SLAC-PUB-16666

CALCULATING POINT-CHARGE WAKEFIELDS FROM FINITE LENGTH BUNCH WAKE-POTENTIALS*

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Abstract

Starting from analytical properties of high frequency geometric impedance we show how one can accurately calculate short bunch wake-potentials (and even point-charge wakefields) from time domain calculations performed with a much longer bunch. In many practical instances this drastically reduces the need for computer resources, speeds up the calculations, and improves their accuracy. To illustrate this method we give examples for 2D accelerator structures of various complexities.

INTRODUCTION AND MAIN IDEA

Knowledge of wakefields, and in particular geometric wakefields, is critically important for studies of accelerator beam dynamics. While analytical solutions are known for a number of simple geometries, detailed wakefield calculations for realistic vacuum chamber components is typically done utilizing time domain EM solvers. These, by design, calculate the fields due to finite length bunches, and one is forced to use extremely fine mesh (small fraction of the bunch length) to compute wakes at small distances. This is where the wakes are usually dominated by singularities, so that a wake-potential due a bunch of rms length σ scales as $W^{\sigma}(z) \propto \sigma^{-q}$, q>0 [1]. Utilizing fine meshes has obvious implications for computer memory requirements as well as calculation speed.

To get around the fine mesh requirement we propose to take care of the wake singularities by representing the wake-potential as a sum of two parts,

$$W^{\sigma}(z) = W^{\sigma}_{\text{model}}(z) + D^{\sigma}(z), \tag{1}$$

where $W^{\sigma}_{\text{model}}(z)$ is the short-bunch asymptotic model (that includes all the singularities at $\sigma \to 0$), and function $D^{\sigma}(z)$ that remains finite for arbitrarily short bunches. Furthermore, we suggest that for bunches shorter than certain length σ_0 (usually correlated with the smallest dimension of the structure) the σ-dependence of $D^{\sigma}(z)$ is rather weak, so it can be dropped,

$$D^{\sigma}(z) \approx D(z), \quad 0 \le \sigma \le \sigma_0 . \tag{2}$$

Thus, if we pick the initial bunch length to be σ_0 (or shorter) and calculate the corresponding wake-potential $W^{\sigma_0}(z)$ with an EM solver, we can then set

$$D(z) = W^{\sigma_0}(z) - W^{\sigma_0}_{\text{model}}(z) . \tag{3}$$

For bunches, shorter than σ_0, we now have

$$W^{\sigma}(z) \approx W^{\sigma}_{\text{model}}(z) + D(z), \; 0 \le \sigma \le \sigma_0 . \tag{4}$$

In the limit of vanishing bunch length we can write for the point-charge wakefield

$$W^{\delta}(z) \approx W^{\delta}_{\text{model}}(z) + D(z). \tag{5}$$

For this method to be practical, we need to show that 1) $W^{\sigma}_{\text{model}}(z)$ is easily identifiable, tractable, and it captures <u>all</u> singularities; 2) with some reasonable choice of σ_0 Eq. (2) holds true. With the help of time domain EM code ECHO [2], we show in the rest of this paper, that both of these statements are correct for a large number of 2D accelerator structures of the collimator and cavity type. We also show that for these structures a short bunch wake can be accurately reconstructed from EM solver results for a much longer bunch.

COLLIMATOR-LIKE GEOMETRY

For a collimator chamber that transitions from radius a to smaller radius b and back, the point-charge wakefield and the wake-potential due to a Gaussian bunch of rms length σ, are given, respectively, by

$$W^{\delta}_{o}(z) = k_o \delta(z) , \tag{6}$$

$$W^{\sigma}_{o}(z) = k_o (2\pi)^{-1/2} \sigma^{-1} \exp\left(-\tfrac{z^2}{2\sigma^2}\right), \tag{7}$$

where Z_0 is the free space impedance, and

$$k_o = -Z_0 c \, Log(a/b)/\pi . \tag{8}$$

In our sign convention $W^{\sigma}(z) < 0$ corresponds to the energy loss of the particle with longitudinal coordinate z; z<0 is the head of the bunch. Eq. (7) implies that the minimum of the wake-potential diverges for short bunches as $W^{\sigma}_{o}(0) \propto \sigma^{-1}$.

