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TOWARD VERIFICATION OF LARGE-SCALE HOMOGENEITY IN COSMOLOGY*

M. HOSSEIN PARTOVI
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

and

BAHRAM MASHHOON
Institut für Theoretische Physik, Universität zu Köln
D-5000 Cologne 41, Federal Republic of Germany

ABSTRACT

Motivated by the desirability of direct observational verification of the large-scale homogeneity of the universe, and relying on assumptions deemed to be well supported by the existing observational data, we develop local models with radial inhomogeneities as a first step toward a detailed framework for comparison with observations. Various observational quantities are expanded in powers of the redshift. While the leading terms generally coincide with the corresponding Friedmann-Robertson-Walker quantities, the higher-order contributions are modified by the presence of inhomogeneities. It is concluded that in as much as the next-to-leading terms (such as the deceleration parameter) in such an expansion are generally poorly determined by present observations, the inhomogeneous models developed here are locally indistinguishable from the standard homogeneous models. A possible exception may be the measurement of redshift evolution, should this become feasible with sufficient accuracy.

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I. INTRODUCTION AND SUMMARY

The standard hot big bang cosmology, founded on the cosmological principle and the primeval fireball, provides a remarkably simple and successful description of the evolution of the universe (recent reviews include Dolgov and Zel'dovich 1981; Steigman 1979; a standard reference is Weinberg 1972). Aside from providing a globally consistent space-time description which is in accord with cosmological observations, notably the Hubble expansion (Tammann, Sandage and Yahil 1980), it provides a scenario of primordial nucleosynthesis (reviewed in Schramm and Wagoner 1977) which is consistent with the observed light-element abundances and, more importantly, has received a decisive confirmation in the discovery of the cosmic microwave radiation. Recent developments in connection with grand unified theories of particle interactions, particularly with respect to a mechanism of baryon asymmetry generation in the early universe (Dolgov and Zel'dovich 1981), as well as the more speculative schemes (Guth 1982) to account for the horizon and flatness puzzles (Dicke and Peebles 1979), have served to integrate further cosmology and the standard model into the main body of research on the fundamental structure of matter. Needless to say, the standard model is not without its difficulties, as can be seen in the various reviews cited above.

A particularly striking aspect of the standard model, and one which is the concern of the present work, is the limited extent of verification which the cosmological principle has received from observations, especially in regard to the issue of large-scale homogeneity in the deep universe. This is a well-recognized circumstance (see, e.g., Kristian and Sachs 1966; Peebles 1980; Dicke and Peebles 1979; Ellis 1980), and one which is principally a consequence of the meager amount of reliable information that can be extracted from difficult cosmological observations. In view of the enhanced significance of cosmology, particularly in relation to the physics of very high energies, it seems worthwhile to continue the effort toward establishing a more reliable observational basis for the cosmological principle. The pioneering work in this direction is the important paper of Kristian and Sachs (1966), and more recent work in the same spirit is described in Ellis (1980).

The present work seeks to examine the large-scale homogeneity of the universe within a framework which incorporates only what may be considered to be well supported by observations. Thus we are considering a limited region of the universe in our neighborhood, corresponding to a limited range of redshifts, say less than a half or so, disregarding thereby the global aspects of space-time structure. It is then expedient, as well as adequate, to expand the various quantities of interest in powers of the redshift, as suggested by Kristian and Sachs (1966), and to retain only the first correction beyond the leading term. Stated crudely, the leading quantities (in each term) will normally coincide with those of the Friedmann-Robertson-Walker (FRW) universe, while the corrections represent deviations thereof. In this way, one is able to parametrize an important portion of the presently observable universe in a fairly model-independent way.

To implement the above ideas, we first observe that the observational evidence supporting isotropy, particularly the isotropy of the cosmic microwave background, the source distributions, and the Hubble flow (Peebles 1980; MacCallum 1979), is sufficiently compelling to render the assumption of a spatially isotropic space-time (centered about us) a reasonable working hypothesis.[1] In contrast to isotropy, there is little direct

1. This is not to imply that isotropy is uniquely implied by the evidence; see Ellis (1980) in this connection.

evidence in support of (large-scale, radial) homogeneity, a property which is usually argued for on the basis of the Copernican principle (Peebles 1980). While the latter is a reasonable and aesthetically appealing assumption, it certainly does not constitute acceptable evidence. We shall therefore allow for radial inhomogeneities as a diagnostic technique to study the issue of large-scale homogeneity. It is perhaps worth emphasizing that the use of such essentially local models to confront observations is not to be confused with suggestions of anthropocentric global models for which there is neither observational evidence nor a reasonable *a priori* justification. Further motivation for investigating large-scale inhomogeneities is provided by the recent discovery of voids (Kirshner *et al.* 1981; Gregory and Thompson 1982; Doroshkevich, Shandarin, and Zel'dovich 1982).

Having thus assumed a spatially isotropic metric in our neighborhood, we add the assumption that (the averaged-out) cosmic matter may be represented by a perfect fluid obeying a physically reasonable equation of state of the form $p = p(\rho)$, or $p = 0$, where ρ and p represent the density and the pressure. These two cases will be referred to as I and II. They cover the equations of state usually considered, case II being appropriate where pressure is negligible, and case I where it is not.[2] Finally, in expanding the pro-

2. Note that the pressure-free limit of case I corresponds to a constant density, a situation which is neither relevant, nor equivalent to case II.

perties of the universe in our neighborhood, we are assuming our world line to be nonsingular and the metric sufficiently differentiable thereabout.

A number of authors have considered a variety of spatially isotropic, but radially inhomogeneous models in the past (e.g., Lemaître 1933; Tolman 1934; Bondi 1947; Dodson 1972; Roeder 1975). With very few exceptions (Roeder 1975), these are proposed as global models of the universe. As stated before, the present work differs from the above-cited ones in that it relies on spatially isotropic metrics primarily as model-independent parametrizations of our observable neighborhood. *Indeed we consider the relaxation of the assumption of isotropy and the inclusion of angular inhomogeneities a logical next step in our approach.*

The conclusions we reach in comparing the detailed results of our models with observations indicate that there is no direct evidence excluding large-scale inhomogeneities, in accord with general expectations (Kristian and Sachs 1966; Ellis 1980). Indeed the changes caused by these inhomogeneities in various observational quantities are to some degree manifested by a redefinition of the usual deceleration parameter. Mainly for this reason, any probe of large-scale homogeneity will have to rely on a fairly accurate determination of this parameter. The possibility of measuring changes in redshifts (Davis and May 1978) offers a hope in this connection.

