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Bootstrapping a Five-Loop Amplitude from Steinmann Relations

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The analytic structure of scattering amplitudes is restricted by Steinmann relations, which enforce the vanishing of certain discontinuities of discontinuities. We show that these relations dramatically simplify the function space for the hexagon function bootstrap in planar maximally supersymmetric Yang-Mills theory. Armed with this simplification, along with the constraints of dual conformal symmetry and Regge exponentiation, we obtain the complete five-loop six-particle amplitude.

INTRODUCTION

To "bootstrap" generally refers to solving a problem via an ansatz constrained by symmetries and physical principles. This is naturally most successful in very special theories such as low-dimensional integrable models, but it has also proved powerful for conformal field theories in arbitrary dimensions. The hexagon function bootstrap [1, 2] is a perturbative version aimed at solving a scattering problem in a four-dimensional quantum field theory: the planar limit of $\mathcal{N} = 4$ super Yang-Mills (SYM). While scattering amplitudes in this theory are interesting in their own right, the methods developed to solve them have often had broader applicability, for example to computing amplitudes in QCD for scattering at the Large Hadron Collider.

The hexagon function bootstrap exploits the idea that, order by order in perturbation theory, the first nontrivial amplitude in planar $\mathcal{N} = 4$ SYM, the six-point amplitude, "lives" within a relatively small space of functions, which can be parametrized by a finite set of coefficients. This rigidity means that information from physical limits, such as when two gluons become collinear, or in a high-energy (Regge) limit, often suffices to fix the result. In turn this generates new predictions, a fact which has led to much fruitful interplay with the pentagon operator-product-expansion program [3, 4, 5, 6].

The aim of this Letter is to point out that the relevant space of hexagon functions is far smaller than previously thought. This is due to constraints stemming from the classic work of Steinmann [7], which restrict the analytic structure of scattering amplitudes in any quantum field theory. We show that, when combined with Regge exponentiation and the so-called final-entry condition [8], this restriction makes it possible to bootstrap the six-gluon amplitude to at least 5 loops *without any external input*. Analogous constraints can be exploited for n-particle scattering with $n > 6$.

HEXAGON STEINMANN FUNCTIONS

We consider the scattering amplitude for six gluons (or other partons) in the planar limit of $\mathcal{N} = 4$ SYM. A priori, such an amplitude can depend, in four spacetime dimensions, on 8 Mandelstam invariants. Dual conformal symmetry of this model restricts the nontrivial dependence to be on 3 cross-ratios [9, 10]

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad v = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad w = \frac{s_{34}s_{61}}{s_{345}s_{234}}, \tag{1}$$

where $s_{i\dots k} \equiv (p_i + \dots + p_k)^2$ are Mandelstam invariants. The same symmetry forces the four- and five-particle amplitudes to be essentially trivial, which is why we concentrate on six particles. It has been conjectured that the amplitude, which is a transcendental function of these three variables, lives in a restricted space of "hexagon" functions [1]. These are iterated integrals with singularities generated by logarithms of the nine letters [11]

$$\mathcal{S} \ = \ \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}, \tag{2}$$

where

$$y_u = \frac{1+u-v-w-\sqrt{\Delta}}{1+u-v-w+\sqrt{\Delta}}, \quad \Delta = (1-u-v-w)^2 - 4uvw,$$

and cyclic rotations act as

$$C: \quad u \to v \to w \to u, \quad y_u \to 1/y_v \to y_w \to 1/y_u, \tag{3}$$

while parity acts as $u_i \to u_i$, $y_i \to 1/y_i$. These letters arise naturally as projectively invariant combinations of momentum twistors [12], variables that make manifest the dual conformal symmetry. Multiple zeta values $\zeta_{q_1,q_2,\dots}$ with positive indices q_i also appear.

