BOSONS

In all of the previous sections, we have been primarily interested in gauge theories with exact chiral symmetries. The main thrust of the discussion has been to find the physics which determines the pattern of chiral symmetry breaking and to compute what that pattern should be. Now I would like to change my emphasis slightly, to study a different, but closely related, problem. In the real world (such as we see it) strong interaction theories do not occur in isolation. The fermions which have strong interactions also have weak and electromagnetic interactions; they also have masses generated by small chiral symmetry breaking perturbations. It is then appropriate to pose the following question: Given a pattern of χSB for a certain gauge system, how is this system affected by the presence of a small symmetry-breaking perturbation? Most of the physics I will discuss in relation to this question was discovered in the 1960's and is contained already in [3,4,5]; I will, however, use a more modern language and an emphasis which reflects the recent application of these ideas to weak-interaction theory [76,77,78,79].

One usually thinks of small perturbations as having only a small effect on the overall structure of a theory. This is emphatically not true for systems with chiral symmetry breaking. Systems with exact chiral symmetries generally leave a large vacuum degeneracy and a number of massless Goldstone bosons. Important qualitative conclusions about the behavior of these systems depend on how the degeneracy of vacuum states is broken and what small masses these bosons eventually acquire as the result of symmetry breaking perturbations. It is these issues that I wish to address.

As a preface to this study, however, it will be useful to survey again the symmetry-breaking patterns we have found in previous sections, at least for fermions in real representations of the gauge group, in order to make clear the presence of degenerate vacuum states and to count Goldstone bosons. In the process, I will introduce some notation which will be useful in our discussion of the effects of explicit symmetry breaking.

For a gauge theory containing n multiplets of fermions in paired complex representations $(r+\bar{r})$ of the gauge group, the chiral symmetry is $G = SU(n) \times SU(n) \times U(1)$. In our previous discussion, we assumed that all fermions acquire equal dynamical masses. The fermions, then, acquired a mass term

$$\epsilon^{\alpha\beta}\bar{\psi}_{\alpha a i}\psi_{\beta}{}^{aj}\Sigma^{i}{}_{j} + \text{h.c.} \tag{8.1}$$

where a is a color index, i,j are flavor indices of $SU(n)_L$ and $SU(n)_R$, respectively, and

$$\Sigma^{i}{}_{j} = \Sigma\delta^{i}{}_{j} \tag{8.2}$$

The condensate (8.2) is preserved by a subgroup $H = SU(n) \times U(1)$ of G.

Because G is an exact symmetry of this theory, however, we are required to consider a larger class of possible condensates. Any condensate which can be transformed to the form (8.2) by a transformation in G must be energetically equivalent to (8.2). The class of such Σ's parameterizes a manifold of degenerate vacuum states. We can construct this class of Σ's explicitly by subjecting (8.1) to a general SU(n) × SU(n) × U(1) transformation:

$$\psi_{\alpha a i} \to \psi_{\alpha a k}(V)^k{}_i e^{i\alpha} \qquad \psi^{aj}{}_\beta \to e^{-i\alpha}(V')^j{}_k \psi_\alpha{}^{ak} \tag{8.3}$$

where V, V′ are SU(n) transformations. This gives a mass term of the same form, but with

$$\Sigma^i{}_j = (V^i{}_k V'^k{}_j)\Sigma = (U^i{}_j)\Sigma \tag{8.4}$$

Apparently, the manifold of degenerate vacua is isomorphic to SU(n).

In general, if a theory with a global symmetry G possesses a vacuum state $|\Psi_0\rangle$ which respects only a subgroup H of G, the action of G on this state generates a manifold of degenerate vacua. Any given state in this manifold is unchanged by a group of transformations isomorphic to H. Hence the set of degenerate vacua is isomorphic to the coset space G/H. For the case we considered above

$$\frac{G}{H} = \frac{SU(n) \times SU(n) \times U(1)}{SU(n) \times U(1)} = SU(n) \tag{8.5}$$

which checks our conclusion from (8.4). Transformations in G/H correspond to directions of variation of Σ in which the effective action is level at its minimum. Quantizing the excitations along these directions produces zero mass particles — Goldstone bosons — one for each orthogonal direction in G/H.

Let me introduce some notation to describe this situation. Label the generators of G as $\{G_a\}$, the generators of H (a subset of the G_a) as $\{T_i\}$, and the orthogonal generators of G, the generators of G/H, as $\{X_z\}$. Throughout sections 8 and 9, I will use indices a,b,c to denote

G's, i,j,k to denote T's, and x,y,z to denote X's. Normalize these generators to

$$\mathrm{Tr} G_a G_b = \delta_{ab} \tag{8.6}$$

To each generator of G corresponds a symmetry current

$$J^{\mu}{}_a(x) = \bar{\psi}_L \gamma^{\mu} G_a \psi_L(x) \tag{8.7}$$

I will also use G_a, T_i, X_z to denote the charges constructed from the currents (8.7).

Let us choose one of the degenerate vacua as a reference point; denote this state by $|0\rangle$. With respect to this state, the $\{T_i\}$ are the generators of G which satisfy $T_i|0\rangle = 0$. The set of degenerate vacua may then be written:

$$\{\exp(i\alpha_z X_z)|0\rangle\} \tag{8.8}$$

Goldstone's theorem insists that each current involving an X_z can create a single massless boson π_z from the vacuum. Let us write the amplitude for creation of a boson with momentum p as

$$\langle \pi_y(p)|J^{\mu}{}_z(0)|0\rangle = -ip^{\mu} f_{yz} \tag{8.9}$$

f_{yz} is a set of constants with the dimensions of mass. For the gauge model we discussed at the beginning of this section, the X_z span the generators of SU(n) = H. The X_z therefore correspond to a single irreducible representation of H, and symmetry insists that f_{yz} should be diagonal:

$$\langle \pi_y(p)|J^{\mu}{}_z|0\rangle = -ip^{\mu} f_{\pi} \delta_{yz} \tag{8.10}$$

The mass f_{π} appears ubiquitously in Goldstone boson dynamics.*

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*My normalization convention implies f_{π} = 93 MeV in the familiar strong interactions.

All of the more specific results we have found so far apply to the case in which r is a complex representation of the gauge group. What if r is real, so that r is equivalent to $\bar{r}$ [80,81]? In this case, the system we described earlier may be recast as a system of 2n (L-) fermions belonging to r. The chiral symmetry is thus enlarged to G = SU(2n). To determine H, however, even under the assumption that all fermions acquire equal dynamical masses, it is necessary to consider two distinct cases. If r is equivalent to $\bar{r}$, there exists an invariant with two indices in r. This invariant may be either symmetric or antisymmetric. I will differentiate these two cases by referring to r as a strictly real or pseudoreal representation, respectively.

In the case of a strictly real representation, the invariant may be diagonalized to δ^{ab}, then the condensate induced by χSB may be written

$$\epsilon^{\alpha\beta}\psi_{\alpha a i}\psi_{\beta b j}\delta^{ab}\Sigma^{ij} + \text{h.c.} \tag{8.11}$$

Σ^{ij} must be symmetric; for equal masses,

$$\Sigma^{ij} = \Sigma\cdot\delta^{ij} \quad . \tag{8.12}$$

We can identify H as the subgroup of unitary transformations on i,j which preserves (8.12): H = O(2n). The $\{X_z\}$ form a single irreducible representation of O(2n), the traceless symmetric tensor representation, so the matrix f_{yz} in (8.9) reduces as before to $f_\pi\cdot\delta_{yz}$.

In the case of a pseudoreal representation, the two-index invariant may be brought to the form

$$E^{ab} = \left(\begin{array}{c|c} & 1 \\ \hline -1 & \end{array}\right) \tag{8.13}$$

The mass term

$$\epsilon^{\alpha\beta}\psi_{\alpha a i}\psi_{\beta b j}E^{ab}\Sigma^{ij} \tag{8.14}$$

requires an antisymmetric Σ^{ij}; for equal masses, one must set

$$\Sigma^{ij} = \Sigma \cdot E^{ij} \tag{8.15}$$

H must be the group of transformations on i,j which preserves E; this is the symplectic group: $H = Sp(2n)$. The $\{X_z\}$ form the traceless antisymmetric tensor representation of Sp(2n), so that, again $f_{yz} = f_\pi \cdot \delta_{yz}$. The case of complex r fits naturally inside each of these two cases according to the decomposition:

$$\begin{array}{ccc} SU(2n) & \supset & SU(n) \times SU(n) \times U(1) \\ \downarrow & & \downarrow \\ O(2n) \text{ or } Sp(2n) & \supset & SU(n) \times U(1) \end{array} \tag{8.16}$$

All three scenarios have the property that there exists a parity operator P satisfying

$$P^2 = 1 \quad , \qquad P\, T_i\, P = +T_i \quad , \qquad P\, X_z\, P = -X_z \tag{8.17}$$

so that the symmetry breaking respects parity. (In mathematical terms, in each case, G/H is a symmetric space [82].) This parity invariance will be a useful constraint on our later analysis.

8.2 VACUUM ENERGETICS

Let us now discuss, in general terms, the effect of a small symmetry-breaking perturbation on the patterns of symmetry-breaking described in the previous section. Call this perturbation ΔH, and denote a given vacuum state by

$$|\alpha\rangle = \exp(i\alpha_z X_z)|0\rangle \tag{8.18}$$

Then, to leading order in perturbative theory, the energy shift of each of the degenerate vacua is given by:

$$\Delta E(\alpha) = \langle\alpha|\Delta H|\alpha\rangle = \langle 0|\exp[-i\alpha_z X_z](\Delta H)\exp[i\alpha_z X_z]|0\rangle \tag{8.19}$$

This expression generally depends on α and breaks the degeneracy of the vacuum states. The minimum of $\Delta E(\alpha)$ obeys

$$i \frac{\partial}{\partial \alpha_y} \Delta E(\alpha) = \langle \alpha | [X_y, \Delta H] | \alpha \rangle = 0 \tag{8.20}$$

It is useful to rearrange our coordinates so that the minimum occurs at the state $|0\rangle$, that is, at $\alpha_y = 0$.

