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A Two Fluid Description of the Quantum Hall Soliton

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Abstract

We show that the Quantum Hall Soliton constructed in [1] is stable under small perturbations. We find that creating quasiparticles actually lowers the energy of the system, and discuss whether this indicates an instability on the time scales relevant to the problem.

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1 Introduction and Review

In [1] a configuration of branes and strings – the Quantum Hall Soliton – was discussed with low energy dynamics similar to those of condensed matter systems displaying the fractional Quantum Hall effect. The configuration consists of a spherical $D2$-brane wrapping K flat $D6$-branes. For topological reasons K fundamental strings must stretch from the $D6$-branes at the center to the spherical $D2$-brane. The string ends on the $D2$-brane play the role of the electrons. The magnetic flux quanta are N $D0$-branes dissolved in the $D2$-brane. The filling factor is therefore given by

$$\nu = \frac{K}{N}. \tag{1.1}$$

Several phenomena that occur in Quantum Hall systems can be modelled qualitatively in terms of the strings and branes involved in the configuration.

In [1], it was shown how to describe the background magnetic field in terms of an incompressible fluid of $D0$-branes using Matrix Theory. In this paper, we follow [2] and show how to model the electrons as a charged fluid moving with the $D0$-brane fluid. The resulting two-fluid description allows us to investigate further aspects of the Quantum Hall Soliton dynamics at low energies. The effective action describing the two interacting fluids involves two gauge fields coupled together with a scalar field controlling the size of the soliton. In the two-fluid picture, quasiparticles may be thought of as vortices in the electron fluid.

It was shown in [1] how the soliton can be stabilized in the near horizon geometry of the $D6$-branes. It was found that there is a characteristic energy scale associated with the low energy dynamics of the soliton. The stability of the configuration with respect to a variety of perturbations was discussed and it was found that in the large N limit the soliton is stable under such perturbations. An issue that was left open is the stability of the soliton under non-spherically symmetric perturbations of the configuration. A preliminary investigation of the stability of the soliton with respect to non-spherically symmetric perturbations was carried out in [6]. The description of the soliton dynamics in terms of the two fluids allows us to investigate thoroughly the stability of the configuration. It is found that the system is stable under such perturbations. The energy scale for such oscillations is of the same order as the characteristic energy scale identified in [1].

Finally, we note that the creation of quasiparticles (vortices in the electron fluid with charge equal to the filling fraction) actually lowers the energy of the system. Naively, this seems like an instability. However, in the small perturbations regime we are working in there is a static solution with any number of quasiparticles, and a conservation law prevents the creation or annihilation of quasiparticles. There are many higher order terms in the Lagrangian which we ignore because we assume small fluctuations, so we can only speculate about whether these terms will lead to an instability on a time scale relevant to the problem. We discuss this matter further in the stability section.

Please note that the construction discussed in this paper is different from the more recent construction of Hellerman and Susskind [3]. Roughly speaking, the electrons and the $D0$-branes swap roles in the two constructions.

2 The Two Fluids

In this section, we follow [2] and describe the Quantum Hall Soliton dynamics in terms of two coupled fluids, one describing the $D0$-branes and the other describing the string ends. The fluid descriptions are valid for large N and K and at distances bigger than the microscopic length scales of the problem. These are the string length scale l_s and the magnetic length. As we saw in [1], the magnetic length is of order the string scale. We focus on a flat $D2$-brane substrate first and generalize to the spherical brane configuration appropriate for the Quantum Hall Soliton later.

2.1 The $D0$-brane Fluid

When a $D0$-brane enters a $D2$-brane, it dissolves into magnetic flux. The density of the $D0$-branes is equivalent to a magnetic field on the membrane while the particle currents result in an electric field. To see the precise connection, we recall that in $2+1$ dimensions the field strength $F_{\mu\nu}$ is dual to a 3-vector J^μ

$$J^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}. \tag{2.1}$$

Thus,

$$J^0 = B = \frac{1}{2}\epsilon^{0ij}F_{ij} = F_{12} \tag{2.2}$$

is the density of the $D0$-branes η and

$$J^i = BV^i = \epsilon^{i0j}F_{0j} = -\epsilon^{ij}E_j \tag{2.3}$$

are the particle currents ηV^i. Here, V^i may be thought of as the velocity field of the fluid particles. The Bianchi identity for the field strength

$$\epsilon^{\mu\nu\sigma}\partial_\mu F_{\nu\sigma} = 0 \tag{2.4}$$

becomes the continuity equation for the $D0$-brane fluid

$$\partial_\mu J^\mu = \partial_t \eta + \partial_i(\eta V^i) = 0. \tag{2.5}$$

In order to desribe the $D2$-brane dynamics, one usually employs the static gauge in which the worldvolume co-ordinates $\xi^0,\ \xi^1,\ \xi^2$ are set equal to the embedding fields $X^0,\ X^1,\ X^2$ along the directions parallel to the brane. In this gauge, the co-ordinate freedom of the problem–worldvolume diffeomorphisms–is completely fixed. The dynamics is then described by a $U(1)$ gauge field $A_\mu(X)$–or in terms of the gauge invariant quantities $B(X)$ and $E_i(X)$–in addition to the embedding fields along the directions transverse to the brane. From the point of view of the fluid picture, the meaning of this gauge is clear: we use co-ordinates $X^0 = t,\ X^1,\ X^2$ fixed in space and the equations of motion of the fluid are expressed in terms of the density and the velocity fields, $\eta(X) = B(X)$ and $V^i(X) = -\epsilon^{ij}E_j(X)/B(X)$. This is what is usually called the "Eulerian description" of the fluid.