Branch cuts for massless scattering amplitudes start only at vanishing values of the Mandestam invariants, $s_{i\dots k} = 0$. Consequently, there is a canonical Riemann

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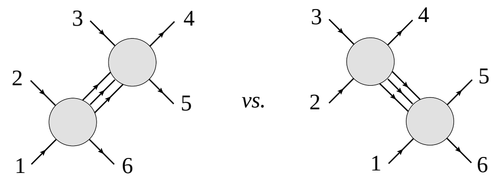

FIG. 1. Illustration of the channels s_{345} and s_{234} for $3 \to 3$ kinematics. The discontinuity in one channel should not know about the discontinuity in the other channel.

sheet on which the amplitude is analytic in the positive octant $u, v, w > 0$. This constraint is included in the definition of hexagon functions. It implies a "first-entry" condition [13]: discontinuities associated with the letters $(1-u) = 0$ or $y_u = 0$ are not visible in the canonical Riemann sheet; however, they can be exposed after analytic continuation. The physical interpretation of the restriction (2) is that, even after analytic continuation along an arbitrary complex path, the only possible branch points remain those characterized by $\mathcal{S}$.

The focus of this Letter is the Steinmann relations, which state that an amplitude A can have no double discontinuities in overlapping channels [7]. Using the correspondence between discontinuities and cut diagrams via the Cutkosky rules [14], overlapping channels correspond to cut lines that intersect. Thus for example the channels s_{345} and s_{234} overlap, which leads, schematically, to:

$$\text{Steinmann relation:} \quad \text{Disc}_{s_{345}}\left(\text{Disc}_{s_{234}} A\right) = 0, \tag{4}$$

illustrated in figure 1.

We focus on three-particle invariants s_{ijk} because these can change sign along fairly generic codimension-1 surfaces in the space of external momenta. The relation can therefore be probed with real external momenta. (In contrast, massless thresholds in two-particle invariants s_{ij} occur at phase space boundaries where other invariants may change sign; it is unclear to the authors how to extract putative constraints from these thresholds beyond the Regge limit [15].) For functions of the cross-ratios u, v, w, the discontinuity with respect to s_{234} can be computed by rotating v, w by a common phase, as follows from eq. (1). The Steinmann relation (4) thus implies that the following combination is analytic in a neighborhood of $r = \infty$:

$$0 = \text{Disc}_{r=\infty}\left[A(ru, ve^{i\pi}, re^{i\pi})) - A(ru, ve^{-i\pi}, re^{-i\pi})\right],$$

where $u, v > 0$ (and $r > 0$ before taking the discontinuity). The reason why $r = \infty$ appears is that the three-particle invariants appear in the denominators of eq. (1).

Focusing on the region where all three cross-ratios are large and combining this condition with its permutations, we obtain an equivalent but more practical statement: the amplitude must be expressible as a sum of terms with singularities in only one three-particle channel:

$$A = \sum_k \left[a_k^u \log^k\left(\frac{u}{vw}\right) + a_k^v \log^k\left(\frac{v}{wu}\right) + a_k^w \log^k\left(\frac{w}{uv}\right)\right], \tag{5}$$

with the $a_k^{u,v,w}$ analytic around $u = v = w = \infty$.

THE STEINMANN BASIS TO WEIGHT 4

A complete basis of 88 hexagon functions at transcendental weight 4 was originally constructed in ref. [16]. The Steinmann relations imply that only a subspace is physically relevant, a subspace sufficiently small that it can be described in this Letter. We begin with weight 1, where the first entry condition allows only elementary logarithms: $\log u$, $\log v$, $\log w$. To build the higher weight basis, we use the fact that all derivatives of a Steinmann function also obey the Steinmann relations.

The derivative of a weight-k hexagon function F has the form [17]

$$dF = \sum_{i=1}^{9} F^i \, d\ln \mathcal{S}_i \,, \tag{6}$$

where F^i are weight-$(k-1)$ hexagon functions and $\mathcal{S}_i \in \mathcal{S}$ in eq. (2). We thus make an ansatz (6) for the derivatives of F where the F^i are Steinmann functions. For the ansatz to represent a function, the partial derivatives must commute ("integrability condition"). Once this condition is solved, the analyticity and Steinmann properties simplify dramatically. It suffices to impose the following constraints, which serve only to fix a few coefficients of zeta-values of weight $(k-1)$ and $(k-2)$:

- F^{1-u}, F^{y_v} and F^{y_w} must vanish at $(u, v, w) = (1, 0, 0)$ [2, 17].
- The s_{234}-discontinuity of $F^u + F^{1-u} + F^w + F^{1-w}$ must vanish at $(u, v, w) = (+\infty, 0, -\infty)$.