Near the minimum, the effective action curves upward. The effect of ΔH is indicated in fig. 21: Level directions of the effective action, corresponding to directions in G/H, become directions with small but nonzero curvature. Since the potential has positive curvature near $\alpha = 0$, the Goldstone bosons acquire a mass matrix of the form:

$$\begin{aligned}(m^2)_{yz} &= C \cdot \frac{\partial^2}{\partial \alpha_y \partial \alpha_z} E(\alpha) \\ &= -C \, \langle 0 | [X_y, [X_z, \Delta H]] | 0 \rangle \quad .\end{aligned} \tag{8.21}$$

It's not hard to find the normalization factor C, but it is not a one-line argument; I will refer you to the paper of Dashen in which this formula first appeared [83]. The result is

$$(m^2)_{yz} = - \frac{1}{f_\pi^2} \langle 0 | [X_y, [X_z, \Delta H]] | 0 \rangle \quad . \tag{8.22}$$

In the usual strong interactions, the most important symmetry-breaking perturbation is the quark mass term. This perturbation gives rise, for example, to pion masses of the form

$$m_\pi^2 \propto \frac{(m_u + m_d)}{f_\pi^2} \tag{8.23}$$

where m_u, m_d are the u and d quark masses. In theories of dynamical breaking of the weak interaction symmetry, the most important perturbations come from weak and electromagnetic (and color) gauge boson exchange. This latter situation is a bit more interesting because it turns inward on itself: The weak interactions determine their own pattern of symmetry breaking by their choice of a vacuum orientation. I will devote the rest of this section to studying the physics of this situation. This analysis has been given in full generality only relatively recently [76,81,84], though it originated in the classic calculation of Das, Guralnik, Mathur, Low, and, Young of the π^+-π^0 mass difference [85].

Let me begin by defining the problem a bit more carefully. One may couple additional gauge bosons to a theory with global symmetry G by promoting some subgroup G_W of G to a group of local symmetries. The gauge bosons then couple to some subgroup of G. The spontaneous breaking of G picks out another subgroup, H. In this situation, we find that the relative alignment of these subgroups may have physical meaning. The geometric relation between G_W and H is sketched in fig. 22. Gauge bosons coupled to charges in region I will be undisturbed and remain massless; gauge bosons in region II will find themselves coupled to broken symmetries and will receive mass from the Higgs mechanism. However, the overlap of G_W and H is dynamically determined: It depends on which of the initially degenerate vacua $|\alpha\rangle$ is preferred by the perturbations induced by G_W exchanges. Thus, even if these additional gauge bosons are weakly coupled, they have a dramatic effect on the qualitative structure of the model. I would like to find a criterion which determines the preferred vacuum state, and, thereby, the pattern of G_W breaking.

To begin, let me write the G_W couplings in the form

$$\delta L = A^A{}_\mu \bar{\psi}\gamma^\mu G_A \psi = A^A{}_\mu J^\mu{}_A \tag{8.24}$$

where the G_A and, generally, representation matrices with capital indices, are defined to absorb normalization factors and coupling constants. For any given vacuum $|0\rangle$, I can choose the generators T_i so that $T_i|0\rangle = 0$, then I can decompose each G_A into T and X parts. Denote this decomposition by

$$G_A = T_{I(A)} + X_{Z(A)} \qquad \text{or} \qquad J^\mu{}_A = J^\mu{}_{I(A)} + J^\mu{}_{Z(A)} \tag{8.25}$$

Using this notation, we may write the perturbation due to one-gauge boson exchange, to leading order in $A^A{}_\mu$ coupling constants, as

$$\Delta H = -\frac{1}{2}\int d^4x\ \Delta^{\mu\nu}(x)\ T[J_{\mu A}(x)J_{\nu A}(0)] \tag{8.26}$$

where $\Delta^{\mu\nu}$ is the free gauge boson propagator. The expectation value of (8.26) is the free gauge boson propagator. The expectation value of (8.26) in the particular vacuum $|0\rangle$ is given by

$$\Delta H = -\frac{1}{2}\int d^4x\ \Delta^{\mu\nu}(x)\ \langle 0|TJ_{\mu A}(x)J_{\nu A}(0)|0\rangle \tag{8.27}$$

To evaluate this, imagine decomposing G_A as in (8.25). In the schemes of χSB discussed in the previous section, the product of two T's or two X's contains only one invariant. This allows us to simplify products of currents

$$\langle 0|TJ_{\mu i}J_{\nu j}|0\rangle = \langle J_T J_T\rangle\ \delta_{ij} = \langle J_T J_T\rangle\ \mathrm{Tr}(T_i T_j)$$

$$\langle 0|TJ_{\mu x}J_{\nu y}|0\rangle = \langle J_X J_X\rangle\ \delta_{xy} = \langle J_X J_X\rangle\ \mathrm{Tr}(X_x X_y) \tag{8.28}$$

The quantities in brackets are H-invariant amplitudes; their dependence on μ,ν has been suppressed. To these relations, we may add the constraint

$$\langle 0|TJ_{\mu i}J_{\nu x}|0\rangle = 0 \tag{8.29}$$

which follows from parity (eq. (8.17)). Equations (8.28) and (8.29) allow us to simplify the expectation value in (8.27) as follows:

$$\langle 0|TJ_{\mu A}J_{\nu 1}A|0\rangle = \langle 0|TJ_{\mu I(A)}J_{\nu I(A)}|0\rangle + \langle 0|TJ_{\mu Z(A)}J_{\nu Z(A)}|0\rangle$$
$$= \langle J_T J_T\rangle\ Tr(T_{I(A)}T_{I(A)}) + \langle J_X J_X\rangle\ Tr(X_{Z(A)}X_{Z(A)})$$
$$= \langle J_T J_T\rangle\ Tr(T_{I(A)}G_A) + \langle J_X J_X\rangle\ Tr(X_{Z(A)}G_A)$$
$$= \langle J_T J_T\rangle\ Tr(G_A G_A) + \langle (J_X J_X - J_T J_T)\rangle\ Tr(X_{Z(A)}G_A) \tag{8.30}$$

The first term in the last line of (8.30) is an invariant independent of the vacuum orientation. Thus we may rewrite (8.27) as

$$E(0) = E_0 + \left\{1/2 \int d^4x\ \Delta^{\mu\nu}\ \langle J_{\mu T}J_{\nu T} - J_{\mu X}J_{\nu X}\rangle\right\}\cdot Tr(X_{Z(A)})^2 \tag{8.31}$$

where E_0 is independent of the vacuum orientation. The projection of G_A to $X_{Z(A)}$ clearly depends upon the relative orientation of G_W and H. The expectation value of (8.26) in any other vacuum $|\alpha\rangle$ is given by an expression of the same form, but with $X_{Z(A)}$ replaced by the broken part of G_A with respect to that vacuum. Let me note that the quantity in brackets is quite plausibly positive: The positivity of this quantity is roughly the statement that the lightest particle created by $J_{\mu T}$ (a vector meson) is lighter than the lightest axial-vector meson created by $J_{\mu X}$; the point has been argued out with care by Preskill [84].

Our final result is that the preferred vacuum is the one which minimizes

$$Tr(X_{Z(A)})^2 \tag{8.32}$$

This criterion has an instructive physical interpretation. Let us think a bit about the physics of the Higgs mechanism and the mass generation for the G_W gauge boson. The G_W bosons receive mass at leading order in perturbation theory if the current-current vacuum expectation value shown in fig. 23 has the form:

$$\langle 0|TJ_{\mu A}(-k)J_{\nu B}(k)|0\rangle \xrightarrow[k\to 0]{} -im^2{}_{AB}\cdot g_{\mu\nu} \tag{8.33}$$

However, this matrix element is transverse; hence, (8.33) implies the more complete form [86]:

$$\langle 0|TJ_{\mu A}(-k)J_{\nu B}(k)|0\rangle \xrightarrow[k\to 0]{} -i(m^2{}_{AB})\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right] \tag{8.34}$$

The $g_{\mu\nu}$ term is generally hard to isolate, but the pole can be produced only by a massless particle created by one current and annihilated by the other. The only candidate is the Goldstone boson. Inserting π_y as an intermediate state in the current-current matrix element, we have:

$$\begin{aligned}\langle 0|TJ_{\mu A}J_{\nu B}|0\rangle \underset{k\to 0}{\approx} & \langle 0|J_{\mu A}(-k)|\pi_y\rangle \frac{i}{k^2}\langle\pi_y|J_{\nu B}(k)|0\rangle \\ &= (ik_\mu f_\pi \, Tr(G_A X_y)) \frac{i}{k^2} (-ik_\mu f_\pi \, Tr(X_y G_B)) \\ &= i\frac{k_\mu k_\nu}{k^2} f_\pi{}^2 \, Tr(X_{Z(A)} X_{Z(B)})\end{aligned} \tag{8.35}$$

So, in any given vacuum, the gauge bosons acquire a mass matrix

$$m^2{}_{AB} = f_\pi{}^2 \, Tr(X_{Z(A)} X_{Z(B)}) \tag{8.36}$$

The preferred vacuum is then the one which minimizes $Tr(m^2)$. This criterion, that the elementary fermions should condense in such a way as to break G_W as little as possible, is reminiscent of the MAC criterion which we discussed in section 2.3. Indeed, a bit of rearrangement shows that they are identical [84].