To illustrate the above remarks, we will summarize here some characteristic results of our analysis (see §IIIc for details). The metric forms for cases I (corresponding to any physically reasonable equation of state $p = p[\rho]$) and II (corresponding to pressure-free matter) are found to be

$$\mathcal{J}^I = -\left(1 + \frac{1}{2}\alpha r^2 + ...\right)^2 dt^2 + S^2(t)\left[\left(1 + \frac{1}{2}\beta r^2 + ...\right)^2 dr^2 + r^2\left(1 + \frac{1}{2}\gamma r^2 + ...\right)^2 d\Omega^2\right],$$

and

$$\mathcal{F}^{II} = -dt^2 + S^2(t)\left[(1 + \delta r + ...)^2 dr^2 + r^2\left(1 + \frac{1}{2}\delta r + ...\right)^2 d\Omega^2\right],$$

where α, β, γ, and δ are functions of time, and triple dots represent higher-order terms in r. For case I, β-3γ is the spatial curvature, with $d(\beta - 3\gamma)/dt = -2\alpha H$, where H is the Hubble parameter defined in terms of S in the usual way. In both cases, inhomogeneity is characterized by a dimensionless function of time C, where

$$(SH)^2 C^I = -\alpha, \qquad (SH)^2 C^{II} = S\, d\delta/dt.$$

The luminosity distance is then given for both cases by

$$d_L = \frac{z}{H} + \frac{1}{2H}(1 - q - C)z^2 + O(z^3),$$

where the deceleration parameter q is defined in terms of S in the usual manner. For case I, the relation of C to density and pressure distributions is given by

$$q - C^I = \frac{1}{2}\Omega^I\left(1 + \frac{3p}{\rho}\right),$$

where Ω is density in units of the critical density, and all quantities on the right-hand-side of the equation (as well as of the equation below) refer to measurements by a comoving observer at $r = 0$. In case II, the analogous expression is

$$C^{II} = \frac{3}{4}\,\frac{1 - (1+q)T}{1 - \frac{3}{2}T}\,\frac{1}{SH}\,\frac{1}{\Omega^{II}}\,\frac{\partial \Omega^{II}}{\partial r},$$

where T is the "age of the universe" in units of H^{-1}, and Ω is defined as above. While C is in principle an observationally measurable quantity, its actual determination will at the very least be subject to the difficulties encountered in the measurement of q.

The existence of a non-zero C, even as a local phenomenon, obviously has important consequences. For example, the expressions for luminosity distance and density distribution for case I illustrate the possibility that the two conventional methods of determining the deceleration parameter q may in fact be measuring two different quantities q - C and q + C. Since C is restricted only by $C^I < q$, the significance of the existing determinations of q could thereby be drastically altered.

This paper is organized as follows: In §II a number of observationally relevant results are derived for spatially isotropic metrics in general. Section IIIa is concerned with expanding these relations in powers of the redshift, and §IIIb with developing the properties of models I and II in a detailed manner. A number of these results are summarized in §IIIc. Finally in §IV a brief evaluation and discussion of these results is presented.

II. OBSERVATIONAL RELATIONS FOR SPATIALLY ISOTROPIC METRICS

In this section we shall consider a number of quantities related to cosmological observations and derive suitable relations for them in the context of a universe described by a spatially isotropic metric. Further assumptions appropriate to models I and II will not be introduced until the following section, so that the results to be derived here are generally valid for a spatially isotropic space-time. The underlying formalism is well known (particularly useful sources are Kristian and Sachs 1966 and Ellis 1971), our task here being the application of the general formalism to the case at hand.

a) Redshift and Related Quantities

The interpretation of redshift measurements proceeds from the assumption of light propagation along null geodesics according to the laws of geometrical optics (as deduced from Maxwell's equations) and properties of the underlying space-time geometry. For a spatially isotropic universe, the space-time metric in terms of comoving cosmic coordinates $x^\mu = (t, r, \vartheta, \varphi)$ may be written as[3]

3. Unless otherwise specified, we shall use the notation of Mashhoon and Partovi (1979).

$$\mathcal{J} = -a^2 dt^2 + b^2 dr^2 + R^2 d\Omega^2, \tag{1}$$

where a, b, and R are non-negative functions of t and r. These metric coefficients will be assumed to be sufficiently differentiable functions of their arguments.

In the limit of geometrical optics, the propagation of radiation may be described in terms of the wave vector k_μ which satisfies the (null) geodesic equations

$$k^\nu k_{\mu;\nu} = 0, \quad \left(k_\mu k^\mu = 0\right), \tag{2}$$

with the light rays described parametrically by

$$\frac{dx^\mu(\varsigma)}{d\varsigma} = k^\mu(\varsigma), \tag{3}$$

where ς is a suitable affine parameter.

In the particular situation at hand, equations (2) and (3) will be used to describe the radial propagation of radiation from a comoving source S to the comoving observer O located at $r = 0$. The emission and observation events can be specified by (t_s, r_s) and $(t_0, r_0 = 0)$, respectively. The redshift z corresponding to radiation emitted by S and observed by O is given by

$$1 + z = \frac{u_\lambda(x_s)k^\lambda(x_s)}{u_\lambda(x_0)k^\lambda(x_0)}, \tag{4}$$

where $u_\nu(x)$ denotes the comoving four-velocity at the point x. If we now adopt the redshift z as the path parameter for the null rays, we find that the coordinates along a radial ray must satisfy

$$\frac{dt(z)}{dz} = -(1+z)^{-1} b \left(\frac{\partial b}{\partial t} - \frac{\partial a}{\partial r}\right)^{-1}, \tag{5}$$

and

$$\frac{dr(z)}{dz} = -\frac{a}{b}\,\frac{dt(z)}{dz}. \tag{6}$$

Thus the observer, having detected some radiation at time t_0 to have a redshift equal to z, will reconstruct the emission event from the integral equations

$$t_s = t^*(t_0, z) = t_0 - \int_0^z \frac{d\bar{z}}{1+\bar{z}} b[t^*(t_0, \bar{z}), r^*(t_0, \bar{z})]/D(t_0, \bar{z}), \tag{7}$$

and

$$r_s = r^*(t_0, z) = \int_0^z \frac{d\bar{z}}{1+\bar{z}} a[t^*(t_0,\bar{z}), r^*(t_0,\bar{z})]/D(t_0,\bar{z}), \tag{8}$$

where

$$D(t_0, z) = \frac{\partial}{\partial t^*} b(t^*, r^*) - \frac{\partial}{\partial r^*} a(t^*, r^*). \tag{9}$$