Cyclic rotations of these conditions are implied.

Following this procedure, at weight 2 we find 7 elements: the constant ζ_2 and two cyclic orbits containing

$$K_{1,1}^u \equiv \text{Li}_2(1-1/u), \quad L_2^u \equiv \tfrac{1}{2}\left[\log^2(u) + \log^2(v/w)\right]. \tag{7}$$

The naming convention will be explained shortly. Already, the Steinmann relations' impact is noticeable: without it there would be three additional functions, $\log^2 u$, $\log^2 v$ and $\log^2 w$, which do not satisfy eq. (5).

At weight 3, the basis contains 17 elements, the 5 cyclic 3-orbits of

$$\begin{aligned}
K_3^u &\equiv \tfrac{1}{3!}\log^3(1/u) + \tfrac{1}{2}\log(1/u)\log^2(v/w), \\
K_{2,1}^u &\equiv \text{Li}_2(1/u)\log(1/u) - 2\text{Li}_3(1/u) + 2\zeta_3, \\
K_{1,2}^u &\equiv K_{1,1}^u \log(v/w), \qquad K_{1,1,1}^u \equiv -\text{Li}_3(1-1/u), \\
\zeta_2 K_1^u &\equiv \zeta_2 \log(1/u),
\end{aligned} \tag{8}$$

the constant ζ_3, and a single parity-odd element: the six-dimensional scalar hexagon integral $\tilde{\Phi}_6$ [17, 18].

At this stage we see that the functions in eqs. (7)-(8) depend nontrivially on only u, apart from simple powers of $\log(v/w)$. We can construct $3 \times 2^{k-1}$ similar elements at weight k, as follows. We start from "seeds" which trivially satisfy eq. (5):

$$\begin{aligned} K^u_k(u, \tfrac{v}{w}) &\equiv \frac{1}{2 \cdot k!}\left[\log^k\left(\frac{v}{uw}\right) - \log^k\left(\frac{uv}{w}\right)\right], \\ L^u_k(u, \tfrac{v}{w}) &\equiv \frac{1}{2 \cdot k!}\left[\log^k\left(\frac{v}{uw}\right) + \log^k\left(\frac{uv}{w}\right)\right]. \end{aligned} \tag{9}$$

We then construct nontrivial functions as a simple generalization of harmonic polylogarithms (HPLs) [19] with argument $x = 1/u$, by integrating the seeds from the base point $u = \infty$. Using this base point automatically maintains the Steinmann relations. The constraint of analyticity for $u > 0$ is enforced by recursively removing values at $u = 1$:

$$K^u_{i,\ldots}(u, \tfrac{v}{w}) \equiv \sum_j c_j L^u_j + \int_0^{1/u} \frac{dx}{1-x} \frac{\log^{i-1}(\frac{1}{ux})}{(i-1)!} K^u_{\ldots}(\tfrac{1}{x}, \tfrac{v}{w}), \tag{10}$$

where the zeta-valued coefficients c_j are chosen uniquely to make the total vanish at $u = 1$. Without the c_j, the recursive definition would be identical to that of HPLs with argument $x = 1/u$, which makes it straightforward to express the K^u as combinations of HPLs. At weights 2 and 3, this definition agrees with the examples given.

Defining K^v, K^w, L^v and L^w as cyclic images of K^u, L^u, the K functions with positive indices do generate $3 \times 2^{k-1}$ linearly independent elements. There is one exception: the three $K^{u,v,w}_k$ for even weight k are linearly dependent, so for even k we use $L^{u,v,w}_k$ instead.