8.3 AN AMUSING EXAMPLE

To clarify the workings of vacuum orientation, and to illustrate its potential importance, let us work through an example of the phenomenon within a simple model of dynamically broken weak interactions. Let me imagine a gauge theory with two flavors of Dirac fermions U,D; the corresponding multiplet of L-fermions is

$$\psi^A = (U_L,\ D_L,\ U^\dagger{}_R,\ D^\dagger{}_R) \quad . \tag{8.37}$$

Let us gauge, with a weak coupling constant, the SU(2) symmetry which links U_L and D_L, so that (8.24) takes the explicit form

$$\delta L = W^A{}_\mu \bar{\psi}\gamma^\mu G_A \psi \tag{8.38}$$

with

$$G_A = \left(\begin{array}{c|c} g/2\ \sigma^A & \\ \hline & 0 \end{array}\right) \tag{8.39}$$

and A = 1,2,3. (For those interested in realism, this is the standard model of weak interactions, with $\sin^2\theta_W = 0$.)

If U and D belong to a complex representation of the strong-interaction gauge group, the chiral symmetry is $G = SU(2) \times SU(2) \times U(1)$. In this case, G_W coincides with the factor $SU(2)_L$ of G, and there is no freedom of vacuum alignment. The condensate

$$\Sigma\cdot(\bar{U}U+\bar{D}D) = \Sigma\cdot(\epsilon^{\alpha\beta}[U_{L\alpha}U^\dagger{}_{R\beta}+D_{L\alpha}D^\dagger{}_{R\beta}] + h.c.) \tag{8.40}$$

breaks G_W completely; all other condensates related to (8.40) by G transformations may be brought into the form of (8.40) by G_W gauge transformations.

With respect to the condensate (8.40):

$$T_i = \left(\begin{array}{c|c} 1/2\ \sigma^i & \\ \hline & -1/2\ \sigma^{iT} \end{array}\right) \qquad X_z = \left(\begin{array}{c|c} 1/2\ \sigma^z & \\ \hline & 1/2\ \sigma^{zT} \end{array}\right) \tag{8.41}$$

where i,z = 1,2,3. The generators (8.39) may be decomposed into

$$T_{I(A)} = \frac{g}{4}\begin{pmatrix} \sigma^A & \\ & -\sigma^{AT} \end{pmatrix} \qquad X_{Z(A)} = \frac{g}{4}\begin{pmatrix} \sigma^A & \\ & \sigma^{AT} \end{pmatrix} \tag{8.42}$$

and, using (8.36), we may construct the mass matrix of the G_W bosons

$$m^2{}_{AB} = \frac{g^2 f_\pi{}^2}{4}\,\delta_{AB} \tag{8.43}$$

The three SU(2) bosons become the massive mesons $W^+{}_\mu, W^-{}_\mu, W^0{}_\mu$ which mediate the charged and neutral currents.*

If U and D belong to a real representation, there are physically distinct possibilities for the vacuum orientation, and we must decide among them. Consider first the case of a strictly real representation. Equation (8.40) is still a possible condensate; however, we can construct another condensate which preserves at least the component A = 3 of G_W:

$$\Sigma\cdot\epsilon^{\alpha\beta}[U_{L\alpha a}D_{L\beta b}+U^\dagger{}_{R\alpha a}D^\dagger{}_{R\beta b}]\delta^{ab} + \text{h.c.} \tag{8.44}$$

In the vacuum corresponding to (8.44), $W^3{}_\mu$ acquires no mass. However, the components A = 1,2 of G_W are still spontaneously broken. Since (8.44) is symmetric under interchange of U_L and D_L, it represents a condensate of isospin 1, while (8.40) carries isospin 1/2 under the gauged SU(2). One can show that (8.44) implies $T_{I(A)} = 0$ for A = 1,2; this leads to

$$f_\pi{}^2\ \mathrm{Tr}(X_{Z(A)}X_{Z(B)}) = f_\pi{}^2\ \mathrm{Tr}(G_A G_B) = \frac{g^2 f_\pi{}^2}{2}\,\delta_{AB} \tag{8.45}$$

for A = 1,2. Thus

$$\mathrm{Tr}\ m^2 = g^2 f_\pi{}^2 \quad , \tag{8.46}$$

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*For a more realistic version of this model, see [87,88].

a larger value than that obtained from (8.43). The condensate (8.40) still gives the preferred vacuum orientation in this model.

If U and D belong to a pseudoreal representation, however, the situation is quite different [81]. The condensate

$$\Sigma \cdot \epsilon^{\alpha\beta}[U_{L\alpha a}D_{L\beta b}+U^{\dagger}_{R\alpha a}D^{\dagger}_{R\beta b}]E^{ab} + h.c. \tag{8.47}$$

is antisymmetric under the interchange of U and D. This condensate thus carries isospin 0; it preserves all of the weakly gauged symmetries. Thus, (8.47) allows all three W^{A}_{μ} to remain massless. Since, for this choice,

$$Tr(m^2) = 0 \tag{8.48}$$

Equation (8.47) represents the preferred vacuum orientation.

Apparently, this last case, which seems to differ little in its construction from the previous two, realizes the gauged SU(2) in a completely different way. Whereas the previous two cases may be extended to plausible models of the weak interactions, the physics of the third case makes this impossible there.

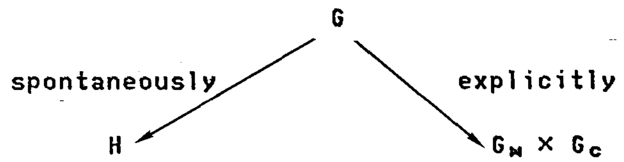
Chapter 9.

PHENOMENOLOGICAL LAGRANGIANS AND THEIR APPLICATIONS

9.1 THE FATE OF THE GOLDSTONE BOSONS

In the previous section, we began a discussion of the effect on a given pattern of χSB of explicit symmetry-beaking perturbations. We considered, in particular, the effects of weakly gauging a subgroup G_W of the chiral symmetry group. We saw that this symmetry-breaking perturbation orients the broken symmetry vacuum; we learned how to determine this orientation and showed how the vacuum orientation determines the masses of gauge bosons coupled weakly to the chiral currents. In this section, I would like to pursue the physics of this system further, to study the dynamics of the Goldstone bosons. I will focus, in particular, on the question of what masses these particles acquire from the symmetry-breaking perturbation. At the end of this section, I will briefly indicate how this physics generalizes to chiral gauge theories with almost massless fermions.

Let me first discuss the basic systematics of the Goldstone boson spectrum. In section 8.2 and in fig. 22, we discussed the overlap of the subgroups G_W and H of G, and the physical consequences of this overlap. We might also single out another subgroup of G for our attention: Let G_C be the subgroup of elements of G which commute with all the generators of G_W. Then the coupling of gauge bosons to the currents of G_W breaks G explicitly to $G_W \times G_C$. The complete pattern of symmetry breaking in the theory with G_W weakly gauged is then:



(9.1)

Each spontaneously broken generator of G has associated with it a Goldstone boson which is massless to zeroth order in the G_W couplings. These bosons are divided into three classes according to the relation of the corresponding generators to the generators of G_W and G_C; this division is indicated pictorially in fig. 24 [76]. The transformations in regions I and II remain exact symmetries even when the effects of the perturbation are included. The Goldstone bosons of region I, however, are absorbed by the gauge bosons which acquire mass through the Higgs mechanism. The Goldstone bosons of region II remain exactly massless bosons in the final theory. The bosons in region III, however, do not correspond to exact symmetries in the full theory; these bosons acquire masses of order $g\Lambda$, where g is the coupling constant of the gauge group G_W, and Λ is the mass scale of the strong interactions producing the original χSB. Weinberg calls these particles "pseudo-Goldstone bosons" [89].

Let us try to compute the masses of these bosons, to leading order in g^2. One way to do this would be to use the result (8.31) for $E(\alpha)$ in conjunction with Dashen's formula (8.22). This procedure gives for the pseudo-Goldstone boson mass matrix

$$m^2{}_{yz} = \frac{1}{f_\pi{}^2}\,\frac{\partial^2}{\partial\alpha_y\partial\alpha_z}\,E(\alpha) = M^2\left\{\frac{\partial^2}{\partial\alpha_y\partial\alpha_z}\,\mathrm{Tr}[(X_{Z(A)})^2]_\alpha\right\} \tag{9.2}$$

where

$$M^2 = \frac{1}{f_\pi{}^2}\int d^4x\;\Delta^{\mu\nu}\;\langle J_{\mu T}J_{\nu T} - J_{\mu X}J_{\nu X}\rangle \tag{9.3}$$

This strategy is worked out in detail in [81,84]. I would prefer, however, to obtain this spectrum of masses by a different route. This second technique will not be quite as powerful as the first for this particular application; it will only be able to compute the mass matrix $(m^2)_{yz}$ up to an overall scale. However, this technique will be applicable to a broader class of problems than the energetic considerations of section 8; it will also give us a different point of view from which to survey the dynamics of Goldstone bosons.

9.2 THE FORMALISM OF PHENOMENOLOGICAL LAGRANGIANS

In the analysis just given, and in the whole of section 8, we studied Goldstone bosons by working out the shape of the energy surface as a function of vacuum orientation. I would now like to change my perspective slightly, and try simply to write a phenomenological description of particles and their interactions in a theory with broken chiral symmetries. In constructing this description, I will concentrate on the lightest particles of the theory and their interactions at low energy. I will also build in the pattern of spontaneous symmetry breaking $G \to H$. I must allow the most general possible interactions consistent with these restrictions to appear in the phenomenological Lagrangian. But these simple restrictions, properly applied, turn out to be remarkably powerful constraints, which fix all of the low-energy dynamics in terms of a few easily recognized parameters. The power of this approach was first demonstrated some time ago by Weinberg [90] and Schwinger [91]; the philosophy of the method has recently been discussed from a modern perspective by Weinberg [92].

