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Geometric Representation of Fundamental Particles' Masses

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A geometric representation of the lowest lying inertial masses of the known particles (N=299) was introduced by employing a Riemann Sphere facilitating the interpretation of the N masses in terms of a single, hypothetical particle we call the Masson (M). Geometrically, its mass is the radius of the Riemann Sphere. Dynamically, its derived mass is near the mass of the only stable hadron regardless of whether it is determined from all N particles or only the hadrons, the mesons or the baryons separately. Ignoring all other properties of these particles, it is shown that the eigenvalues, the polar representation θ_ν of the masses on the Sphere, satisfy the symmetry $\theta_\nu + \theta_{N+1-\nu} = \pi$ within less than 1% relative error. These pair correlations form 6 distinct clusters. A function can be established whose zeros are a good approximation to the full set of masses $\{\theta_\nu\}$.

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Spanning from zero to more than 100GeV, we introduce a geometric representation of the masses of particles allowing us to posit a generating particle - the Masson (pronounced as one does the Muon). Associated with it, there is a generating function whose zeros are the normalized masses of the N known particles[1, 2]. These masses can then be projected onto a 2D Riemann Sphere[3] of radius equal to the mass of the Masson that is determined by imposing the equivalent of a minimum action criterion; throughout this study whenever we refer to mass, the intention is to the *inertial mass*. We do *not* consider antiparticles because there has never been a fermion discovered that did not lead to the discovery of its corresponding antiparticle as first implied by Dirac.[4]

The only particle we fully understand is the photon with zero mass that must move at the speed of light because there is no rest frame to measure the mass explicitly based on $m/\sqrt{1-\beta^2}$. Thus, while we know how to determine the extreme, in general, we do not know the fundamentals underlying the other values beyond the work of Brodsky and collaborators[5] based on QCD.

It is important to clarify here that we are working only with direct mechanical observables i.e. the mass. We do know, according to Sommerfeld[6], that it is not associated with the charge alone. He pointed out that given a macroscopic charge of finite radius and mass, the energy associated with the two is different. His approach was simple: denoting by $E_{\rm EM}^{(\rm rest)}$ the electrostatic energy of the charged particle when at rest and subtracting this energy from the electric and magnetic energy when the particle is in motion $E_{\rm EM}^{(\rm motion)}$, it was shown that the difference does not equal the kinetic energy of the particle.

Here we introduce a geometric (polar θ_ν) representation of the N masses on a Riemann Sphere. This allows us to interpret them in terms of a single particle, the Masson, that may be in one of the N states and whose mass M we take as the radius of the Sphere as shown in Figure 1.

Ignoring the other properties of these particles, it is shown that these values satisfy the symmetry $\theta_\nu + \theta_{N+1-\nu} = \pi$ within less than 1% relative error. These eigenvalues form at least 6 clusters suggestive of a "Periodic" Chart of the Fundamental Particles. This mapping is not unique but was chosen for its simplicity whereas others might reveal further relationships.

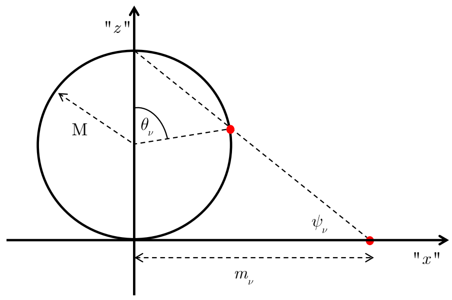

FIG. 1: The mass of a particle is marked on the axis (red-dot). Projection of the mass of the particle on the Riemann Sphere, whose radius represents the mass of the Masson M, is uniquely determined by the polar angle θ_ν.