At weight 4, the Steinmann basis contains the 8 3-orbits generated by:

$$L^u_4, \; K^u_{1,3}, \; K^u_{2,2}, \; K^u_{3,1}, \; K^u_{1,1,2}, \; K^u_{1,2,1}, \; K^u_{2,1,1}, \; K^u_{1,1,1,1}.$$

The iterative construction also generates 5 "non-K" functions: 3 parity-even functions — the integral $\Omega^{(2)}$ [16, 17] and its cyclic permutations — plus 2 parity-odd functions. Ten more functions come from multiplying ζ_2, ζ_3 and ζ_4 by the lower-weight Steinmann functions listed earlier. In summary, at weight 4 there are 39 physically relevant Steinmann functions, to be contrasted with 88 in the original hexagon function space.

This gap increases rapidly with higher weights, as evidenced by the first two lines of table I, which was generated by implementing the construction iteratively. The paucity of Steinmann functions is because the space is not a ring: the product of two Steinmann functions is generically *not* an allowed function.

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
0. Functions	(10,10)	(82,88)	(639,761)	(5153,6916)	(???,???)
1. Steinmann	(7,7)	(37,39)	(174,190)	(758,839)	(3105,3434)
2. Symmetry	(3,5)	(11,24)	(44,106)	(174,451)	(???,???)
3. Final-entry	(2,2)	(5,5)	(19,12)	(72,32)	(272,83)
4. Collinear	(0,0)	(0,0)	(1,1)	(3,5)	(9,15)
5. Regge	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

TABLE I. Free parameters remaining after applying each constraint, for the 6-point (MHV,NMHV) amplitude at L loops.

APPLICATION TO TWO LOOPS

Before using the Steinmann basis to help bootstrap the hexagon amplitude, we comment on the subtraction and renormalization of its infrared divergences. Indeed, as usual in the presence of divergences, it is the renormalized quantities which matter physically.

A particularly convenient renormalization scheme in the SYM model is to divide by the so-called BDS ansatz [20]. This soaks up the dual conformal anomaly, leaving a remainder which depends only on the cross-ratios u, v, w, and furthermore vanishes in soft and collinear limits [10]. However, in order to preserve the Steinmann relation (4), it is critical to divide only by quantities which are free of three-particle discontinuities. This singles out the so-called BDS-like ansatz [2, 21] $\mathcal{R}'_6$:

$$\mathcal{R}'_6 \equiv \mathcal{M}_6^{\text{bare}} / \mathcal{M}_6^{\text{BDS-like}}. \tag{11}$$

In fact, the amplitude is a function of the helicity of all 6 particles, in a way which can be neatly encoded in so-called R-invariants [12, 22]. In this Letter we thus deal with bosonic functions $\mathcal{E}$, E and $\tilde{E}$ which encode all the information and correspond to suitable components of the MHV and NMHV BDS-like remainders. Schematically, $\mathcal{R}'_6 \simeq \mathcal{E} \oplus E \oplus \tilde{E}$. The relations to the more conventional BDS MHV remainder ($\mathcal{R}_6$) and NMHV ratio function ($V, \tilde{V}$), defined for example in ref. [2] (to which we refer for further details), are:

$$e^{\mathcal{R}_6} \equiv \mathcal{E} e^{-\frac{1}{4}\Gamma_{\text{cusp}}\mathcal{E}^{(1)}}, \quad V \equiv E/\mathcal{E}, \quad \tilde{V} \equiv \tilde{E}/\mathcal{E}, \tag{12}$$

where $\frac{1}{4}\Gamma_{\text{cusp}} = g^2 - 2\zeta_2 g^4 + \ldots$ is the cusp anomalous dimension, known exactly as a function of the coupling $g^2 \equiv \frac{g^2_{\text{YM}} N_c}{16\pi^2}$ [23]. We stress that while $\mathcal{E}$, E and $\tilde{E}$ obey the Steinmann relations, $\mathcal{R}_6$, V and $\tilde{V}$ do not: the space of Steinmann functions is not a ring.