Eqs. (6-8) were derived a long time ago by various authors. Recently it was shown [3] that this example is a particular case of the so-called optical regime, where the wakes of all collimator-like structures (including 3D) have the same z-dependence as is shown above. Furthermore, in this asymptotic regime one can find exact impedances and wakefields of very complicated collimators (which amounts to calculating the value of geometry-dependent coefficient k_o, for which the recipe was given in [3]). This makes us believe that what we illustrate below for axially symmetric case is applicable to 3D. Meanwhile through the rest of this section we will assume that $W^{\sigma}_{\text{model}}(z)$ is given by Eqs. (7-8).

We investigated in detail a family of collimator-like geometries; their radial dependence along the collimator is shown in Fig. 2 (inset). In the short bunch limit all of them have the same asymptotic wake given by Eqs.(7-8).

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Basic Energy Sciences, under Contract No. DE-AC02-76SF00515.

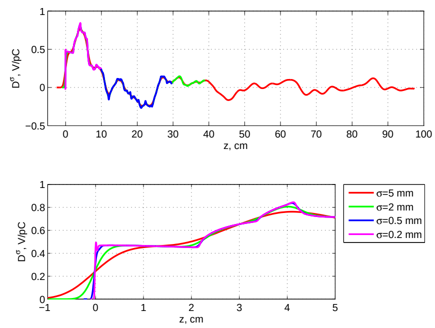

Figure 1: Function $D^{\sigma}(z)$ for 3-step collimator.

First, in Fig. 1, we present the calculations for 3-step collimator (geometry shown in Fig. 2 inset in red dash). Specifically, we plot the function $D^{\sigma}(z)$ defined in Eq. (1), i.e. the wake-potential (calculated with ECHO) with the optical model subtracted. Clearly, the plots in Fig. 1 do not show any singular behaviour as σ gets shorter. Furthermore, for the range of the bunch lengths shown, the traces basically overlap for most values of *z*, which confirms that $D^{\sigma}(z)$ is weakly dependent on σ, i.e. Eq. (2) holds true. However, at certain "corner" points ($z \approx 2.2$ cm, 4.2 cm, etc.), $D^{\sigma}(z)$ has discontinuous derivatives, so strictly-speaking σ–>0 is needed to exactly match this behaviour. As long as we are not interested in the derivatives of the wake at these points, then $D^{\sigma}(z)$ calculated with a reasonably short bunch (i.e. σ < 2 mm) clearly provides a very good accuracy near these locations as well.

Fig. 2 shows $D^{0.5\text{mm}}(z)$ for the set of collimators. For *z*<1 cm (which happens to be the smallest radial step among this set) all $D^{0.5\text{mm}}(z)$ curves converge to the same linear trend already discussed above. Thus, for fixed *a* and *b*, the short-range $D^{\sigma}(z)$ is geometry-independent. This observation is important, since this simple short range behaviour (straight line) can be accurately calculated with a fairly long bunch, i.e. σ_0=0.5 mm. Furthermore, σ_0 can be predicted in advance for most geometries; it simply needs to be short compared to the characteristic dimensions of the structure.

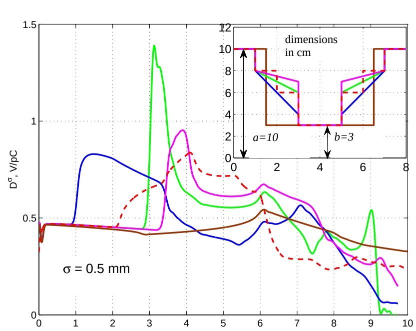

Figure 2: Functions $D^{\sigma}(z)$ for a family of collimators.