Let us try first to construct one example of such a phenomenological Lagrangian; we can discuss the question of its uniqueness later. We may assume that, in a theory with spontaneous xSB, the only light particles in the theory are the Goldstone bosons. These Goldstone bosons may be described by fields which are coordinates on the coset space G/H; these bosons are, in fact, precisely the quantized excitations along these coordinate directions. It is therefore easy to write a Lagrangian which has the Goldstone bosons π_y as its fundamental fields and which is invariant under G transformations; one can simply write

$$L = \int d^4x \, \frac{1}{2} \, g^{xz}(\pi/F) \, \partial_\mu \pi_x \partial^\mu \pi_z \tag{9.4}$$

where $g^{xz}(x)$ is the metric on G/H in the chosen coordinates and F is a constant with dimension of mass. F is presumably of order Λ, the mass scale of the strong interactions which induce xSB. For the special case of the breaking pattern SU(n) × SU(n) → SU(n), G/H = SU(n) may be parameterized by unitary matrices

$$U = \exp\left(i \, \frac{\pi \cdot t}{F} \right) \tag{9.5}$$

where t is an SU(n) generator. In these coordinates, an invariant Lagrangian is given by:

$$L = \int d^4x \, \frac{F^2}{2} \, \mathrm{tr}[\partial_\mu U^\dagger \partial^\mu U] \tag{9.6}$$

The SU(n) × SU(n) global symmetry acts on U according to

$$U \to V_L \; U \; V_R^\dagger \tag{9.7}$$

where V_L and V_R are unitary matrices corresponding, respectively, to $SU(n)_L$ and $SU(n)_R$ transformations; it is easily seen that (9.6) is invariant to (9.7).

One should note, however, that the transformation (9.7) (and, more generally, the action of G on the coordinates π_y of (9.4)) acts, near $\pi_y = 0$, as a translation of π_y:

$$\pi_y \to \pi_y + \alpha_y + \cdots \tag{9.8}$$

The expansion about $\pi = 0$, then, is an expansion about a state of spontaneously broken symmetry. For example, the expansion of (9.6) about $U = 1$, or $\pi = 0$, respects only the SU(n) subgroup of the full global symmetry on which $V_L = V_R$. This emphasizes the fact that the coordinates π are precisely the right fields with which to describe physics at low energies.

Now that we have written one candidate for a phenomenological Lagrangian, let us construct the most general such Lagrangian. For the moment, I will restrict myself to the SU(n) × SU(n)-invariant case and use the coordinates (9.5). Since $U^\dagger U = 1$, (9.6) is actually the most general invariant coupling with two derivatives. Adding terms with higher derivatives, we can construct

$$L = \int d^4x \left\{ \frac{F^2}{2} \operatorname{tr}[\partial_\mu U^\dagger \partial^\mu U] + A_1 \operatorname{tr}[\partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U] + A_2 \operatorname{tr}[\partial_\mu U^\dagger \partial^\nu U \partial^\mu U^\dagger \partial_\nu U] + (6 \text{ derivatives}) + \cdots \right\} \tag{9.9}$$

A_1 and A_2 are unknown dimensionless parameters. But consider the consequences of expanding (9.9) in powers of (π/F) and using this Lagrangian to compute scattering amplitudes at center-of-mass energies E much less than F, or much less than Λ. Any vertex appearing in the second or third term of (9.9) is smaller than the corresponding vertex from the first term by a factor

$$A_i \frac{E^2}{F^2} \approx \frac{E^2}{\Lambda^2} \qquad (9.10)$$

Thus, if E is much less than Λ we need keep only the first term of (9.9); this term restricts the dynamics of Goldstone bosons completely once the value of F is specified.

It is clearly useful to extend this argument to more general patterns of symmetry breaking, and to include the effects of symmetry-breaking perturbations. To do this, it is easiest to first think more systematically about the best choice of coordinates. A beautiful set of coordinates, which works for any coset space G/H, was constructed and applied to this problem by Callan, Coleman, Wess, and Zumino (CCWZ) [93,94]. Let me now describe their formalism. (This formalism has also been reviewed in [4].)

To understand the choice of coordinates made by CCWZ, it is useful to think about how the underlying fermion fields ψ_{ri} and ψ_{rj} behave under chiral transformations. In general, ψ_{ri} and ψ_{rj} have completely different transformation laws under G. However, in any phenomenological description of the broken-symmetry state, we would like to be able to write a mass term linking $\bar{\psi}_{ri}$ and ψ_{ri}. Such a mass term would be permitted by invariance under H; however, we would like to construct L to be invariant under the full group G. The problem may be phrased more generally, in terms of a general matter field ϕ_0 of the original theory: If ϕ_0 transforms according to some representation R of G, this representation R will split into several irreducible representations R_i of H. These representations will correspond to particle multiplets under H which might be given very different masses as the result of the spontaneous symmetry breaking. To describe the components of ϕ_0 phenomenologically, we must replace ϕ_0 by a field ϕ whose transformation law under general transformations in G does not mix up the component representations R_i.

This requirement seems at first sight paradoxical. However, it can be met by defining at each point a local vacuum orientation and referring $\phi_0(x)$ to this orientation: Let us represent the local orientation of the vacuum by an element of G:

$$\exp[i\Pi_y(x)X_y] \tag{9.11}$$

where $\Pi_y(x)$ is a (dimensionless) field. We will see later that $\Pi_y(x)$ is proportional to the Goldstone boson field. Using (9.11), ϕ_0 can be written in the factorized form:

$$\phi_0(x) = \exp[i\Pi_y(x)X_y]\phi(x) \tag{9.12}$$

Now we can factor a general G transformation of ϕ_0 into two parts: A motion in H which acts on ϕ, and which does not mix H-representations within ϕ, and a motion in G/H which changes the local vacuum orientation. Represent the transformation

$$\phi_0 \rightarrow \exp[i\alpha_a G_a]\phi_0 \tag{9.13}$$

by writing

$$\begin{aligned}\exp[i\Pi_y X_y]\phi &\rightarrow \exp[i\alpha_a G_a]\ \exp[i\Pi_y X_y]\phi \\ &= \exp[i\Pi'_y(\alpha,\Pi)\cdot X_y]\ \exp[i\mu_i(\alpha,\Pi)\cdot T_i]\phi \end{aligned} \tag{9.14}$$

To obtain the second line, I hve used the fact that a general element of G may be decomposed uniquely as a transformation in H followed by transformation into G/H. Equation (9.14) suggests that we consider ϕ as the field which represents ϕ_0 in the phenomenological Lagrangian and assign to this field and the auxiliary fields Π_y the following transformation laws:

$$\Pi_y \rightarrow \Pi'_y(\alpha,\Pi)$$

$$\psi \rightarrow \exp[i\mu_i(\alpha,\Pi)T_i]\psi \qquad (9.15)$$

This is a nonlinear representation of G which does not mix H-representations within ψ. Under a pure H transformation ($\alpha_a G_a = \alpha_i T_i$), (9.15) simplifies to

$$\Pi_y X_y \rightarrow \exp[i\alpha_i T_i]\ (\Pi_y X_y)\ \exp[-i\alpha_i T_i]$$

$$\psi \rightarrow \exp[i\alpha_i T_i]\psi \qquad (9.16)$$

so that ψ,Π transform linearly under H. Π_y transforms like the charge X_y. Coleman, Wess and Zumino [93] have proven that any nonlinear representation of G which becomes linear on a subset H may be brought into the form of (9.15) by a change of coordinates.

Because (9.15) is a nonlinear transformation, $\partial_\mu \Pi_y$ has a very complicated transformation law. We can find an object with a simpler transformation law by starting from $\partial_\mu \psi_0$, which transforms linearly under G. Let me first introduce the notation.

$$\exp[-i\Pi\cdot X]\ \partial_\mu(\exp[i\Pi\cdot X]) = i\partial_\mu \Pi_y(D_{yz}(\Pi)X_z + E_{yi}(\Pi)T_i) \qquad (9.17)$$

The quantity on the left is a generator of G; I have expanded it in terms of generators, defining expansion coefficients D(Π) and E(Π). Using (9.17), we may expand

$$\begin{aligned}\partial_\mu \psi_0 &= \partial_\mu(\exp[i\Pi\cdot X]\psi) \\ &= \exp[i\Pi\cdot X]\ (\{i\partial_\mu \Pi_y D_{yz}(\Pi)X_z\}\psi \\ &\quad + \{(\partial_\mu + i\partial_\mu \Pi_y E_{yi}(\Pi)T_i)\psi\})\end{aligned} \qquad (9.18)$$

The quantity in parentheses transforms under G like ψ in (9.15). It is consistent to restrict ψ to be nonzero for only one of the R_i, but then the two terms in brackets will belong to different H representations in R. They must therefore transform independently with this transformation law. Thus

$$D_\mu \Pi_z = \partial_\mu \Pi_y \cdot D_{yz}(\Pi) \rightarrow (\exp[i\mu_i T_i])_{zx}(\partial_\mu \Pi_y \cdot D_{yx}(\Pi)) \tag{9.19}$$

under (9.15), where T_i in (9.19) is taken in the representation of H to which the $\{X_y\}$ belong.

We can now build Lagrangians invariant to G by constructing H-invariant combinations of $D_\mu \Pi_y$. Since a transformation in G/H translates Π_y as in (9.8) by a global parameter α_y, Π_y may not appear without a derivative in a G-invariant Lagrangian except as a part of $D_{yz}(\Pi)$. Thus, if the Π_y belong to a single irreducible representation of H (a situation to which I will specialize for the remainder of my discussion), the most general invariant Lagrangian containing no more than two derivatives is

$$L = \frac{1}{2} f_\pi^2 (D_\mu \Pi_y)^2 \tag{9.20}$$

For the moment, simply regard f_π as a coefficient with the dimensions of mass; I will show later, though, that it is the same as the coefficient f_π which appears in (8.10). Possible G-invariant terms with four or more derivatives are smaller than (9.20) by the factor (9.10) and may be ignored.

