# Geometric Representation of Fundamental Particles' Masses 

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#### Abstract

A geometric representation of the lowest lying masses of the quarks, leptons, hadrons and gauge bosons ( $\mathrm{N}=299$ ) was introduced by employing a Riemann Sphere facilitating the interpretation of the N masses in terms of a single, hypothetical particle we call the Masson (M) which itself might be in one of the N eigenstates. Geometrically, its mass is the radius of the Riemann Sphere. Dynamically, its derived mass is near the mass of the only stable hadron regardless of whether it is determined from all N particles or only the hadrons, the mesons or the baryons separately. Ignoring all other properties of these particles, it is shown that the eigenvalues, the polar representation $\theta_{\nu}$ of the masses on the Sphere, satisfy the symmetry $\theta_{\nu}+\theta_{N+1-\nu}=\pi$ within less that $1 \%$ relative error. These pair correlations include the pairs $\theta_{\gamma}+\theta_{\text {top }} \simeq \pi$ and $\theta_{\text {gluon }}+\theta_{\text {Higgs }} \simeq \pi$ as well as pairing the three weak gauge bosons with the three neutrinos. The eigenvalues form 6 distinct clusters and a function can be established whose zeros are a good approximation to the full set of masses $\left\{\theta_{\nu}\right\}$.


Spanning from zero to more than 100 GeV , we introduce a geometric representation allowing us to posit a generating particle - the Masson (pronounced as one does the Muon). Associated with it, there is a generating function whose zeros are the masses of the N known particles ${ }^{1,2}$. These masses are then projected onto a 2D Riemann Sphere ${ }^{3}$ of radius equal to the mass of the Masson that is determined by imposing the equivalent of a minimum action criterion; throughout this study whenever we refer to mass the intention is to the inertial mass.

The only particle we understand is the photon with zero mass that must move at the speed of light because there is no rest frame to measure the mass explicitly based on $m / \sqrt{1-\beta^{2}}$. Thus, while we know how to determine the extreme, in general, we do not know the fundamentals underlying the other values. However, we do know, according to Sommerfeld ${ }^{4}$, that it is not associated with the charge alone. He pointed out that given a macroscopic charge of finite radius and mass, the energy associated with the two is different. His approach was simple: denoting by $E_{\mathrm{EM}}^{(\mathrm{rest})}$ the electrostatic energy of the charged particle when at rest and subtracting this energy from the electric and magnetic energy when the particle is in motion $E_{\mathrm{EM}}^{(\text {motion })}$, it was shown that the difference does not equal the kinetic energy of the particle.

Here we introduce a geometric (polar $\theta_{\nu}$ ) representation of the N masses on a Riemann Sphere. This allows us to interpret them in terms of a single particle, the Masson, that may be in one of the N eigenstates and whose mass M we take as the radius of the Sphere as shown in Figure 1. Ignoring the other properties of these particles, it is shown that the eigenvalues satisfy the symmetry $\theta_{\nu}+\theta_{N+1-\nu}=\pi$ within less than $1 \%$ relative error. These eigenvalues form at least 6 clusters suggestive of a "Periodic" Chart of the Particles. This mapping is not unique but was chosen for its simplicity whereas others might be expected to reveal additional relationships.

Because the range of the N masses spans over many orders of magnitude, we introduced a compact representa-
tion based on the "Riemann Sphere" shown in Figure 1. The masses are organized in ascending order along the horizontal axis " $x$ ". A circle of radius M has its center at $\mathrm{x}=0, \mathrm{z}=\mathrm{M}$ and the intersection of the straight-line, connecting the top of the circle with $\mathrm{z}=0, \mathrm{x}=m_{\nu}$ defines a unique angle $\theta_{\nu}$ given by

$$
\begin{equation*}
\theta_{\nu}=2 \arctan \left(\frac{2 \mathrm{M}}{m_{\nu}}\right) \tag{1}
\end{equation*}
$$

This transformation represents the projection of any one of the masses on the circle whose radius we attribute to the mass of the Masson. The latter is established next based on the experimental data and a minimal action criterion. To establish M , the angle $\theta_{\nu}$ is organized in ascending order and we define the interval-spread of any two adjacent angles as

$$
\begin{equation*}
\mathcal{E}(\mathrm{M})=\frac{1}{\pi} \sqrt{\frac{1}{N+1} \sum_{\nu=0}^{N}\left(\theta_{\nu+1}-\theta_{\nu}\right)^{2}} \tag{2}
\end{equation*}
$$