The situation becomes more complicated for slowly tapered collimators. However, recently we used a similar approach [4] (although in the frequency domain) to reconstruct accurate short bunch wake-potentials of a slowly tapered collimator from band-limited impedance.

CAVITY-LIKE GEOMETRY

For cavity-like structures the singular parts of the wakes are given by the diffraction model [1],

$$W_d^{\delta}(z) = k_d\, z^{-1/2} \quad (z > 0), \tag{9}$$

$$W_d^{\sigma}(z) = k_d\, \sigma^{-1/2} f\left(z/\sigma\right), \tag{10}$$

where $k_d = -Z_0 c\, \pi^{-2} a^{-1} \sqrt{g/2}$, $\quad (11)$

$$f(s) = e^{-s^2/4}\sqrt{|s|}\left(\mathrm{I}_{-1/4}(\tfrac{s^2}{4}) + \mathrm{sign}(s)\,\mathrm{I}_{1/4}(\tfrac{s^2}{4})\right),$$

and $\mathrm{I}_{\pm 1/4}(\frac{s^2}{4})$ are the Bessel functions. The minimum of the wake-potential occurs at $z \approx 0.76\sigma$, and it diverges $\propto \sigma^{-1/2}$ for short bunches. Below we assume that $W_{\text{model}}^{\sigma}(z)$ is given by $W_d^{\sigma}(z)$, defined by Eq. (10).

We first consider a simple *g*=1 cm long cylindrical cavity of rectangular profile; the cavity and outer pipe radii are *b*=5 cm, *a*=1 cm (Fig.4 inset, solid brown). ECHO-calculated wake-potentials are shown in Fig 3.

The short-range wake-potentials clearly conform to the diffraction model, i.e. their minima scale $\propto \sigma^{-1/2}$. However, similarly scaling peaks occur at multiples of *2g*; particles at these locations gain energy. Subtracting the diffraction model from the calculated wake-potential results in a function that, in the limit of $\sigma \to 0$, is still singular at these points (see Fig. 4, "Rectangular" traces).

For this simple geometry, one could expand the diffraction model to analytically describe the wake-potential at these locations [5], so that one can then eliminate all singularities by subtracting this expanded model. For now we omit this exercise, but rather point out that singularities at multiples of *2g* are due to reflections of the diffracted wave at the cavity sidewalls. These singularities disappear for realistic cavities with smooth corners and/or non-parallel sidewalls.

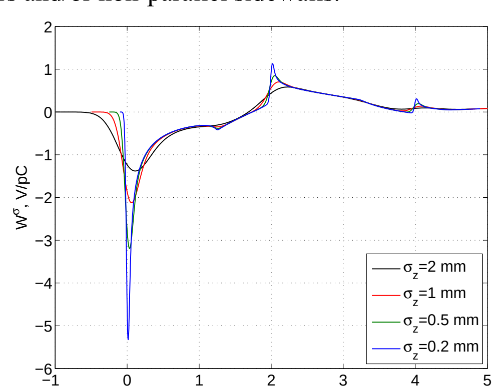

Figure 3: Wake-potentials for rectangular profile cavity.

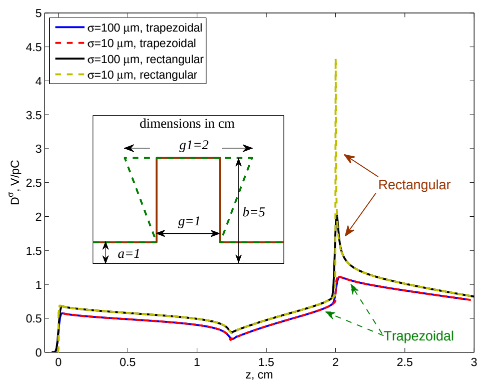

Figure 4: $D^{\sigma}(z)$ for rectangular and trapezoidal cavities.

For example, if the inner corners of this cavity are replaced with 90 degree arc segments of radius r =1 mm, then, after reaching the value of ~1.3 V/pC at $\sigma \approx 200\ \mu\text{m} \approx r/2\pi$, the $W^{\sigma}(z \approx 2\text{cm})$ peak stops growing with decreasing σ.