It is not difficult to extend the global G invariance of (9.20) to a local invariance by applying the trick used in (9.18) to the gauge-covariant derivative, $(\partial_\mu - iA_{\mu A} G_A)\phi_0$. Let us define F(Π) and H(Π) by

$$\exp[-i\Pi \cdot X](G_A) \exp[i\Pi \cdot X] = F_{Ay}(\Pi) X_y + H_{Ai}(\Pi) T_i \tag{9.21}$$

Then we may decompose

$$(\partial_\mu - iA_{\mu A} G_A)\phi_0 = \exp[i\Pi \cdot X] \left(i\left\{\partial_\mu \Pi_y \cdot D_{yz}(\Pi) - A_{\mu A} F_{Az}(\Pi)\right\} X_z \phi + \left\{\partial_\mu - i(\cdots) T_i\right\}\phi\right) \tag{9.22}$$

From (9.22), we can identify the chiral- and gauge-covariant derivative

$$\mathscr{D}_\mu \Pi_z = (\partial_\mu \Pi_y D_{yz}(\Pi) - A_{\mu A} F_{Az}(\Pi)) \tag{9.23}$$

If we then write

$$L = \frac{1}{2} f_\pi^2 (\mathscr{D}_\mu \Pi_z)^2 \tag{9.24}$$

we have constructed a Lagrangian locally gauge-invariant under the symmetries associated with the $A_{\mu A}$.

Let us now work out a few terms of the chiral Lagrangian more explicitly. It is not hard to see that, to leading order in Π,

$$\begin{aligned} D_{yz}(\Pi) &= \delta_{yz} + \cdots \\ F_{Az}(\Pi) &= \mathrm{Tr}(G_A X_z) + \cdots \end{aligned} \tag{9.25}$$

If we define

$$\pi_y = f_\pi \Pi_y \quad , \tag{9.26}$$

so that the boson kinetic energy term is conventionally normalized, the gauge-invariant phenomenological Lagrangian (9.24) takes the form

$$L = \frac{1}{2} (\partial_\mu \pi_y - f_\pi \,\mathrm{Tr}(G_A X_y) A_{\mu A})^2 + \cdots \tag{9.27}$$

By setting G_A equal to X_z, so that $A_{\mu A}$ couples to the current $J_{\mu z}$, we can check that (9.27) does yield precisely the expression (8.10) for the amplitude that $J_{\mu z}$ creates a Goldstone boson. Thus f_π has been correctly identified. The gauge field mass matrix (8.36) is also displayed manifestly in (9.27).

The higher-order terms in the phenomenological Lagrangian contain multi-pion and pion-gauge boson interactions. In parity-invariant theories such as those of section 8.1, these interactions may be represented compactly if one introduces a bit more notation. Denote the structure constants of G by writing

$$[G_a, G_b] = t_{abc} G_c \quad . \tag{9.28}$$

We might then define

$$(t\cdot\Pi)_{ab} = t_{axb}\Pi_x \qquad (9.29)$$

Equation (9.29) is a representation of $(\Pi\cdot X)$ in the adjoint representation of G. Then, classifying the various terms in (9.17) and (9.21) by parity, one can verify that

$$D_{yz}(\Pi) = \left[\frac{\sin t\cdot\Pi}{t\cdot\Pi}\right]_{yz} \qquad F_{Az}(\Pi) = (\cos t\cdot\Pi)_{Az} \qquad (9.30)$$

The matrix functions are defined as power series. Inserting (9.30) into (9.23) gives all of the pion-gauge boson vertices of the theory explicitly, at least to leading order in E^2/Λ^2.

The Lagrangian (9.24) is not yet, however, a complete description of the low-energy dynamics of the perturbed gauge theory. One important effect is still missing: the masses generated for pseudo-Goldstone bosons by exchange of the weak gauge bosons $A_{\mu A}$. It is true that these masses are seen to be induced when one computes radiative corrections to the Lagrangian (9.24). Thinking about radiative corrections, however, only emphasizes the problem: The radiative corrections to (9.24) contain a number of quadradically ultraviolet divergent contributions, the first of which are shown in fig. 25. Since these contributions do not vanish at E = 0 and do not respect the global G symmetry, they are not of the form of any π interaction in (9.24). We should represent these effects by a set of counterterms. We need to know, then, how to construct such counterterms.

The one constraint that we have on the structure of these counterterms is that they have a definite transformation property under G: The graphs of fig. 25 transform under G in the same way as the perturbation (8.26) induced by one-gauge-boson exchange — as a symmetric tensor with two indices in the adjoint representation of G. The philosophy of phenomenological Lagrangians suggests that we should simply construct all functions of Π with this transformation law containing no derivatives; the counterterms for fig. 25 should then be an arbitrary linear combination of these functions. To perform this construction, we must first answer the following question: How do we construct a function of Π which transforms linearly under G according to a specific irreducible representation R?

This question is readily answered by making use of the CCWZ coordinates [94]. Let us first try to construct such a function using both Π and ϕ fields. Imagine decomposing R into representations R_i of H. Choose a ϕ field which transforms under some particular R_i. Then we can reconstitute a ϕ_0 field by writing

$$\phi_0 = \exp(i\Pi\cdot X)\phi \tag{9.31}$$

If Π,ϕ transform according to (9.15), (9.31) transforms linearly under G according to R. But now notice that, if R_i is a singlet under H, ϕ is not transformed by (9.15), and we may omit it. Thus, if R contains in its decomposition a singlet $\underline{1}$ of H, the object

$$\exp[i\Pi\cdot X]\cdot\underline{1} \tag{9.32}$$

transforms linearly under G according to R.

The product of currents $[J_{\mu A}J_{\nu B}]$ which appears in (8.26) belongs to a representation which contains two H-invariants, δ_{ij} and δ_{xy}. However,

$$\delta_{ij} + \delta_{xy} = \delta_{ab} \tag{9.33}$$

is a G-invariant; X_y, acting on (9.33), gives zero. Hence, the most general function of Π's which transforms linearly under G like (8.26) is [95]

$$\delta L = \frac{K}{2} (\exp[it\cdot\Pi])_{Ai} (\exp[it\cdot\Pi])_{Aj} \delta_{ij} \tag{9.34}$$

I have used the representation (9.29) of $(\Pi\cdot X)$. If the vacuum alignment is chosen properly (in particular, if (8.20) is satisfied), (9.34) may be seen to contain no linear terms in Π. The terms quadratic in Π are:

$$(\delta L)_2 = \frac{K}{2} \left[-\frac{1}{2} ((t\cdot\Pi)^2{}_{Ai}\delta_{iI(A)} + (t\cdot\Pi)^2{}_{Aj}\delta_{jI(A)}) + (t\cdot\Pi)_{Ai}(t\cdot\Pi)_{Aj}\delta_{ij} \right] \tag{9.35}$$

$I(A)$ and $Z(A)$ denote the decomposition (8.25). Replacing Π using (9.26), setting $K = M^2 f_\pi{}^2$, and writing out the commutators explicitly, (9.35) may be cast into the form

$$(\delta L)_2 = -\frac{1}{2} (m^2)_{yz} \pi_y \pi_z \tag{9.36}$$

where

$$(m^2)_{yz} = M^2[\mathrm{Tr}(X_y[T_{I(A)},[T_{I(A)},X_z]]) - \mathrm{Tr}(X_y[X_{Z(A)},[X_{Z(A)},X_z]])] \tag{9.37}$$

This result for the pseudo-Goldstone boson mass matrix may be shown [81,84] to be identical to (9.2). The coefficient M^2 may then be identified with (9.3). Except for the question of determining this parameter, the phenomenological Lagrangian formalism has allowed us to simply write down the correct result, without the necessity of a detailed computation.

I should make a few comments on the form of (9.37). The first set of commutators is just proportional to

$$g^2 C_2(r_W) \tag{9.38}$$

where r_W is the G_W representation to which the Goldstone bosons belong. This factor is just what one would expect from one-gauge-boson exchange.

The second term is more complex. (It is simplified in [81].) Each of the two traces is a positive matrix; thus, if we interpret $(m^2)_{yz}$ as the curvature of the effective action, we see that unbroken G generators stabilize the vacuum, and broken G generators destabilize it, in accordance with the physics we discussed in section 8.2.

The full chiral Lagrangian, (9.24) plus (9.34), contains many multiparticle vertices. It is worth asking whether we can use it to do higher-order computations. This seems at first sight problematical, because the Lagrangian is formally non-renormalizable. Even so, one can obtain sensible results from loop graphs if one makes use of the presence of a natural cutoff, the scale Λ. Chadha and I have calculated some of the one-loop corrections to the π mass matrix (9.37) in models of dynamical symmetry breaking [95]. The quadratic divergences in this calculation have the same structure as the mass term above and may be removed by adjusting K in (9.34). The logarithmically divergent terms give new group-theoretic structures. These new structures transform like $(\Delta H)^2$. We should, properly, represent these contributions by writing terms of the most general possible structure for this transformation law, to appear with coefficients of order $(g^2)^2$ in the phenemonological Lagrangian. However, the logarithmically divergent terms which arise from perturbation theory contain an extra infrared enhancement factor of the form

$$\log(\Lambda^2/m_\pi^2) = \log(1/g^2) \quad . \qquad (9.39)$$

One can hope that these terms, of order $g^4 \cdot \log(1/g^2)$, are the dominant corrections;* using this assumption, one can predict the leading

*In doing this, we follow the philosophy of chiral perturbation theory [5,96].

corrections to the lowest order mass formula.

9.3 AN EXAMPLE WITH LIGHT FERMIONS

In addition to providing insight into the dynamics of light bosons, the phenomenological Lagrangian method can also be used to explore the dynamics of light fermions. We have seen in section 5 that it is common in chiral gauge theories for a multiplet of fermions to be kept massless by symmetry constraints. By analogy to the physics of Goldstone bosons which we have just discussed, one can easily imagine that weak gauging of some chiral symmetries in such a model could give these fermions small masses. In this section, I will sketch the analysis of a very simple model, due to Dimopoulos and Susskind [97], in which fermion masses appear in this way. My presentation must unfortunately be rather brief; a more detailed analysis of this model may be found in [98]. Other models in which light fermion masses arise in this way have been discussed by Weinberg [53], Nilles and Raby [99], and Sikivie [100].