M is the value that minimizes this functional; $\theta_{\nu=0}=0$ and $\theta_{\nu=N+1}=\pi$ represent the upper and lower limits of the masses in this polar representation. For the case


FIG. 1. The mass of a particle is marked on the axis (reddot). Projection of the mass of the particle on the Riemann Sphere, whose radius represents the mass of the Masson M, is uniquely determined by the polar angle $\theta_{\nu}$.
of a single particle represented by an angle $\theta$, there are two intervals: $\theta-0$ and $\pi-\theta$ so the intervals spread is proportional to $\theta^{2}+(\pi-\theta)^{2}$ with a minimum at $\theta=\pi / 2$ implying that the radius of the sphere is half the mass of the particle i.e. $\mathrm{M}=m / 2$ or, equivalently, the particle's mass is twice the mass of the Masson: $\mathrm{m}=2 \mathrm{M}$.


FIG. 2. Spread of intervals for the N particles as a function of M . The dominant minimum is calculated numerically and occurs at $\mathrm{M}[\mathrm{MeV}]=1003$ near the lowest lying baryon mass.

Now we can introduce the particles. The spread of their intervals in Figure 2 clearly shows resonance-like behavior. The absolute minimum, occurring at 1003 MeV , we take to be the mass of the Masson. The Riemann Sphere is illustrated in Figure 3 for this value. Two facts are evident - first, as anticipated, most of the particles are located in the $\theta \sim \pi / 2$ region and, second, close to zero and $\pi$ there are voids although these are not symmetrically disposed nor correlated in any obvious way.

With the polar representation established, we can study some features based on it. To begin, consider only the hadrons $(N=281)$. If we were to establish the Masson based on the hadrons alone, its mass would be only slightly reduced to $\mathrm{M}^{(\mathrm{H})}=962.2 \mathrm{MeV}$. Moreover, if we attribute a separate Masson to baryons $(N=121)$ and to mesons ( $N=160$ ) the corresponding masses would be $\mathrm{M}^{(\mathrm{B})}=1094 \mathrm{MeV}$ and $\mathrm{M}^{(\mathrm{M})}=964 \mathrm{MeV}$. All of these and esp. $\mathrm{M}^{(\mathrm{H})}$ and $\mathrm{M}^{(\mathrm{M})}$ are close to both M as well as to the only stable hadron mass, the nucleon $\mathrm{N}(940)$. Also, there are more mesons than baryons even though their confined quarks(2) are fewer than for the baryons(3). Their corresponding "intervals spread", similar to Figure 2 for all particles, gave a single comparable minimum.

Another perspective on the polar representation can be obtained by ordering the $\left\{\theta_{\nu}\right\}$ in ascending order and plotting them as a function of the normalized index $\nu$ (quantum number) as the red squares in Figure 4. For comparison, the $N$ zeros of the Legendre polynomial of order $N=281$ are organized in ascending order and given by the black diamonds $\left[P_{N}\left(\cos \zeta_{\nu}\right)=0 ; \nu=1,2, \ldots . N\right]$. While the latter is virtually linear, the former has a more complex structure with distinct "band-gaps" in the range $\nu<0.2 N$ and $\nu>0.9 N$.


FIG. 3. Projection of the masses of all 299 particles where the mass of the Masson is determined from the requirement that the spread of the intervals in Figure 2 is minimal. Light particles $(\theta \sim \pi)$ are the gamma, gluon and neutrinos. The heavy ones $(\theta \sim 0)$ the gauge-particles, Higgs and top quark.

Two observations may be made: (1) if the absolute value of the argument of the Legendre polynomial is larger than unity the behavior is hyperbolic and the function has no zeros in this range. This is consistent with the existence of band-gaps. (2) Having in mind that the argument of the Legendre polynomial $(\cos \theta)$ varies between -1 and 1 , we consider another function which is defined in this range (tanh) and we calculate the zeros of $P_{N}\left[\tanh \left(3.46\left(\pi / 2-\theta_{\nu}^{(M)}\right)\right)\right]=0$ which are represented by the green squares in Figure 4. In the range $0.2<\nu / N<0.9$ these zeros approximate the polar representation of the masses $(\theta)$ with an accuracy of $0.07 \%$ being defined as $100 \times\left\langle\left[1-\theta_{\nu}^{(M)} / \theta_{\nu}\right]^{2}\right\rangle_{\nu}$.