Another example is a cavity with non-parallel sidewalls, like the "trapezoidal" one shown in Fig. 4, inset. Its wakes at multiples of 2g are not singular, as is clear from Fig. 4. $D^{10\mu\text{m}}(z)$ and $D^{100\mu\text{m}}(z)$ are almost exactly equal. In a separate set of calculations, we confirmed that these singularities are absent for other values of g1≠g, all the way down to g1=0.

With regards to the short-range behaviour of $D^{\sigma}(z)$ for either cavity, we observe that it is qualitatively similar to that discussed earlier for the collimator case. A positive, almost linear and featureless trend continues to z~1 cm, the distance easily predictable from cavity dimensions.

Since for trapezoidal cavity $D^{\sigma}(z)$ is non-singular everywhere, and it is σ-independent for σ<~1 mm, $D^{100\mu\text{m}}(z)$ clearly allows for an accurate calculation of the wake-potential of shorter bunches.

As a final and more practical example we calculate a very short bunch wake for 9-cell TESLA structure. Principle dimensions are 115.4 mm period, 70 mm iris diameter, and 1.036 m total length [6].

In this case the asymptotic wake model is unknown, although we expect σ-scaling similar to the diffraction model. To obtain an accurate $W^{\sigma}_{\text{model}}(z)$, we pick reasonably short σ1 and σ2 and calculate $W^{\sigma 1}(z)$ and $W^{\sigma 2}(z)$ with ECHO. Then we assume the equation

$$W^{\sigma}(z) = \kappa W^{\sigma}_{d}(z) + D(z), \tag{12}$$

to hold exactly for σ=σ1 and σ=σ2, with constant κ.

Solving for κ by direct elimination of $D(z)$ results in z-varying κ. However, near z=0 (the region important for modelling short-bunch wakes) this dependence is almost flat. Thus we set $\kappa = (W^{\sigma 1} - W^{\sigma 2})(W^{\sigma 1}_{d} - W^{\sigma 2}_{d})^{-1}\Big|_{z=0}$.

Now $D(z)$ follows directly from Eq. (12), and wake-potential of a shorter bunch is calculated from Eq. (4).

A wake-potential for 3µm rms bunch reconstructed in this manner from σ1=100µm and σ2=30µm ECHO calculations is plotted in Fig. 5. It shows perfect agreement with the direct ECHO calculation for σ=3µm bunch. A similar agreement was also found for the long-range wake-potentials.

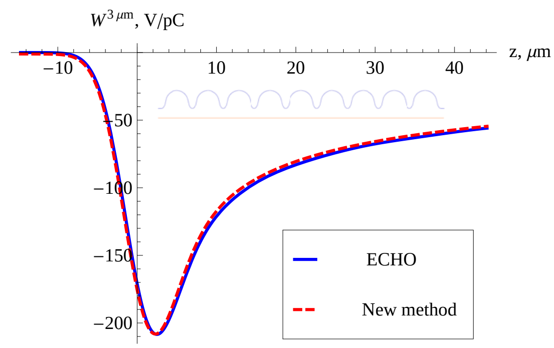

Figure 5: TESLA cavity wakes for 3µm bunch calculated directly and from σ1=100µm and σ2=30µm calculations.

CONCLUSIONS

We describe preliminary results of a new method that allows us to accurately obtain longitudinal wakefields of short bunches by adding a long-bunch result from an EM solver and a singular analytical wake model. In the future this work will be generalized to 3D geometries as well. Similarly, the method should be equally applicable to the calculations of transverse wakefields.

Periodic structures with a significant number of periods ($N \geq a^2/\sigma L$, where L is the period length) have not been considered so far. They have asymptotic wakefields that differ from the examples described above. We believe this method is applicable to such geometries as well, as long as correct asymptotic solutions are used.

We acknowledge the support of US DOE under contracts DE-AC02-98CH10886 and DE-AC02-76SF00515.

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