Specifically, the masses are organized in ascending order along the horizontal axis "x". A circle of radius M has its center at x=0, z=M and the intersection of the straight-line, connecting the top of the circle with z=0, x=m_ν defines a unique angle θ_ν on the sphere given by

$$\theta_\nu = 2\arctan\left(\frac{2\rm M}{m_\nu}\right). \quad (1)$$

This transformation represents the projection of any one of the masses on the circle whose radius we attribute to the mass of the Masson; it preserves the order we organized the masses. Next we establish M based on the experimental data and a minimal action criterion. Keeping in mind that the masses were organized organized in ascending order, and we define the interval-spread of any two adjacent angles as

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$$\mathcal{E}\left(\mathrm{M}\right)=\frac{1}{\pi}\sqrt{\frac{1}{N+1}\sum_{\nu=0}^{N}\left(\theta_{\nu+1}-\theta_{\nu}\right)^{2}}\ . \tag{2}$$

M is the value that *minimizes* this functional; $\theta_{\nu=0}=0$ and $\theta_{\nu=N+1}=\pi$ represent the upper and lower limits of the masses in this representation. For the trivial case of a *single* particle its mass is represented by an angle θ and there are two intervals: $\theta-0$ and $\pi-\theta$ so the intervals spread is proportional to $\theta^2+(\pi-\theta)^2$ with a minimum at $\theta=\pi/2$ implying that the radius of the sphere is half the mass of the particle i.e. M$=m/2$ or, equivalently, the particle's mass is twice the mass of the Masson.

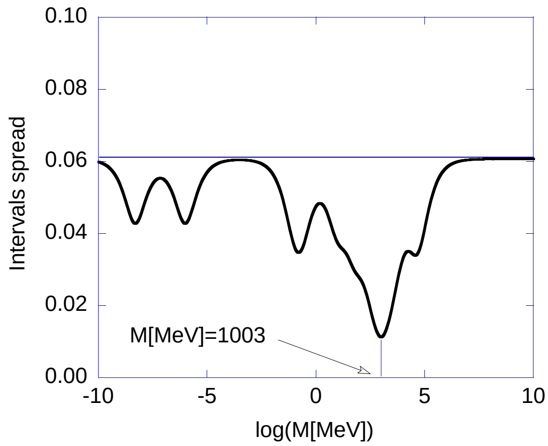

FIG. 2: Spread of intervals for the N=299 particles as a function of M. The dominant minimum is calculated numerically and occurs at M[MeV] = 1003 near the lowest lying baryon mass.

When considering the entire ensemble of particles (ν=1,2,...N) their spread intervals in Figure 2 clearly shows resonance-like behavior. The absolute minimum, occurring at 1003 MeV, we take to be the mass of the Masson. The 2D Riemann Sphere is illustrated in Figure 3 for this value. Two facts are evident – first, as anticipated, most of the particles are located in the $\theta\sim\pi/2$ region and, second, close to zero and π there are voids although these are not symmetrically disposed nor correlated in any obvious way.

Repeating the same procedure only for hadrons or baryons or mesons separately, leads only to a relatively small deviation from the value of the Masson's mass. To begin, consider only the hadrons ($N=281$). If we were to establish the Masson based on the hadrons alone, its mass would be only slightly reduced to $\mathrm{M}^{(\mathrm{H})}$=962.2 MeV. Moreover, if we attribute a separate Masson to baryons ($N=121$) and to mesons ($N=160$) the corresponding masses would be $\mathrm{M}^{(\mathrm{B})}$= 1094 MeV and $\mathrm{M}^{(\mathrm{M})}$=964 MeV. All of these and especially $\mathrm{M}^{(\mathrm{H})}$ and $\mathrm{M}^{(\mathrm{M})}$ are close to both M as well as to the only stable hadron mass, the nucleon N(940). Also, there are more mesons than baryons even though their confined quarks(2) are fewer than for the baryons(3). Their corresponding "intervals spread", similar to Figure 2 for all particles, gave a *single* comparable minimum.

Another perspective on the polar representation of the masses can be obtained by ordering the $\{\theta_\nu\}$ in ascending order and plotting them as a function of the normalized index ν (quantum number) as the red squares in Figure 4. For comparison, the N zeros of the Legendre polynomial of order $N=299$ are organized in ascending order and given by the black diamonds $[P_N\left(\cos\zeta_\nu\right)=0;\ \nu=1,2,....N]$. While the latter is virtually linear, the former has a more complex structure with distinct "band-gaps" in the range $\nu<0.2N$ and $\nu>0.9N$.