Let us describe a concrete example, the bootstrap of $\mathcal{E}$ at two loops. We begin by applying the following:

1. $\mathcal{E}$ is a hexagon Steinmann function

2. $\mathcal{E}$ is parity-even and dihedrally symmetric

3. The collinear limit to leading power is universal:

$$\lim_{v\to 0} \mathcal{E} = e^{-\frac{1}{4}\Gamma_{\rm cusp}(L_2^v + 2\zeta_2)} + \mathcal{O}(\sqrt{v}\,\ln^{L-1} v).$$

In the weight 4 Steinmann space, no linear combination vanishes in all three collinear limits. Therefore the two-loop MHV amplitude is fully determined by just the above three conditions! Loop-expanding using $\mathcal{E} = \mathcal{E}^{(0)} + g^2\mathcal{E}^{(1)} + g^4\mathcal{E}^{(2)} + \ldots$, the result at tree-level is $\mathcal{E}^{(0)} = 1$, at one loop

$$\mathcal{E}^{(1)} = K^u_{1,1} + K^v_{1,1} + K^w_{1,1}\,, \tag{13}$$

and at two loops

$$\mathcal{E}^{(2)} = (1+C+C^2)\left[\Omega^{(2)} - K^u_{1,2,1} - 4K^u_{1,1,1,1} - \zeta_2 K^u_{1,1}\right] + 8\zeta_4. \tag{14}$$

This result agrees completely with refs. [11, 16].

For MHV at higher loops, and for NMHV, we imposed an additional "final-entry" condition, obtained by considering the action of the $\bar{Q}$ generator of dual superconformal transformations [8]. The MHV final-entry condition is simply $\mathcal{E}^{1-u} = -\mathcal{E}^u$, plus the cyclic relations. Similarly, the differential of the NMHV BDS-like remainder is spanned by the 18 elements listed in eq. (3.10) of ref. [2]. These conditions almost completely determine the higher-loop amplitudes; we need information from only one more limit.

REGGE EXPONENTIATION AND BOOTSTRAP

In the multi-Regge limit of $2 \to 4$ gluon scattering, the four outgoing gluons are strongly ordered in rapidity. The cross-ratios have the limits $u \to 1$, $v, w \to 0$, but on an analytically continued Riemann sheet which ensures nontrivial Lorentzian kinematics. This limit has been thoroughly analyzed for both MHV and NMHV amplitudes [15, 24, 25, 26, 27, 28]. Amplitudes exponentiate in terms of Fourier-Mellin variables ν, m which are conjugate to the transverse plane coordinates, schematically:

$$\mathcal{E}(\nu, m, vw) \xrightarrow{\text{Regge}} \Phi(\nu, m) \times \left(-1/\sqrt{vw}\right)^{\omega(\nu,m)} \tag{15}$$

where the Regge trajectory ω vanishes at tree level and Φ is an "impact factor". Exponentiation implies that terms with $\log^2(vw)$ or higher powers of the large logarithm are predicted by the multi-Regge limit at lower loops.

Remarkably, through five loops such terms suffice to fix all remaining parameters and uniquely determine $\mathcal{E}$, E, and $\tilde{E}$! Terms with $\log(vw)$ or lower were not needed, but rather led to predictions for the next loop order, enabling a pure bootstrap with no external information. The constraints are summarized in table I.

With $\mathcal{E}$, E, and $\tilde{E}$ fixed through five loops we can evaluate them numerically on a variety of lines in cross-ratio space. Figure 2 shows the remainder function on the line

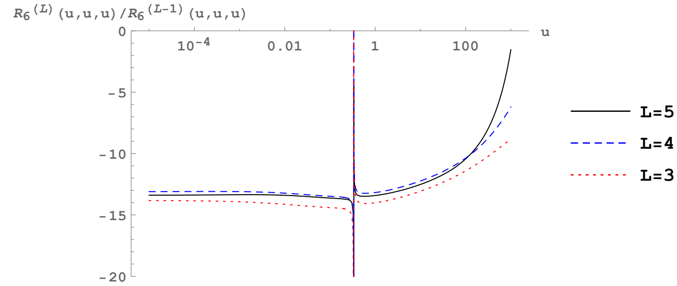

FIG. 2. The remainder function $\mathcal{R}_6$, evaluated at ratios of successive loop orders L on the line $u = v = w$. The spike is an artifact due to $\mathcal{R}_6^{(L)}(u,u,u)$ crossing zero very close to $u = 1/3$ at each loop order.