The next step is to determine the luminosity distance of the source as a function of t_0 and z. As usual, the (bolometric) luminosity distance is defined in terms of the power output of the source (absolute luminosity, L) and the corresponding measured flux (apparent luminosity, ℓ) by

$$4\pi \, {d_L}^2 = L/\ell. \tag{10}$$

Using conservation of flux (Etherington 1933; Kristian and Sachs 1966), it follows that in the present case

$$d_L = (1+z)^2 \, R(t_s, r_s) \,. \tag{11}$$

Stated in terms of t_0 and z, this is

$$d_L(t_0, z) = (1+z)^2 \, R[t^*(t_0, z), r^*(t_0, z)] \,. \tag{12}$$

We have thus arrived at the desired relation between the luminosity distance and the redshift, for a given (observer) time t_0, from which the customary magnitude-redshift relation may be derived.

A related measure of distance, d_A, referred to as the angular-diameter distance, is defined in terms of the angular diameter of the source. If a source of angular size θ has a proper diameter d, then d_A is defined to be d/θ. Since for small θ, d is given by $R\theta$, we see that

$$d_A(t_0, z) = R(t^*, r^*) \,. \tag{13}$$

Alternatively, assuming a known proper diameter, we can write

$$\theta(t_0, z) = d/R(t^*, r^*) \,. \tag{14}$$

A simple calculation shows that for sources of a given proper diameter d, the angular diameter (at a given t_0) reaches a minimum at redshift $\hat{z}$ (corresponding to coordinates $\hat{t}^*$ and $\hat{r}^*$) where

$$0 = \left[\frac{1}{a}\frac{\partial R}{\partial t} - \frac{1}{b}\frac{\partial R}{\partial r}\right]_{t=\hat{t}^*, r=\hat{r}^*}. \tag{15}$$

This is the cosmological lens effect considered by Klauder *et al.* (1958) in the context of certain (closed) FRW models. Observational searches for such effects have encountered certain difficulties, however (Sandage 1961).

Equation (15) for the minimum redshift surface may be rewritten, using equation (42) of §IIIb, as the condition

$$2m(\hat{t}^*, \hat{r}^*) = R(\hat{t}^*, \hat{r}^*), \tag{16}$$

where $m(t^*, r^*)$ represents the total mass inside a sphere bounded by sources of redshift z as observed at time t_0. The hypersurface represented by equation (16) may be called the *cosmological apparent horizon.* This should be compared with the notion of apparent horizon that arises in the study of gravitational collapse (cf. equation [92] and the paragraph preceding it in Mashhoon and Partovi 1979, hereafter referred to as MP1).

Another potentially useful quantity related to the redshift is its evolution, defined to be the temporal change in the observed redshift of a given source. While there is considerable uncertainty regarding the feasibility of such measurements, the fact that they can provide information free of evolutionary effects makes them attractive enough so as to merit serious attention (Sandage 1962; McVittie 1962; Ebert and Trümper 1975; Davis and May 1978; Rüdiger 1980; Lake 1981).

Referring to equations (7) and (8), we can fix r^* and regard z as a function of t_0. The resulting function $z(t_0)$ then describes the redshift corresponding to a given source being observed over a period of time. The quantity of interest is $dz(t_0)/dt_0$, which is obtained from equations (7) and (8) to be

$$\frac{dz(t_0)}{dt_0} = -\frac{\partial r^*(t_0, z)}{\partial t_0} \bigg/ \frac{\partial r^*(t_0, z)}{\partial z}. \tag{17}$$

This equation is therefore a differential characterization of redshift evolution for a spatially isotropic space-time.

b) Number Counts

Here we consider those observations which seek to determine the number of sources in a specified range of redshifts and magnitudes. Let n be the (proper) volume density of a given class of objects at the space-time point (t,r). Alternatively, the source coordinates may be specified by (t_0, z). Then the number of such objects within the solid angle $d\omega$ with observed redshifts in the interval $(z, z+dz)$ is given by

$$-\left[u_\mu \frac{dx^\mu}{dz} R^2 n\right]_s d\omega\, dz\,, \tag{18}$$

where, as suggested by the notation, the quantity in the brackets is to be evaluated at the source. Substituting for u_μ and dx^μ/dz, the above quantity reduces to

$$(bR^2 n)_s\, d\omega\, dr(z) = \left(\sqrt{\Gamma}\, n\right)_s d\vartheta d\varphi\, dr(z)\,, \tag{19}$$

where Γ is the determinant of the spatial part of the metric.

For definiteness, we shall continue the discussion for optical sources, excluding radio measurements (which, however, can be similarly treated). Accordingly, we take $n(t_0, z|L)dL$ to be the (proper) volume density of sources with absolute luminosities in the range $(L, L+dL)$. Since the apparent luminosity (= observed flux) from such sources will be equal to $L/[4\pi d_L{}^2(t_0, z)]$, the number of sources in the redshift interval $(z, z+dz)$ having an apparent luminosity greater than ℓ will be equal to

$$\int_0^\infty dL\, \Theta\left[L - 4\pi\, \ell\, d_L{}^2(t_0, z)\right] I(t_0, z|L)\, dz\,, \tag{20}$$

where Θ represents the step function and

$$I(t_0, z|L) = 4\pi\, (bR^2)_s \frac{dr(z)}{dz}\, n(t_0, z|L)\,. \tag{21}$$

Finally, the number of sources with a redshift less than z and an apparent luminosity greater than ℓ is given by

$$N(< z, > \ell) = \int_0^\infty dL \int_0^\infty d\bar{z}\, \Theta\Big[L - 4\pi\, \ell\, {d_L}^2(t_0, \bar{z})\Big] \Theta(z - \bar{z})\, I(t_0, \bar{z}|L)\,. \quad (22)$$