The model of Dimopoulos and Susskind is built upon a strongly interacting SU(3) gauge theory containing one 6 and seven $\bar{3}$ representations of L-fermions. The 6 is the $\{2\}$ of SU(3); according to eq. (5.12), this model is anomaly-free. The chiral symmetry of this theory is $G = SU(7) \times U(1)$. Let me break chiral symmetries and generate masses following the methods of section 5. I will assume that chiral symmetry is broken by the condensate

$$\{2\} \times \bar{3} \to 3 \tag{9.40}$$

The corresponding mass term with maximal global symmetry is:

$$\epsilon^{\alpha\beta}\psi_{\alpha}{}^{ab}\psi_{\beta a,i}\delta^{i}{}_{b}\cdot\Sigma + \text{h.c.} \tag{9.41}$$

where a,b = 1,2,3 are color indices and i = 1,...,7 is a flavor index. Note that only three of the $\bar{3}$'s acquire mass. Equation (9.41) breaks the SU(3) gauge symmetry completely; it also breaks down the SU(7) × U(1) flavor symmetry. But, as in the examples of section 5, a considerable amount of global symmetry remains. The SU(4) flavor symmetry which acts on the components i = 4,5,6,7 of $\psi_{a,i}$ is undisturbed by (9.41). The SU(3) flavor symmetry acting on the components i = 1,2,3 of $\psi_{a,i}$ may be combined with global SU(3) gauge symmetry to produce a global symmetry which is respected by (9.41). The U(1) charge

$$\tilde{Q} = \frac{1}{7}\,(2Q + Q_{(7)}) \tag{9.42}$$

built from the anomaly-free global U(1) charge Q and the SU(7) generator $Q_{(7)}$ given by (5.34), is also preserved by (9.41). The pattern of symmetry-breaking is, then (in the notation of (5.37))

$$[SU(3)] \times SU(7) \times U(1) \rightarrow SU(3) \times SU(4) \times U(1) \tag{9.43}$$

The fermions transform under the unbroken symmetries as

$$\begin{aligned} \{2\} &\rightarrow (\{2\},\ 1,\ (-2)) \\ 7\cdot\bar{3} &\rightarrow (\{\bar{2}\},\ 1,\ (+2)) + ([\bar{2}],\ 1,\ (+2)) + (\bar{3},\ 4,\ (+1)) \end{aligned} \tag{9.44}$$

One should recall that $[\bar{2}] = 3$ of SU(3). The pair of fermions in the $\{2\} + \{\bar{2}\}$ of SU(3) are given mass by (9.41); the remaining fermions are kept massless by symmetry constraints.

We might now try to break the chiral symmetries which protect these massless states by gauging some subgroup of SU(7). A convenient choice is the exceptional group G_2, which has a 7-dimensional fundamental representation. This representation is real, so the theory remains anomaly-free. G_2 is actually the smallest group which contains SU(3) as a proper subgroup; the 7 decomposes neatly under SU(3) according to

$$7 \to 3 + \bar{3} + 1 \qquad (9.45)$$

It will also be useful to note that the adjoint representation of G_2, which is 14 dimensional, decomposes according to:

$$14 \to 8 + 3 + \bar{3} \qquad (9.46)$$

The 8 is, of course, the adjoint representation of SU(3).

The breaking of SU(7) necessarily breaks G_2; the most we can save of G_2 is its SU(3) subgroup. The orientation in which this SU(3) is preserved is thus the preferred vacuum alignment. In this orientation, the 7 of SU(7) decomposes as follows:

$$\begin{array}{llll} 7 \to & (\bar{3},\ 1) & + \ (1,\ 4) & \text{of } SU(3) \times SU(4) \\ & \quad\downarrow & \qquad\downarrow & \\ & \quad\bar{3} & + \quad 3+1 & \text{of } SU(3) \end{array} \qquad (9.47)$$

The U(1) charge $Q_{(7)}$ does not commute with G_2; it is therefore explicitly broken, and so is $\tilde{Q}$. In fact, the only symmetry which is neither explicitly nor spontaneously broken at this stage is the weakly gauged SU(3) group itself. The fermions left massless in (9.44) have the following quantum numbers under this SU(3):

$$(3,\ 1,\ (+2)) \to 3$$

$$(\bar{3},\ 4,\ (+1)) \to 8 + 1 + \bar{3} \qquad (9.48)$$

There are no longer symmetry constraints forbidding mass for any fermion.

We have now demonstrated the possibility that small fermion masses are generated in this model as effects of weak gauge boson exchanges. What we would really like to do, however, is to compute the spectrum of these masses. This can, in fact, be done, using the methodology of phenomenological Lagrangians.

The first step is to write a mass counterterm representing the effects of one-G_2-boson exchange. To do this, we must construct combinations of two massless fermions which transform under H like the one-gauge-boson exchange perturbation, and which therefore have the structure of fig. 26. One may verify that the G_2 currents carry the charges ±1 or 0 under $\tilde{Q}$. Thus, the G_2 exchange can absorb only an amount of charge $\Delta\tilde{Q} = \pm 2$. This means that fig. 26 can only produce a mass term which pairs the $(\bar{3}, 4, (+1))$ of (9.44) with itself. Such a term gives mass only to the 8 and 1 in (9.48). There is, in fact, only one combination of fermions with the required structure; thus, there must be a relation between m_1 and m_8.

The $3 + \bar{3}$ mass may be computed by constructing the phenomenological Lagrangian and computing its perturbative corrections. One can compute at least the contributions to this mass enhanced by infrared logarithms, as I explained at the end of section 9.2. The leading diagrams are shown in fig. 27. The diagram involving a massive G_2 boson W_μ has a logarithmic ultraviolet divergence, which I interpret as

$$\log(\Lambda^2/m_W^2) = \log(1/\alpha_s) \tag{9.49}$$

where α_s is the SU(3) or G_2 coupling constant. The coupling of the W_μ to the 3 is through the analogue of a vector current and so involves no additional parameters; however, the coupling of the W_μ to the 3 occurs via the axial current and so may have a renormalization factor g_A. Hopefully here, as in the familiar strong interactions, this factor is close to 1. The graph involving a π exchange turns out to give zero: The coupling of the π to the 3 vanishes by parity. The result of this computation is the complete spectrum of light masses:

$$m_1 = 2m_8 = O(\alpha_s\Lambda)$$

$$m_3 \doteq \frac{\alpha_s}{2\pi} g_A\, m_8 \log(1/\alpha_s) \tag{9.50}$$

This example, has, I hope, amused you. It has also reviewed all of the major concepts which we have discussed in this course:

— The breaking of chiral symmetries by fermion pair condensation.

— The appearance of massless fermions protected by residual chiral symmetries.

— The perturbation of the pattern of χSB by weak gauge exchanges.

These are all phenomena of importance for understanding the structure of strongly interacting gauge theories of fermions. To varying degrees, they are all in need of further theoretical elucidation. They are also, at the moment, theoretical mechanisms in need of application, whose role in the physics of the fundamental interactions is not at all clear. I hope that these lectures might serve as a starting point for understanding these mechanisms more deeply, and for applying them fruitfully.

ACKNOWLEDGEMENTS

I have been puzzling over chiral symmetry since my days as a graduate student; most of what I have learned about this topic in the intervening time, I owe to discussions and arguments with numerous friends and collegues. I owe a particular debt to John Kogut, Ken Lane, and Ken Wilson, from whom I first learned about chiral symmetry, and to Steven Weinberg and Ed Witten, who have strongly shaped my understanding of this subject. I would also like to thank Orlando Alvarez, Tom Banks, Sidney Coleman, Savas Dimopoulos, Estia Eichten, John Preskill, Stuart Raby, Adam Schwimmer, Michael Stone, Leonard Susskind, Shimon Yankielowicz, and Jean Zinn-Justin for educating me on many of the subjects discussed here. I am grateful to the students of the Les Houches school, and to Ron Shellard, Steve Shenker, and Giorgio Parisi, for their detailed criticism of my lectures, and to Paul Ginsparg, for reading the final text. I thank Bette-Jane Ferandin for her patient typing of the manuscript. Finally, I am grateful to Raymond Stora and Jean-Bernard Zuber for giving me the opportunity to present these lectures in such a stimulating and inspiring environment.