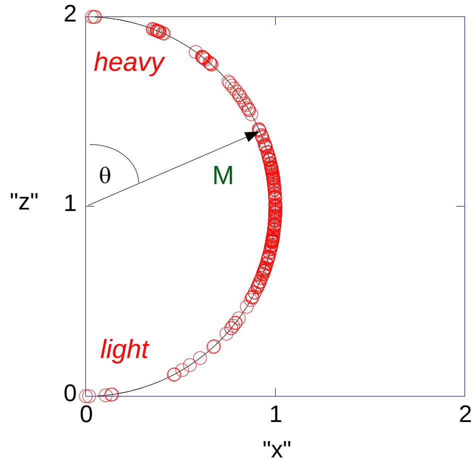

FIG. 3: Projection of the masses of all 299 particles where the mass of the Masson is determined from the requirement that the spread of the intervals in Figure 2 is minimal. Light particles ($\theta\sim\pi$) are the gamma, gluon and neutrinos. The heavy ones ($\theta\sim0$) the gauge-particles, Higgs and top quark.

Four observations may be made: (i) Legendre's function zeroes are *linear* on the index (ν/N). (ii) if the absolute value of the argument of the Legendre polynomial is larger than unity the behavior is hyperbolic and the function has no zeros in this range. This is consistent with the existence of band-gaps. (iii) Having in mind that the argument of the Legendre polynomial ($\cos\theta$) varies between -1 and 1, we consider another function which is defined in this range (tanh) and we calculate the zeros of $P_N\left[\tanh\left(3.46\left(\pi/2-\theta_\nu\right)\right)\right]=0$ which are represented by the green squares in Figure 4. (iv) In the range $0.2<\nu/N<0.9$ the dependence of θ_ν on the index is *linear* with an accuracy of 0.07% being defined as $100\times\left\langle[1-\theta_\nu^{(\mathrm{Linear})}/\theta_\nu]^2\right\rangle_\nu$. This resemblance to the Legendre polynomial zeros hint at the relevance of θ_ν to the dynamics of the Masson.

Furthermore, these results indicate that the θ_ν might be regarded as the eigenvalues of a characteristic polynomial of the Legendre type. In this regard our approach was inspired by the work of Liboff and Wong[7] in con-

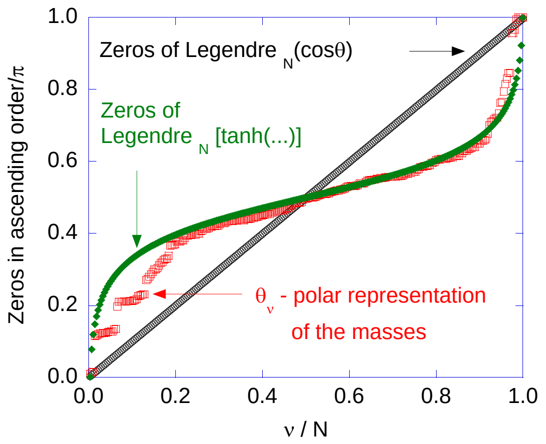

FIG. 4: Red squares represent the masses (θ_ν) in ascending order and the black diamonds the zeros of the modified Legendre function of order $N = 281$ whose argument is the hyperbolic tangent funtion. The green squares are discussed in the text below. The index ν is normalized by N.

nection with their study of the prime numbers and the zeta function.

One of the main results of our approach relies on a property of the Legendre polynomials that the sum of *two zeros* of complementary order ($\nu + \nu' = N + 1$) equals π, or explicitly $\zeta_\nu + \zeta_{N+1-\nu} = \pi$. We have examined to what extent this rule applies to the polar representation of the masses (θ_ν) and found that $\theta_\nu + \theta_{N+1-\nu} = \pi\chi$ with $\chi = 0.958$ within 0.13% relative error defined as

$$\text{Error}[\%] = 100\frac{1}{2N}\sum_{\nu=1}^{N}\left[\frac{\theta_\nu + \theta_{N+1-\nu} - \pi\chi}{\theta_\nu + \theta_{N+1-\nu}}\right]^2 . \quad (3)$$

The factor of 2 in Eq.(3) corrects the fact that each pair of masses is counted twice.