We shall consider two particular relations implied by equation (22) (Weinberg 1972, pp. 455-457). The first is for N($> \ell$), the number of all sources with apparent luminosity greater than ℓ:

$$N(> \ell) = \int_0^\infty dL \int_0^{z_M} d\bar{z}\, I(t_0, \bar{z}|L)\,, \quad (23)$$

where z_M is determined from

$$L = 4\pi\, \ell\, {d_L}^2(t_0, z_M)\,. \quad (24)$$

The second relation is for $N(< z)$, the number of all sources with redshifts less than z:

$$N(< z) = \int_0^\infty dL \int_0^z d\bar{z}\, I(t_0, \bar{z}|L)\,. \quad (25)$$

Let us now consider $I(t_0, z|L)$, and make the assumption (generally not valid, but sufficient for our purposes in §IIIc) that over the range of redshifts occurring in equations (23) and (25), there is neither creation nor annihilation of sources of a given luminosity L. Clearly, this assumption disregards evolutionary effects on source luminosities, a fact which we shall make explicit by denoting the corresponding densities by n^{NE}. With this assumption the flux of these sources will be conserved, and we have the continuity condition

$$\frac{\partial}{\partial x^\mu}\Big(\sqrt{-g}\, n^{NE}\, u^\mu\Big) = \frac{\partial}{\partial t}\Big(\sqrt{\Gamma}\, n^{NE}\Big) = 0\,. \quad (26)$$

Therefore, we can integrate equation (26) and write

$$b(t, r)\, R^2(t, r)\, n^{NE}(t, r|L) = \frac{1}{4\pi} \frac{df(r|L)}{dr}\,, \quad (27)$$

and also

$$f(r|L) = 4\pi \int_0^r d\bar{r}\, b(t,\bar{r})\, R^2(t,\bar{r})\, n^{NE}(t,\bar{r}|L)\,, \tag{28}$$

where we have written the function of integration in a form convenient for later application.

Using equation (27), we can write

$$I = \frac{df}{dr}\frac{dr(z)}{dz}\,, \tag{29}$$

or more explicitly,

$$I(t_0, z|L) = \frac{\partial}{\partial z} f[r^*(t_0, z)|L]\,. \tag{30}$$

Finally, using this result, we have from equations (23) and (25),

$$N^{NE}(< z) = \int_0^\infty dL\, f[r^*(t_0, z)|L]\,, \tag{31}$$

and

$$N^{NE}(> \ell) = \int_0^\infty dL\, f[r^*(t_0, z_M)|L]\,. \tag{32}$$

III. EXPANSIONS IN POWERS OF THE REDSHIFT

In §II a number of quantities related to observations were derived within the space-time represented by the metric (1). Our task here will be the development of an expansion of the properties of this metric (mainly in connection with the null geodesics and the redshift) about the observer, as well as a similar development based on models I and II (introduced in §I) for the detailed properties of the universe. It may be useful to point out here that even though we shall ultimately express observables of interest in terms of coordinate-independent quantities (mainly the redshift), extensive use will be made of coordinate expansions meanwhile.

a) Expansion of the Distance-Redshift Relation

Our starting point here is the development of an expansion in z, up to $O(z^2)$, of the functions t^* and r^* defined by equations (7)-(9). To simplify the notation, we shall henceforth use a dot, respectively a prime, to designate differentiation with respect to t, respectively r, and also use the subscript zero to stipulate that the quantity involved is to be evaluated at $t = t_0$ and $r = z = 0$.

To start, we write from equations (7) and (8),

$$\left.\frac{\partial t^*}{\partial z}\right|_0 = -\left.\frac{b}{D}\right|_0 , \quad \left.\frac{\partial r^*}{\partial z}\right|_0 = \left.\frac{a}{D}\right|_0 , \tag{33}$$

and proceed to evaluate the second derivatives in the same manner. After some algebra, we arrive at

$$\left[\frac{\partial^2 t^*}{\partial z^2} \Big/ \frac{\partial t^*}{\partial z}\right]_0 = -1 - \left\{D^{-2}[\dot{b}D - b(\ddot{b} - \dot{a}')] - ab^{-1}D^{-2}[b'D - b(\dot{b}' - a'')]\right\}_0 , \tag{34}$$

and

$$\left[\frac{\partial^2 r^*}{\partial z^2} \Big/ \frac{\partial r^*}{\partial z}\right]_0 = -1 + \left\{-ba^{-1}D^{-2}[\dot{a}D - a(\ddot{b} - \dot{a}')] + D^{-2}[a'D - a(\dot{b}' - a'')]\right\}_0 . \tag{35}$$

With the expansions of t^* and r^* at hand, we proceed to do the same for equation (12). Let us write that expansion as

$$d_L(t_0, z) = d_L{}^{(0)}(t_0) + d_L{}^{(1)}(t_0)z + \frac{1}{2}\, d_L{}^{(2)}(t_0)z^2 + O(z^3) . \tag{36}$$

Then we have from equation (12),

$$d_L{}^{(0)}(t_0) = R_0 , \tag{37}$$

$$d_L{}^{(1)}(t_0) = \left[2R + \dot{R}\frac{\partial t^*}{\partial z} + R'\frac{\partial r^*}{\partial z}\right]_0 , \tag{38}$$

and

$$d_L^{(2)}(t_0) = \left[2R + 4R^{\bullet}\frac{\partial t^*}{\partial z} + 4R'\frac{\partial r^*}{\partial z}\right.$$

$$\left. + R^{\bullet\bullet}\left(\frac{\partial t^*}{\partial z}\right)^2 + 2R^{\bullet\prime}\frac{\partial t^*}{\partial z}\frac{\partial r^*}{\partial z} + R''\left(\frac{\partial r^*}{\partial z}\right)^2 + R^{\bullet}\frac{\partial^2 t^*}{\partial z^2} + R'\frac{\partial^2 r^*}{\partial z^2}\right]_0 . \tag{39}$$