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FIGURE CAPTIONS

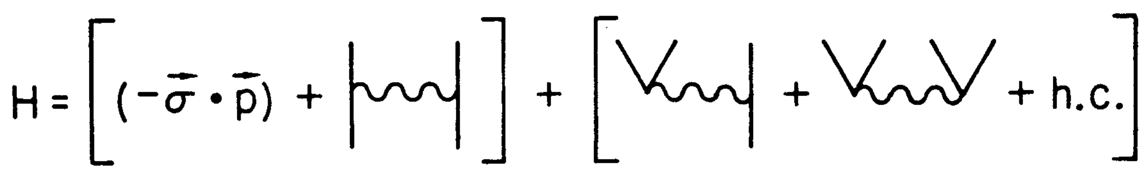

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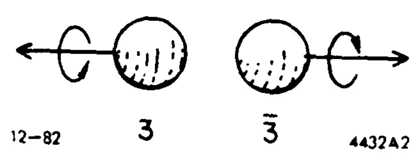
Fig. 1



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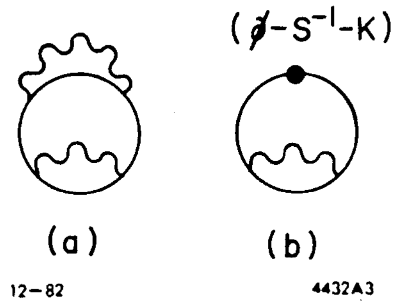

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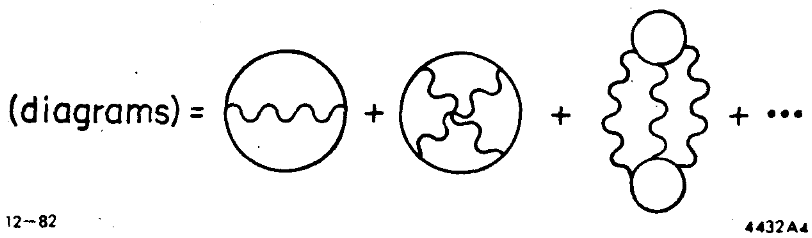

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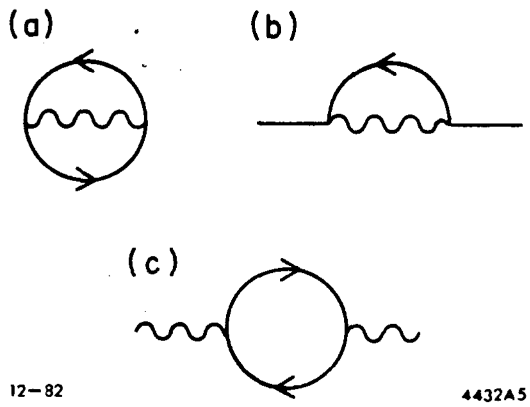

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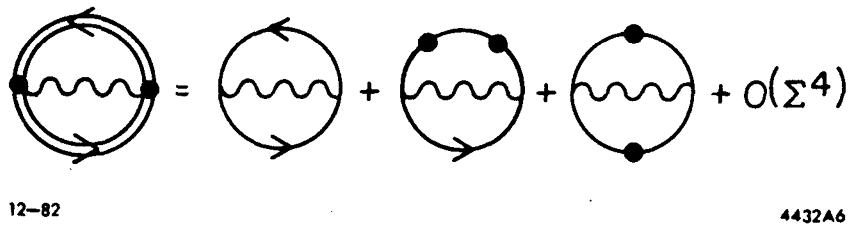

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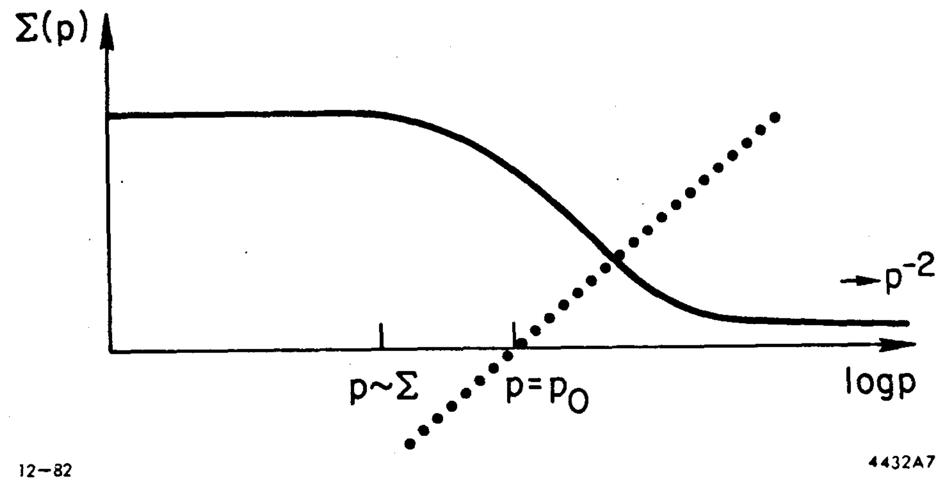

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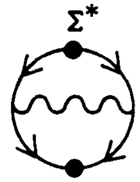

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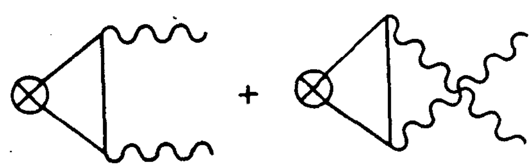

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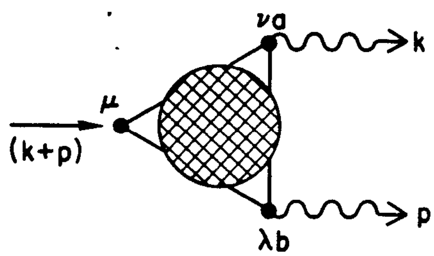
Fig. 9



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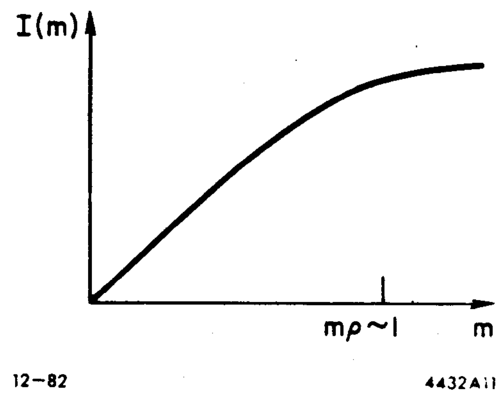

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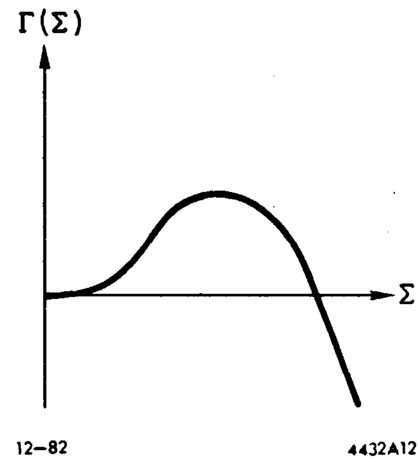

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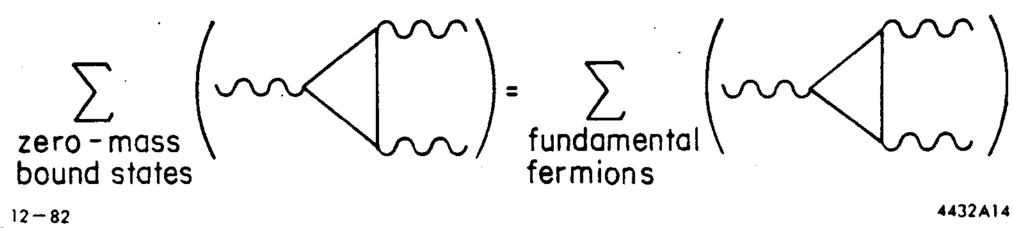

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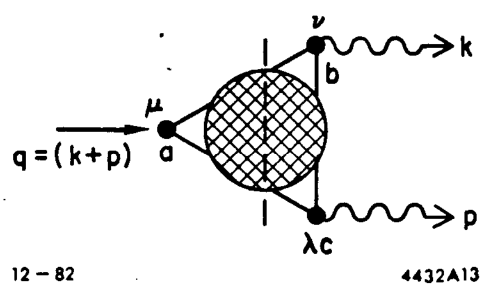

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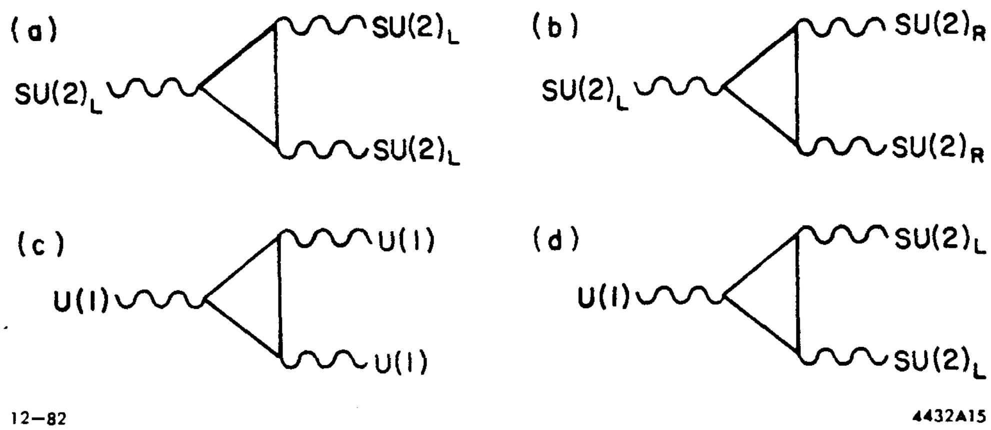
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(c)
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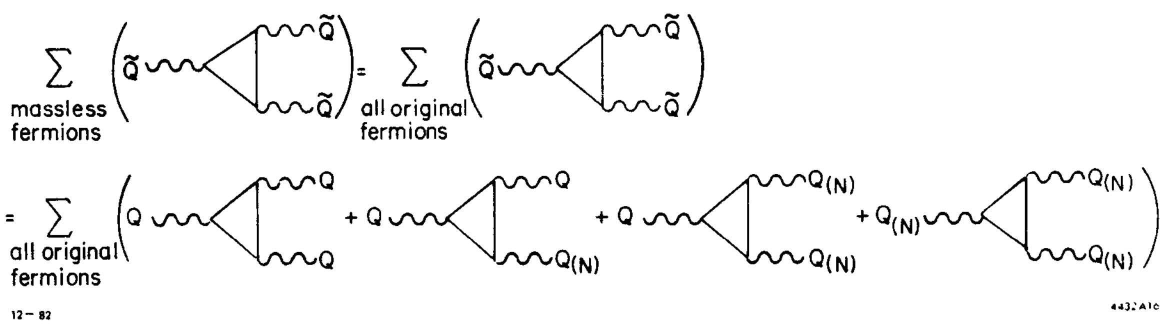
Fig. 15



massless fermions
all original fermions
all original fermions
Q̃
Q
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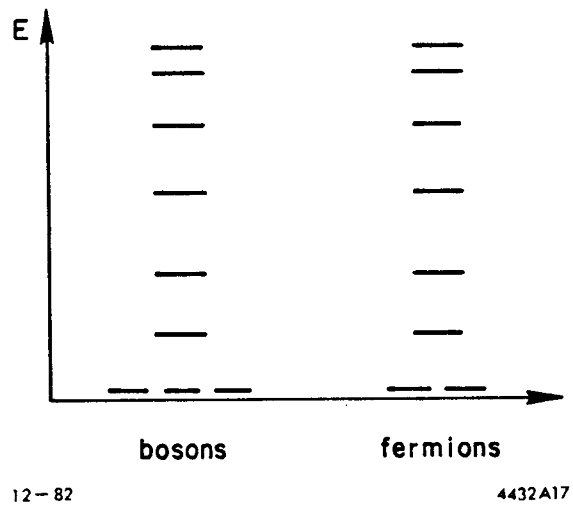
Fig. 16