FIG. 4. Red squares represent the masses $\left(\theta_{\nu}\right)$ in ascending order and the black diamonds the zeros of the Legendre function of order $N=281$. The green squares are discussed in the text below. The index $\nu$ is normalized by $N$.

What these results indicate is that the $\theta_{\nu}$ might be regarded as the eigenvalues of a characteristic polynomial of the Legendre type. Our approach was inspired by the work of Liboff and Wong ${ }^{5}$ in connection with their study of the prime numbers and the zeta function.

Having such a representation in mind, an additional feature is revealed by examining the sum of the eigenvalues. Let us assume that we know the Hamiltonian whose eigen-values are $\theta_{\nu}^{s}$ wherein $s$ is a free parameter to be determined. In many cases of interest, the measurable is given by a term of the form Trace $(H)$ which in turn is proportional to $g(s) \equiv \sum_{\nu} \theta_{\nu}^{s}$. In reality we do not know this Hamiltonian but a rough idea as to its character can be obtained by assuming that $g(s)$ has a minimum. A simple calculation reveals that such a minimum exists for $s \simeq-79 / 150=-0.5267$.

One of the main results of our approach relies on a property of the Legendre polynomials that the sum of two zeros of complementary order $\left(\nu+\nu^{\prime}=N+1\right)$ equals $\pi$, or explicitly $\zeta_{\nu}+\zeta_{N+1-\nu}=\pi$. We have examined to what extent this rule applies to the polar representation of the masses $\left(\theta_{\nu}\right)$ and found that $\theta_{\nu}+\theta_{N+1-\nu}=\pi \chi$ with $\chi=0.958$ within $0.13 \%$ relative error defined as

$$
\begin{equation*}
\operatorname{Error}[\%]=100 \frac{1}{2 N} \sum_{\nu=1}^{N}\left[\frac{\theta_{\nu}+\theta_{N+1-\nu}-\pi \chi}{\theta_{\nu}+\theta_{N+1-\nu}}\right]^{2} \tag{3}
\end{equation*}
$$

The factor of 2 in Eq.(3) corrects the fact that each pair of masses is counted twice. According to the present spectrum of masses [4], this relation implies that the mass of the Higgs and that of the Axion (if observed) would be related $\theta_{\text {Axion }}+\theta_{\text {Higgs }} \simeq \pi$ and that the mass of the electrons neutrino is related to that of the Z -gauge boson $\theta_{\nu_{\mathrm{e}}}+\theta_{Z} \simeq \pi[5]$. However, it should be emphasized that the present estimate of the error is dominated by the light particles with $\theta \sim \pi$ and that it is larger if the deviation is compared to the smallest angle between the two. In fact, due to uncertainty associated with the measurement of many of those masses and especially the neutrinos, comparing to the calculated deviation of $\chi$ from unity, one can hypothesize that $\chi \equiv 1$ or explicitly

$$
\begin{equation*}
\theta_{\nu}+\theta_{N+1-\nu}=\pi \tag{4}
\end{equation*}
$$

For further insight into this result, we plot in Figure 5 the normalized symmetry-pairs $\left(\theta_{\nu}+\theta_{N+1-\nu}\right) / \pi$ as a function of the normalized masses $\left(\theta_{\nu} / \pi\right)$. Several important aspects are reflected in this plot: (i) the pairs linked by Eq.(4) form (at least) six clusters. (ii) The error or deviation from unity is dominated by light particles $(\theta \sim \pi)$. When both particles have similar mass, the deviation is negligible - see the right cluster. (iii) Further splitting is expected when including additional quantum numbers that produce a Riemann hypersphere. (iv) Subject to the condition $\chi \equiv 1$, the error defined above for hadrons is $0.47 \%$, for baryons $0.07 \%$ and for mesons it is $0.63 \%$.


FIG. 5. The normalized symmetry-pairs, $\left(\theta_{\nu}+\theta_{N+1-\nu}\right) / \pi$, as a function of the normalized geometric representation of the masses $\left(\theta_{\nu} / \pi\right)$. These pairs form at least six clusters analogous to a "Periodic Table" for the fundamental particles.