According to the present spectrum of masses [1], this relation implies that the mass of the Higgs and that of the Axion (if observed) would be related $\theta_{\text{Axion}} + \theta_{\text{Higgs}} \simeq \pi$, that the mass of the electrons neutrino is related to that of the Z-gauge boson $\theta_{\nu_e} + \theta_Z \simeq \pi$ and that the $W^{\pm}$ gauge bosons are related similarly to the other neutrinos. However, it should be emphasized that the present estimate of the error is dominated by the light particles with $\theta \sim \pi$ and that it is larger if the deviation is compared to the smallest angle between the two. In fact, due to uncertainty associated with the measurement of many of those masses and especially the neutrinos, comparing to the calculated deviation of χ from unity, one can hypothesize that $\chi \equiv 1$ or explicitly

$$\theta_\nu + \theta_{N+1-\nu} = \pi . \quad (4)$$

For further insight into this result, we plot in Figure 5 the normalized symmetry-pairs $(\theta_\nu + \theta_{N+1-\nu})/\pi$ as a function of the normalized masses (θ_ν/π). Several important aspects are reflected in this plot: (i) the pairs linked by Eq.(4) form (at least) six clusters. (ii) The error or deviation from unity is dominated by light particles ($\theta \sim \pi$). When both particles have similar mass, the deviation is negligible – see the right cluster. (iii) Further splitting is expected when including additional quantum numbers that produce a Riemann hypersphere. (iv) Subject to the condition $\chi \equiv 1$, the error defined above for hadrons is 0.47%, for baryons 0.07% and for mesons it is 0.63%.

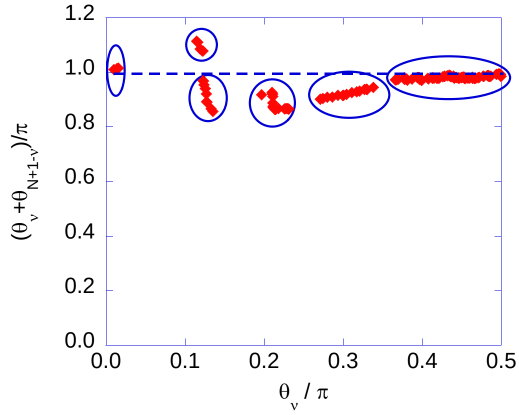

FIG. 5: The normalized symmetry-pairs, $(\theta_\nu + \theta_{N+1-\nu})/\pi$, as a function of the normalized geometric representation of the masses (θ_ν/π). These pairs form at least six clusters analogous to a "Periodic Table" for the Fundamental particles.

Before proceeding it is important to assess whether such small errors are not the result of pure coincidence. For this let us postulate that the Masson has a fixed inertial mass of M_0=1003 MeV and between the two extremes the various particles (299) are randomly distributed. We represent the inertial masses in terms of a random variable $m_\nu\,[\text{MeV}] = 10^{p_\nu}$ wherein p_ν is uniformly distributed $-8 \le p_\nu \le 5 + \log(1.26) = 5.104$. As in the case of the real particles, we employ the transformation in Eq. 1. It is tacitly assumed that the mass of the Masson is not dependent on the specific distribution. Once the θ_ν are established, the error is calculated based on Eq. 3 for $\chi = 1$. Figure 6 illustrates the errors, $7.5 \le \text{Error}[\%] \le 10.5$, associated with this polar representation. For comparison, in the case of using all of the actual particles, its value is 0.225% indicating that the roughly two orders of magnitude (8%) difference is not a result of coincidence.

So far we have focused attention on the kinematics of the Masson. Based on the observations mentioned above we make a few assessments regarding the dynamics of the Masson. Our starting point is the first observation as to the *linear* dependence of the zeros of Legendre polynomials of order N namely, $\xi_\nu \sim \pi\nu/N$ wherein $P_N(\cos\xi_\nu) \equiv 0$. The latter, $\psi_N^{(\text{Leg})}(\theta) = P_N(\cos\theta)$, is a

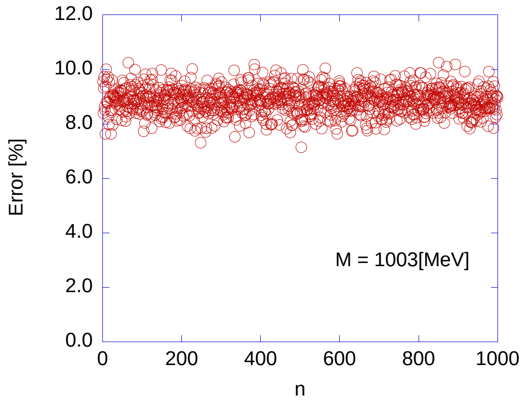

FIG. 6: The errors associated with the polar representation of a random distribution of masses for 1000 different seeds based on using Eq. [3]. For comparison, in the case of using all of the actual particles, its value is 0.225%.