We thus have in equations (33)-(39) an expansion to $O(z^2)$ of the distance-redshift relation for a spatially isotropic metric. In parallel with customary notation, we define

$$d_L^{(1)}(t) = H^{-1}(t)\,, \qquad H_0 = H(t_0)\,, \tag{40}$$

and

$$d_L^{(2)}(t) = H^{-1}(t)[1 - Q(t)]\,; \quad Q_0 = Q(t_0)\,. \tag{41}$$

Anticipating later results, we note that while R_0 vanishes and H(t) turns out to be the same function as in FRW models, the function $Q(t)$ will differ from the customary deceleration parameter, the difference being a consequence of inhomogeneity.

b) Models I and II

As stated in §I, the models to be developed are based on the assumptions (a) spatial isotropy, already incorporated in the metric form $\mathcal{F}$ given in equation (1), and (b) a physically reasonable equation of state obeyed by the cosmic fluid, $p = p(\rho)$ for case I and $p = 0$ for case II, supplemented by assumptions of regularity and smoothness in our neighborhood. As will be seen below, these general assumptions are rather effective in restricting the form of the resulting metrics.

In developing the models, it is very helpful to bear in mind a general result (Mashhoon and Partovi 1980, hereafter referred to as MP2) ensuring that if the assumption of shear-free motion is added to those in (a) and (b) above, then the only physically acceptable solution is the FRW universe.[4] Hence we are assured that the metrics $\mathcal{F}^{I,II}$

4. This is Theorem 1 of MP2; see also the discussion in Section 6 therein. It should be pointed out in this connection that the solutions first discussed by Wyman (1946) and also arrived at in equations (A14) and (A15) of MP2, are excluded on account of their unphysical equation of state.

to be developed below will differ from $\mathcal{J}^{FRW}$ only by virtue of a non-vanishing shear in the former.

We start with the metric $\mathcal{J}$ given in equation (1). For the corresponding field equations, we shall be using equations (8), (14), (15), (19), (20), and (21) of MP1 (for electrically neutral matter). These may be written as follows: Let $m(t,r)$ be a non-negative function defined by

$$\frac{2m}{R} = 1 + \left(\frac{R^{\bullet}}{a}\right)^2 - \left(\frac{R'}{b}\right)^2, \tag{42}$$

and interpreted as the total mass at time t interior to radius r. The comoving coordinate condition may be written as

$$R^{\bullet\prime} = \frac{a'}{a} R^{\bullet} + \frac{b^{\bullet}}{b} R', \tag{43}$$

and the conservation law for energy as

$$-(\rho + p)^{-1} \rho^{\bullet} = \frac{\partial}{\partial t} \ell n(bR^2). \tag{44}$$

The remaining gravitational field equations are equivalent to the following "energy balance" relations:

$$m^{\bullet} = -4\pi p R^2 R^{\bullet}, \tag{45}$$

and

$$m' = 4\pi \rho R^2 R'. \tag{46}$$

Finally, the Euler equation follows as the integrability condition for the latter pair;

$$-(\rho + p)^{-1} p' = \frac{\partial}{\partial r} \ell n\, a\,. \tag{47}$$

To proceed, we use the smoothness assumption to develop the metric coefficients in the radial coordinate;

$$a(t,r) = a(t,0) + a_1(t) r + \frac{1}{2}\, a_2(t) r^2 + O(r^3)\,, \tag{48}$$

$$b(t,r) = b(t,0) + b_1(t) r + \frac{1}{2}\, b_2(t) r^2 + O(r^3)\,, \tag{49}$$

and

$$R(t,r) = R(t,0) + R_1(t) r + \frac{1}{2}\, R_2(t) r^2 + \frac{1}{6}\, R_3(t) r^3 + O(r^4)\,. \tag{50}$$

Clearly, $R(t,0)$ must vanish since the surface area of a sphere of radius r must approach zero as $r \rightarrow 0$. Also, a suitable choice of the temporal coordinate will render $a(t,0) = 1$, a condition which we shall henceforth enforce.

In parallel with the expansions given for the metric coefficients, we may write

$$\rho(t,r) = \rho(t,0) + \rho_1(t) r + \frac{1}{2}\, \rho_2(t) r^2 + O(r^3)\,, \tag{51}$$

and

$$p(t,r) = p(t,0) + p_1(t) r + \frac{1}{2}\, p_2(t) r^2 + O(r^3)\,. \tag{52}$$

Consequently, equations (45) and (46) together with the above expansions imply that

$$m(t,r) = r^3 [\bar{m}(t) + m_1(t) r + O(r^2)]\,. \tag{53}$$

At this point equation (42) may be used in conjunction with the above expansions to give

$$R_1(t) = b(t,0)\,, \tag{54}$$

and

$$R_2(t) = b_1(t)\,. \tag{55}$$

If these latter results are then used with equation (43), there follows

$$a_1(t)\,\overset{\bullet}{R}_1(t) = 0\,. \tag{56}$$

Of the two possibilities implied by this last constraint, $\overset{\bullet}{R}_1(t) = 0$ is unacceptable since it will lead via equations (45), (46), and (43) to a static universe, which we exclude. Therefore the choice $a_1(t) = 0$ is implied, whereupon we use the notation $R_1(t) = S(t)$, and identify the Hubble coefficient on the basis of equations (40), (38), and (33) as

$$H(t) = \frac{d}{dt}\,\ell n\, S(t)\,, \tag{57}$$

which is the well-known result of the standard model.