Hadrons are the absolute majority (281) of the 299 particles we have considered and, because they are composite, being made of different numbers of quarks, gluons and antiquarks, they are distinguishable from those we have somewhat arbitrarily called elementary such as the leptons or quarks. All of the particles are distinguishable as bosons or fermions according to their individual spins. The elementary fermions include two classes - the quarks and leptons whereas the elementary bosons comprise the photon and gluons with $\mathrm{M}=0, \mathrm{~J}=1$. However, one must then ask whether there isn't an $\mathrm{M}=0, \mathrm{~J}=0$ particle such as the Axion related to the Higgs $(\mathrm{M}=125 \mathrm{GeV})$ but this goes beyond our scope. We did not consider antiparticles because there has never been a fermion discovered that did not lead to the discovery of its corresponding antiparticle as first implied by Dirac. ${ }^{6}$

In conclusion, a geometric representation of the $N=$ 281 inertial masses of the reasonably established, lowest lying hadrons was introduced by employing a Riemann Sphere. It allowed us to interpret the $N$ masses in terms of a single entity, the Masson, that might be in one of the $N$ eigenstates. Geometrically, the mass of the Masson was the radius of the Riemann Sphere while its numerical value was closest to the mass of the nucleon, the only stable hadron, regardless of whether it was computed from all of the particles (299), the hadrons (281), or just the mesons (160) or baryons (121) separately.

Ignoring the other properties of these particles, it was shown that the eigenvalues, the polar representation $\theta_{\nu}$, satisfied a symmetry $\theta_{\nu}+\theta_{N+1-\nu}=\pi$ within less than $1 \%$ relative error. A function was established whose zeros were, to good approximation, the polar representation of the masses $\theta_{\nu}$. A rough assessment of the Hamiltonians's character was made by determining that its trace $\sum_{\nu} \theta_{\nu}^{s}$ has a minimum for $s=-0.523$.


FIG. 6. Polar representation of the masses vs the quantum numbers for fermions(blue crosses) and bosons(red circles).

Although we did not include antiparticles in our analysis based on quantum field theory they are important for cosmology where the lack of any apparent antimatter in the universe is an ongoing scientific concern ${ }^{7}$. We did not consider gravity for lack of information notwithstanding a new result on the mass of the graviton $m_{g}<$ $7.7 \times 10^{-17} \mathrm{MeV} / \mathrm{c}^{2}$ i.e. essentially zero ${ }^{8}$. Because the only stable hadron is the relatively heavy nucleon presumably because it contains no antiquarks one sees the weakness of using only classical concepts to understand
the microscopic particle world. We note that different mappings than ours may very well reveal additional relations comparable to Eq. 4.

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${ }^{1}$ Rev. Part. Prop., Phys. Lett. B, Vol. 667/1 (11340) (2008); Rev. Part. Phys., J. Phys. G, Vol. 37, 7A(1-1422) (2010) and Chin. Phys. C, 40, 100001 (2016). The specific Table we used can be found at http://webee.technion.ac.il/people/schachter
/AppendixMassesofFundamentalParticles.pdf
${ }^{2}$ We take the "rest mass" as simply the " mass" - a relativistic invariant with neither "rest" nor subscript "0" attached. The observed masses [1] are understood to be greater than the bare masses due to self interaction contributions. The Axion was included but not the graviton because we did not consider gravity but neither of these has yet been observed.
${ }^{3}$ Our definition is from The Encylopedia of Mathematics.
${ }^{4}$ Sommerfeld A. , Electrodynamics in Lectures on Theoretical Physics, (Academic Press, New York, 1952), pp. 278.
${ }^{5}$ Richard L. Liboff and Michael Wong, "Quasi-Chaotic Property of Prime-Number Sequence", Int. J. Theo. Phys. 37, 3109-3117 (1998).
${ }^{6}$ P. A. M. Dirac, "The Quantum Theory of the Electron", Proc. Roy. Soc.A: Math., Phys. \& Eng. Sci.117, 610 (1928).
${ }^{7}$ Large ( 1 km across) isolated clouds of positrons may have been observed recently but with short lifetimes ( 0.2 s ) - see J.R.Dwyer et al., J. Plasma Physics 81, 475810405 (2015).
${ }^{8}$ B.P. Abbott et al., GW170104: Observation of a 50-SolarMass Binary Black Hole Coalescence at Redshift 0.2, Phys. Rev. Lett., PRL 118221101 (2017).