solution of the Legendre equation

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+N(N+1)\right]\psi_N^{(\mathrm{Leg})}(\theta)=0. \quad (5)$$

By analogy, the polar representation of the masses θ_ν is one of N zeros of $\psi_N^{(\mathrm{Masson})}(\theta)=\Pi_{\nu=1}^{N}(1-\theta/\theta_\nu)$ which is a solution of the "Legendre-like" differential operator

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+V_N(\theta)\right]\psi_N^{(\mathrm{Masson})}(\theta)=0 \quad (6)$$

with $V_N(\theta)$ representing the normalized potential that is responsible for the difference between ξ_ν and θ_ν. Since we know the latter, we may take advantage of the orthogonality of the Legendre polynomials, $\int_0^\pi d\theta\sin\theta P_n(\cos\theta)P_{n'}(\cos\theta) = \delta_{n,n'}/(2n+1)$, to write $\psi_N^{(\mathrm{Masson})}(\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta)$ where a_n can be considered to be known i.e. $a_n = (2n+1)\int_0^\pi d\theta\sin\theta P_n(\cos\theta)\psi_N^{(\mathrm{Masson})}(\theta)$. Contrary to $\psi_N^{(\mathrm{Masson})}(\theta)$, the potential is not known. Yet it is still convenient to represent it in terms of Legendre polynomials $V_N(\theta)=\sum_{n=0}^{\infty}\alpha_n P_n(\cos\theta)$ so α_n is therefore

$$\vec{\alpha}=\mathbf{D^{-1}}\vec{b} \quad (7)$$

with $b_n = a_n n(n+1)/(2n+1)$ and

$$D_{n,m}=\sum_{n'=0}^{\infty}a_{n'}\int_0^\pi d\theta\sin\theta P_n(\cos\theta)P_{n'}(\cos\theta)P_m(\cos\theta) \quad (8)$$

In conclusion, a geometric representation of the $N = 281$ **inertial masses** of the reasonably established, lowest lying hadrons was introduced by employing a Riemann Sphere. It allowed us to interpret the N masses in terms of a single entity, the Masson, that might be in one of the N eigenstates. Geometrically, the mass of the Masson was the radius of the Riemann Sphere while its numerical value was closest to the mass of the nucleon, the only stable hadron, regardless of whether it was computed from all of the particles (299), the hadrons (281), or just the mesons (160) or baryons (121) separately.

Ignoring the other properties of these particles, it was shown that the eigenvalues, the polar representation θ_ν, satisfied the symmetry $\theta_\nu + \theta_{N+1-\nu} = \pi$ within less than 1% relative error. A function was established whose zeros were, to good approximation, the polar representation of the masses θ_ν.

Although we did not include antiparticles in our analysis based on quantum field theory they are important for cosmology where the lack of any apparent antimatter in the universe is an ongoing scientific concern[8]. We did not consider gravity for lack of information notwithstanding a new result on the mass of the graviton $m_g < 7.7\times10^{-17}\mathrm{MeV/c^2}$ i.e. essentially zero[9]. Because the only stable hadron is the relatively heavy nucleon presumably because it contains no antiquarks one sees the weakness of using only classical concepts to understand the microscopic particle world. Finally, we note that different mappings than ours could very well reveal additional relations comparable to Eq. 4.

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[1] Rev. Part. Prop., Phys. Lett. B, Vol. **667/1** (1-1340) (2008); Rev. Part. Phys., J. Phys. G, Vol. **37**, 7A(1-1422) (2010) and Chin. Phys. C, **40**, 100001 (2016). The specific Table we used can be found at http://webee.technion.ac.il/people/schachter /AppendixMassesofFundamentalParticles.pdf

[2] We take the "rest mass" as simply the "mass" – a relativistic invariant with neither "rest" nor subscript "0" attached. The *observed* masses [1] are understood to be greater than the bare masses due to self interaction contributions. The Axion was included but not the graviton because we did not consider gravity even though neither of these has yet been observed.

[3] Our definition is from *The Encylopedia of Mathematics*.

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