At this juncture the two cases I and II will be pursued separately, starting with I, which is characterized by the equation of state $p = p(\rho)$. This functional dependence, together with equations (51) and (52), implies that

$$\left.\frac{d\rho}{dp}\right|_{r=0} = \frac{\overset{\bullet}{\rho}(t,0)}{\overset{\bullet}{p}(t,0)} = \frac{\rho_1(t)}{p_1(t)}\,. \tag{58}$$

On the other hand, we have from equation (47) and the vanishing of $a_1(t)$ the result that $p_1(t) = 0$. But then equation (58) implies that $\rho_1(t) = 0$. The simultaneous vanishing of p_1 and ρ_1 in turn implies that in equation (44)

$$\left.\frac{\partial^2}{\partial r \partial t}\,\ell n\,(bR^2)\right|_{r=0} = 0\,, \tag{59}$$

which in turn implies that

$$\frac{d}{dt}\,\frac{R_2(t)}{R_1(t)} = 0\,. \tag{60}$$

Therefore $b(t,0) = R_1(t) = S(t)$ and $b_1(t) = R_2(t)$, and together with equation (60), these relations allow a redefinition of the radial coordinate, $r \to r - \frac{1}{2}(R_2/R_1)r^2$, so as to render the metric form as follows:

$$\mathcal{J}^I = -\left[1 + \frac{1}{2}\,\alpha(t)r^2 + O(r^3)\right]^2 dt^2$$
$$+ S^2(t)\left\{\left[1 + \frac{1}{2}\,\beta(t)r^2 + O(r^3)\right]^2 dr^2 + r^2\left[1 + \frac{1}{2}\,\gamma(t)r^2 + O(r^3)\right]^2 d\Omega^2\right\}, \tag{61}$$

subject to the condition on the spatial curvature parameter $K(t) = \beta - 3\gamma$,

$$K^{\bullet}(t) = -2H(t)\alpha(t)\,, \tag{62}$$

which expresses the constraint following from the comoving character of the coordinate system. Needless to say, the functions α, β, and γ appearing in $\mathcal{J}^I$ are related to a_2, b_2, and R_3, respectively.

How does $\mathcal{J}^I$ differ from the corresponding homogeneous metric $\mathcal{J}^{FRW}$? The answer can be given on the basis of the general result mentioned before. Since the shear tensor for $\mathcal{J}^I$ is proportional to $[\beta^{\bullet}(t) - \gamma^{\bullet}(t)]r^2 + O(r^3)$ (cf. equation [29] of MP1), it follows that for $\beta^{\bullet}(t) - \gamma^{\bullet}(t) = 0$, $\mathcal{J}^I$ describes shear-free motion to $O(r^2)$. However, since under the transformation $r \to r - \xi r^3$, constant ξ, we have $\beta \to \beta + 6\xi$, $\gamma \to \gamma + 2\xi$, and $\alpha \to \alpha$, the vanishing of $\beta^{\bullet}(t) - \gamma^{\bullet}(t)$ may be taken to imply that of $\beta(t) - \gamma(t)$. Therefore the above-mentioned result implies that for $\beta(t) = \gamma(t)$, the only physically acceptable solution is the homogeneous FRW metric to $O(r^2)$, implying in particular that $\alpha(t) = \beta^{\bullet}(t) = \gamma^{\bullet}(t) = 0$. This conclusion can also be reached in the following direct manner. It can be shown that the metric $\mathcal{J}^I$ given by equation (61) with the conditions $\beta^{\bullet}(t) = \gamma^{\bullet}(t), \alpha(t) \neq 0$ imposed thereupon leads to a solution which is, to $O(r^2)$, the same as those obtained in equations (A14) and (A15) of MP2 and excluded therein because of their unphysical behavior.[4] In other words, for $\beta^{\bullet}(t) = \gamma^{\bullet}(t)$, the vanishing of $\alpha(t)$ is the only possibility with a physically reasonable equation of state.

We conclude our investigation of case I by recording some useful relations. Using equations (42), (44), and (47), one can derive the following:

$$\rho(t,0) = \left[1 + \frac{K(t)}{S^{\bullet 2}(t)}\right]\rho^c(t)\,, \tag{63}$$

$$\rho^{\bullet}(t,0) = -3H(t)[\rho(t,0) + p(t,0)]\,, \tag{64}$$

and

$$\rho(t,0) + 3p(t,0) = 2\left[q(t) + \frac{\alpha(t)}{S^{\bullet 2}(t)}\right]\rho^c(t)\,, \tag{65}$$

where the critical density ρ^c and the deceleration parameter q are defined in the usual manner;

$$\rho^c = \frac{3}{8\pi}H^2\,; \quad q(t) = -S(t)\,S^{\bullet\bullet}(t)/S^{\bullet 2}(t)\,. \tag{66}$$

The inhomogeneities of ρ and p to order $O(r^2)$ are given by

$$\rho_2(t) = -\alpha(t)[\rho(t,0) + p(t,0)]\,[d\rho/dp]_{r=0}\,, \tag{67}$$

and

$$p_2(t) = -\alpha(t)[\rho(t,0) + p(t,0)]\,. \tag{68}$$

Equation (63) implies that the density at $r = 0$ in units of the critical density is greater (less) than unity if the curvature of space is positive (negative), in analogy with *FRW* models. Moreover, equation (65) shows that the rate of charge of the curvature parameter $K(t)$ cannot exceed $-2S^{\bullet}S^{\bullet\bullet}$. It is of interest to note in this connection that a global, shear-free, cosmological model with a time-dependent spatial curvature has recently been studied by Krasiński (1982). In this model, the energy-momentum tensor is that of a perfect fluid which, however, does not satisfy an equation of state.

We now turn to case II.[5] Here we can set $a = 1$ by virtue of the identical vanishing

5. This is the well-known case of a spherically symmetric distribution of dust, the model first discussed by Lemaître (1933) and later by Tolman (1934) and many others. An exact, parametric solution in terms of three arbitrary functions of the radial coordinate is well known.

of pressure (cf. equation [47]), and on the basis of equations (54) and (55) write the metric form as

$$\mathcal{J}^{II} = -dt^2 + S^2(t)\Big\{[1 + \delta(t)r + O(r^2)]^2 dr^2 + r^2[1 + \frac{1}{2}\,\delta(t)r + O(r^2)]^2 d\Omega^2\Big\}\,. \tag{69}$$