(u, u, u). We have also used "hedgehog" variables [29] to generate multiple polylog representations of these functions in one bulk region [30].

Past implementations of the hexagon function bootstrap employed a variety of other constraints, which the Steinmann relations render unnecessary, or relegate to cross checks. For NMHV, the representation in terms of R-invariants has poles at kinematically spurious points that must cancel between different permutations of E and $\tilde{E}$ [16]. Now, after imposing the collinear constraint in table I, the spurious poles cancel automatically. Similarly, for MHV and NMHV the $\bar{Q}$ equation predicts not only final entries, but next-to-final entries; however, again these constraints are satisfied automatically.

For both MHV and NMHV, the pentagon operator product expansion (POPE) [3, 4, 5, 6] served previously as a powerful bootstrap constraint [17, 28]. Now Regge exponentiation is enough to obtain a unique result. Nonetheless, we do check our results against the POPE predictions. We find complete agreement through five loops, to each order in the OPE we have computed (T^1 and T^2F^2 for MHV and T^1 for the (6134) component of NMHV).

In [28], two of the authors conjectured a relationship between the L-loop MHV amplitude and the $(L-1)$-loop NMHV amplitude. Our five-loop MHV amplitude allows us to verify this relation at one more loop order. Expressed in terms of the functions defined in eq. (12), it reads (using the coproduct notation [28])

$$g^2\,(2E - \mathcal{E}) = \mathcal{E}^{y_u,y_u} + \mathcal{E}^{y_w,y_w} - 3\mathcal{E}^{y_v,y_v} - \mathcal{E}^{v,v} - \mathcal{E}^{1-v,v} + 2(\mathcal{E}^{y_u,y_v} + \mathcal{E}^{y_w,y_v}) - \mathcal{E}^{y_u,y_w} - \mathcal{E}^{y_w,y_u}. \tag{16}$$

This relation calls out for explanation.

Remarkably, the space of Steinmann functions appears to be "not much larger" than required to contain $\mathcal{E}$, E and $\tilde{E}$, if we include all derivatives of higher loop amplitudes. Up to at least weight 6, the complete space is needed, apart from certain unexpected restrictions on zeta values. For example, the weight 2 functions found by taking 8 derivatives of $\mathcal{E}^{(5)}$, $E^{(5)}$ and $\tilde{E}^{(5)}$ span a

6 dimensional subspace of the 7 dimensional Steinmann space: $K^u_{1,1}$, $L^u_2 + 2\zeta_2$, plus cyclic; ζ_2 is not an independent element. In an ancillary file, we provide a coproduct representation of this trimmed basis, which suffices to describe $\mathcal{E}$, E and $\tilde{E}$ through five loops. We also give HPL expressions for these functions on the lines $(1, v, v)$ and $(u, 1, 1)$ [30].

CONCLUSION

Leveraging the power of the Steinmann relations, we have bootstrapped six-point scattering amplitudes in planar $\mathcal{N} = 4$ super Yang-Mills through five loops. Loop by loop, these amplitudes are dramatically simpler than one would expect. Crucially, we did not need any external input: all constraints imposed are either general or are fixed by behavior at lower loops. Yet higher loops, or even finite coupling, may well be accessible too.

Unlike other techniques used to calculate in $\mathcal{N} = 4$ SYM, the Steinmann relations apply in general quantum field theories. Their strength here suggests that these often-neglected constraints may have broader applicability, perhaps making similar bootstrap techniques viable in other theories, such as QCD.

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