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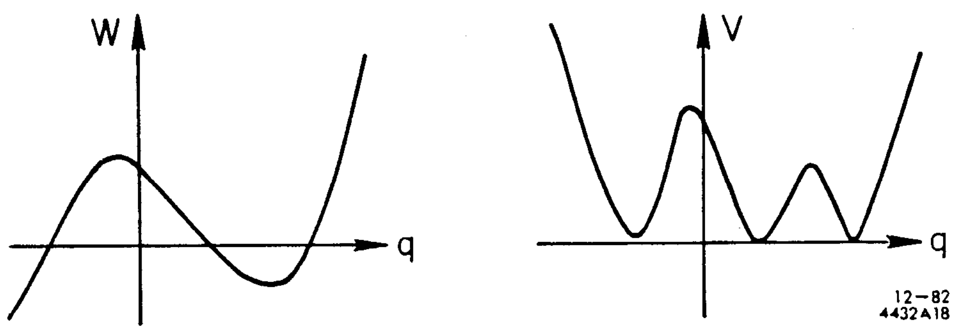

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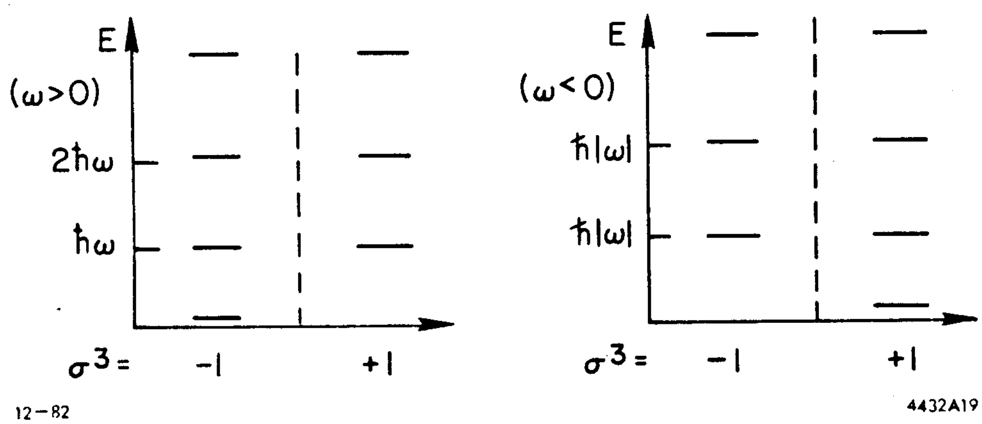

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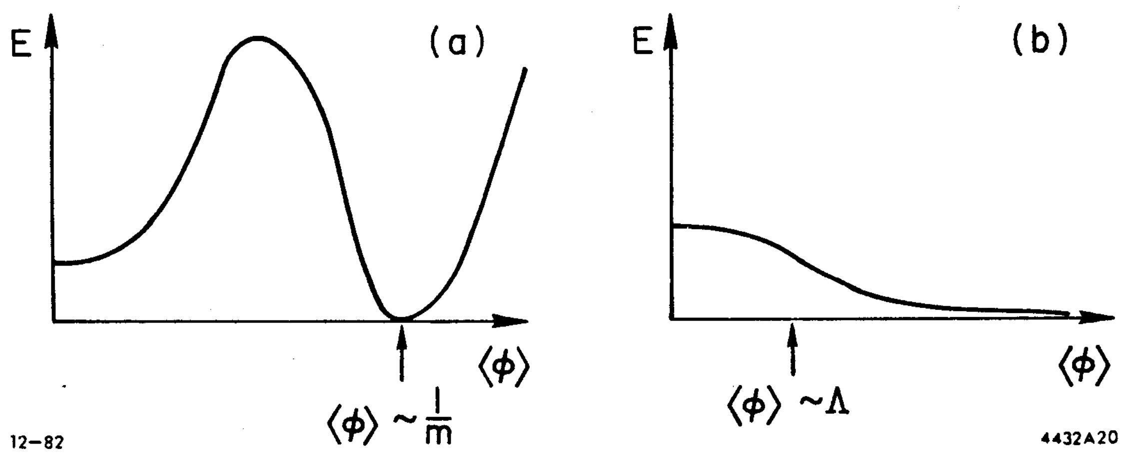

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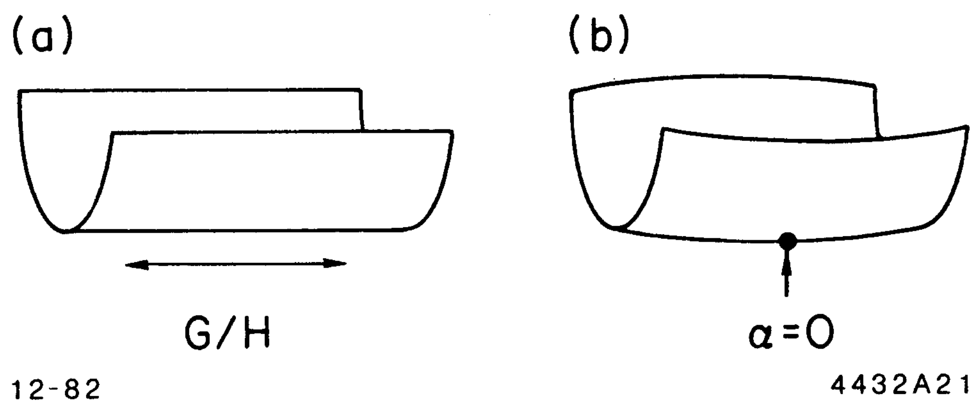

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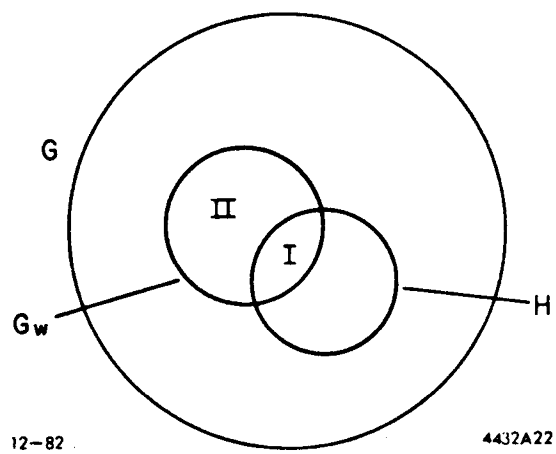

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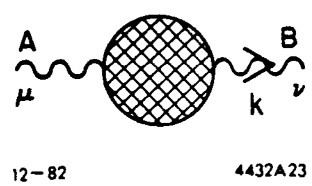

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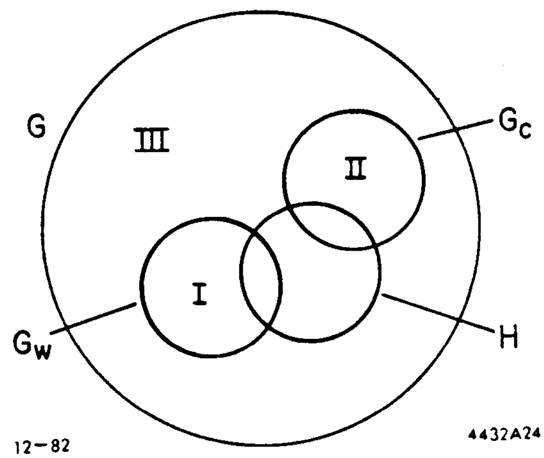

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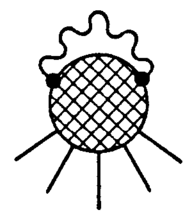

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Fig. 25

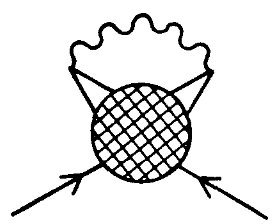

Fig. 26

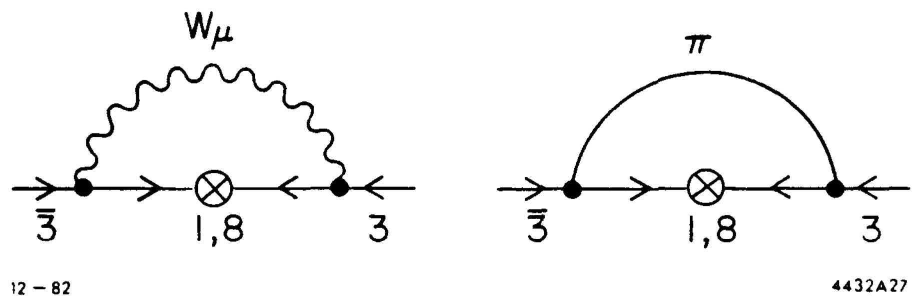

Fig. 27