Clearly, the radial coordinate may be redefined, $r \rightarrow r - \frac{1}{2}\,\delta(t_0)r^2$, so as to render the present value of $\delta(t)$ equal to zero;

$$\delta_0 = \delta(t_0) = 0\,. \tag{70}$$

Furthermore, since $a' = 0$, equation (43) implies that

$$R'(t,r) = G(r)\,b(t,r)\,, \tag{71}$$

where G is a function of integration. Substituting this in equation (42), and taking note of the fact that as a consequence of equation (45), $\bar{m}$ and m_1 in equation (53) reduce to constants here, we find that

$$G^2(r) = 1 - kr^2 - hr^3 + O(r^4)\,, \tag{72}$$

where

$$k = \frac{2\bar{m}}{S(t)} - S^{\bullet 2}(t)\,, \tag{73}$$

and

$$h = -\frac{\bar{m}}{S(t)}[\delta(t) - 3A] - S^{\bullet}(t)\frac{d}{dt}[S(t)\delta(t)]\,. \tag{74}$$

Here k, h, and A are constants, with $A = 2m_1/3\bar{m}$. The constant $\bar{m}$ is in turn related to ρ as in

$$\bar{m} = \frac{4\pi}{3} S^3(t)\, \rho(t,0)\,, \tag{75}$$

and the inhomogeneity in ρ is given by

$$\rho_1(t) = -2[\delta(t) - A]\rho(t,0)\,. \tag{76}$$

To compare $\mathcal{J}^{II}$ to $\mathcal{J}^{FRW}$, we note that shear is here proportional to $\overset{\bullet}{\delta}(t)r$. Hence the vanishing of $\overset{\bullet}{\delta}(t)$ (and therefore also of $\delta[t]$ since $\delta[t_0] = 0$) will, to order O(r), reduce $\mathcal{J}^{II}$ to $\mathcal{J}^{FRW}$. To determine δ, we first note that $S(t)$ satisfies equation (73), and that $\delta(t)$ is determined by

$$\overset{\bullet}{S}(t)\, \overset{\bullet}{y}(t) + \frac{\bar{m}}{S^2(t)}\, y(t) = e\,, \tag{77}$$

where

$$y(t) = [\delta(t) - A]\, S(t)\,, \tag{78}$$

and

$$e = k\,A - h\,. \tag{79}$$

Equations (73) and (77) may be solved jointly to yield

$$\delta(t) - A = \frac{2e}{k}\Big[1 - \frac{3}{2}\,(t - t_1)H(t)\Big]\,, \tag{80}$$

where t_1 is the constant of integration. We choose t_1 to correspond to the time of the primeval singularity so as to make the density contrast ρ_1/ρ vanish at early times.[6] In

6. While this is a customary and reasonable assumption (Peebles 1980), it does nevertheless represent a certain theoretical prejudice. It will not influence the qualitative aspects of our results, however.

other words, we take

$$t - t_1 = \int_0^S dS' \Big[\frac{2\bar{m}}{S'} - k\Big]^{-\frac{1}{2}} = \frac{T(t)}{H(t)}, \tag{81}$$

where $T(t)$ is the age of the universe at time t in units of the Hubble time $H^{-1}(t)$. Of course T may be expressed in terms of H and q (see, e.g., Weinberg 1972, p. 482). A useful relation which follows from equation (80) is

$$\delta^{\bullet}(t)/H(t) = \frac{-3e}{k}\{1 - [1 + q(t)]T(t)\}. \tag{82}$$

c) Observational Relations for Models I and II

Having developed the structure of the metric and the associated quantities for models I and II, we now turn to a restatement of various observational quantities in the form of expansions in z.

In §IIIa, t^*, r^*, and d_L were expanded to $O(z^2)$. The corresponding coefficients can now be evaluated for the metrics $\mathcal{F}^I$ and $\mathcal{F}^{II}$. With the definitions

$$C^I(t) = -\alpha(t)/S^{\bullet 2}(t), \tag{83}$$

and

$$C^{II} = S(t)\,\delta^{\bullet}(t)/S^{\bullet 2}(t), \tag{84}$$

and on the basis of equations (33)-(41) as well as (61) and (69), we find for the quantity $Q(t)$, defined by equations (41), the expression

$$Q(t) = q(t) + C(t), \tag{85}$$

and the corresponding distance-redshift relation

$$d_L(t, z) = \frac{z}{H(t)} + \frac{1}{2}\,\frac{1}{H(t)}\,[1 - Q(t)]\,z^2 + O(z^3), \tag{86}$$

where it is understood here and in the following that these equations are valid for both models provided that quantities appropriate to the model in question are inserted therein.

The corresponding expansions for t^* and r^* read

$$t^*(t,z) = t - \frac{1}{H(t)}\left\{z - \frac{1}{2}\left[2 + Q(t)\right] z^2 + O(z^3)\right\}, \tag{87}$$

and

$$r^*(t,z) = \frac{1}{S(t)H(t)}\left\{z - \frac{1}{2}\left[1 + Q(t) + \Delta(t)\right] z^2 + O(z^3)\right\}, \tag{88}$$

where

$$\Delta^I(t) = 0\,,\ \Delta^{II}(t) = \delta(t)/S^{\bullet}(t)\,. \tag{89}$$

Note that since $\Delta^{II}(t_0) = 0$ as a result of $\delta(t_0) = 0$, the source coordinates (t^*, r^*) corresponding to measurements performed at the present epoch are coordinate-free quantities to $O(z^2)$ for both metrics $\mathcal{J}^{I,II}$ (except for trivial rescalings). In this sense, the coordinate systems adopted for these metrics are canonical at time t_0.

The appropriate relations corresponding to source densities at given redshifts are

$$\rho(t^*, r^*) = \rho(t_0, 0) + \left[\frac{\rho_1(t)}{SH} - \frac{1}{H}\frac{\partial}{\partial t}\rho(t,0)\right]_{t=t_0} z + O(z^2)\,, \tag{90}$$

and similarly for p.[7] Substituting the appropriate quantities from equations (63)-(68)

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7. The $O(z^2)$ corrections are clearly beyond observational reach and will therefore not be considered further.

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and (76)-(82), we arrive at

$$\Omega^I(t_0, z)/\Omega_0^I = 1 + \left[2\left(1 + \frac{q_0}{\Omega_0^I}\right) - \frac{2}{\Omega_0^I} C_0^I\right] z + O(z^2)\,, \tag{91}$$

and

$$\Omega^{II}(t_0, z)/\Omega_0^{II} = 1 + \left[3 + \frac{4}{3}\,\frac{1 - \frac{3}{2}T_0}{1 - (1 + q_0)T_0}\,C_0^{II}\right] z + O(z^2)\,, \tag{92}$$

where, as usual

$$\Omega(t_0, z) = \rho(t^*, r^*)/\rho^c(t)\,. \tag{93}$$

Recall that T_0 is the age of the universe in units of H_0^{-1} for a matter dominated FRW model; it is expected to be of the order of unity. Recall also that for model II, $\Omega_0 = 2q_0$.

Proceeding to redshift evolution,[8] we use equation (17) to obtain

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8. We do not consider $\hat{z}$ (the redshift corresponding to minimum angular diameter) here. This is because its magnitude is typically of order unity, which exceeds the range of validity of our expansions.

$$\frac{d\,\ell n\, z(t)}{dt} = -H(t)\,q(t)\left\{1 + \frac{1}{2}\left[1 + Q(t) + \Delta(t) - \frac{Q^\bullet(t) + \Delta^\bullet(t)}{H(t)\,q(t)}\right] z + O(z^2)\right\}. \tag{94}$$

For model II, this expression can be reduced to

$$\left[\frac{d\,\ell n\, z(t)}{dt}\right]_{t=t_0} = -\,q_0 H_0\left[1 + \left(1 - \frac{1}{2}\,q_0 + \frac{1}{2}\,\frac{1 - (2 - q_0)T_0}{1 - (1 + q_0)T_0}\,C_0^{II}\right) z + O(z^2)\right]. \tag{95}$$

Finally, we turn to number counts and use the results developed in §IIb keeping in mind that in deriving them we disregarded evolutionary effects. Let us parametrize the radial inhomogeneities in n^{NE} by

$$n^{NE}(t, r|L) = n^{NE}(t, 0|L)[1 + \kappa(t|L)r + O(r^2)]\,. \tag{96}$$

Note that although we have indicated a dependence of κ upon t, the structure of the metric form I (II) in fact excludes (restricts) such a dependence.

Using equations (28), (31), and (96), we arrive at

$$N^{NE}(<z) = \frac{4\pi}{3}\left(\frac{z}{H_0}\right)^3 \int_0^\infty dL n^{NE}(t_0, 0|L)\left\{1 - \frac{3}{2}\left[1 + Q_0 - \frac{\kappa(t_0|L)}{2S_0H_0}\right]z + O(z^2)\right\}. \tag{97}$$

To find the analogous expression for $N(>\ell)$, we assume that ℓ is taken sufficiently large so that the two-term expansion (cf. equation [24]),

$$z_M = \left(\frac{LH_0^2}{4\pi\ell}\right)^{1/2}\left[1 - \frac{1}{2}(1 - Q_0)\left(\frac{LH_0^2}{4\pi\ell}\right)^{1/2} + O\left(\frac{LH_0^2}{4\pi\ell}\right)\right], \tag{98}$$

is adequate. Then we may write from equation (32),

$$N^{NE}(>\ell) = \frac{4\pi}{3}\left(\frac{1}{4\pi\ell}\right)^{3/2}\int_0^\infty dL n^{NE}(t_0, 0|L) L^{3/2}$$
$$\times\left\{1 - 3\left[1 - \frac{\kappa(t_0|L)}{4S_0H_0}\right]\left(\frac{LH_0^2}{4\pi\ell}\right)^{1/2} + O\left(\frac{LH_0^2}{4\pi\ell}\right)\right\}. \tag{99}$$

IV. DISCUSSION AND CONCLUSIONS

The principal inquiry of this discussion is: to what extent are the models investigated in this work observationally distinguishable from the homogeneous FRW models? Let us consider the distance-redshift relation given by equation (86), and observce that it differs from its FRW counterpart only in the appearance of $Q_0(= q_0 + C_0)$ instead of q_0.[9] Clearly, the magnitude-redshift data cannot distinguish between these cases, even

9. Recall that we have defined q_0 by equations (66) and Q_0 by the distance-redshift relation.

if it were possible to extract therefrom a reliable determination of q_0 free from evolutionary effects. An independent determination of q_0 is clearly needed. Furthermore, to the extent that the magnitude of C_0 characterizing the effects of the inhomogeneities could easily be comparable to (or even larger than) q_0, a determination of the latter from the Hubble plot could be entirely erroneous, if inhomogeneities do in fact exist.

The inhomogeneities in cosmic matter distribution are given for the two models by equations (91) and (92), both of which reduce to their FRW counterparts upon setting C_0 equal to zero, as they should. Thus a determination of C_0 from these relations requires a fairly accurate determination of the slope of the density versus redshift relation. Needless to say, existing observational data do not provide the requisite accuracy. More conventional tests of homogeneity, of course, rely on number counts, which we consider next.

The relevant results for number counts are given in equations (97) and (99). Here again, observations are essentially limited to the leading term, evolutionary effects being a major source of uncertainty in the study of source counts. Contributions from inhomogeneities simply confound an already confused situation.

Indeed without considering the details briefly discussed above, it is clear that in as much as next-to-leading effects in redshift are poorly determined by observations, models I and II are indistinguishable from a homogeneous universe. By the same token, any method having prospects of improved accuracy in the determination of q_0, for example, could be a discriminating test of homogeneity in the present context. Thus measurements of redshift evolution, should they become feasible, might offer a hope of setting a limit on inhomogeneities (see equation [95]).

Clearly the answer to the question addressed at the beginning of this discussion is a negative one; existing data are not sufficient to rule out large-scale radial inhomogeneities. We have reached this conclusion (which is consistent with general expectations) on the basis of a detailed analysis of models which have been constructed on the basis of observationally well-supported assumptions. At the same time we have arrived at a number of relations (the observationally more relevant ones being those summarized in §IIIc) that could in principle provide ways of setting limits on large-scale radial inhomogeneities.

A next step in the present program is the inclusion of angular inhomogeneities. One would expect to be able to obtain some bounds on these on the basis of the existing data on isotropy. One can also anticipate new complications arising from an interplay between the two kinds of inhomogeneity. These complications and the lack of requisite accuracy in the existing data notwithstanding, we consider such an analysis a worthwhile effort toward the observational verification of the cosmological principle.

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