In fluid dynamics, there is another description of the fluid, called the "Lagrangian description". In this description, we use co-ordinates $\xi^0 = t$ and $\xi^1 = y^1,\ \xi^2 = y^2$ co-moving with the fluid. In this frame, the density of the fluid η is fixed and the currents ηV^i are zero. From the point of view of the $D2$-brane theory, we may work in a frame such that the magnetic field B is fixed to a constant value and the electric field E_i is zero [8]. Equivalently, the field strength $F_{\mu\nu}$ is constant with

$$F_{ij} = B_{ij} \tag{2.6}$$

where B_{ij} is equal to the constant matrix $B\epsilon_{ij}$, and

$$F_{0i} = 0. \tag{2.7}$$

Thus in this particular frame, the gauge field on the membrane can be taken to be fixed and non-fluctuating. For example, we can work in the $A_0 = 0$ gauge and set

$$A_i = -\frac{B}{2}\epsilon_{ij}y^j. \tag{2.8}$$

The dynamical fields are the embedding fields $X^i(y,t)$ along the directions parallel to the brane and the embedding fields along the directions transverse to the brane.

The requirement that the field strength is constant does not completely fix the co-ordinate freedom of the problem. One may find co-ordinate transformations that leave the two form $F_{ij} = B_{ij}$ constant and F_{0i} zero. Infinitesimally, such transformations take the form

$$y'^i = y^i + \Theta^{ij}\partial_j\lambda, \tag{2.9}$$

where

$$\Theta^{ij} = (B^{-1})^{ij} = -B^{-1}\epsilon^{ij} \tag{2.10}$$

and λ is time independent. Such transformations are time independent area preserving diffeomorphisms.

Under a co-ordinate transformation the embedding fields X^i transform as scalars,

$$X'^i(y') = X^i(y). \tag{2.11}$$

Therefore, under infinitesimal area preserving diffeomorphisms, eq. (2.9),

$$\delta X^i(y) = \Theta^{jk}\partial_j\lambda\partial_k X^i = -i\{\lambda, X^i\}, \tag{2.12}$$

where the Poisson bracket between two quantities A and B is defined by

$$\{A, B\} = i\Theta^{ij}\partial_i A\partial_j B. \tag{2.13}$$

For studying small oscillations of the fluid, it is appropriate to introduce the displacement fields $\hat{A}_i(y,t)$ defined by

$$X^i = y^i - \Theta^{ij}\hat{A}_j. \tag{2.14}$$

Then under an infinitesimal area preserving diffeomorphism

$$\delta\hat{A}_i = -\partial_i\lambda + i\{\lambda, \hat{A}_i\}. \tag{2.15}$$

We conclude that the displacement fields are non-commutative gauge fields. Their transformations are the usual non-commutative gauge transformations [5] truncated to first order in Θ. The full non-commutative gauge symmetry can be recovered by replacing the fluid description of the $D0$-branes by their microscopic Matrix Theory description [1][9].

The "Eulerian" and "Lagrangian" descriptions of the $D0$-brane fluid are related to each other by a co-ordinate transformation [8][9]. We review their relation in the appendix.

Here, we note that the ordinary gauge field A_i of the "Eulerian" description is related to the displacement field $\hat{A}_i$ by the Seiberg-Witten transformation between ordinary and non-commutative gauge fields [5].

Chern Simons Couplings. Now let us see the effect of turning on a constant $D0$-brane magnetic field H_2 along the directions of the $D2$-brane. Let the total H_2-flux through the $D2$-brane be $2\pi\mu_6 K$ as in [1]. This flux is sourced by the $D6$-branes. The total magnetic flux due to the dissolved $D0$-branes is $2\pi N$. Thus

$$\frac{H}{B} = \frac{\mu_6 K}{N} = \mu_6 \nu. \tag{2.16}$$

We can work in a gauge in which the RR gauge potential C_1 is given by

$$C_i = -\frac{H}{2}\epsilon_{ij}X^j. \tag{2.17}$$

Now the effect on the worldvolume theory of the $D2$-brane is a Chern-Simons type coupling given by

$$\frac{\mu_2(2\pi\alpha')}{2}\int d^3\xi \epsilon^{\alpha\beta\gamma} C_\mu(X)\frac{\partial X^\mu}{\partial \xi^\alpha} F_{\nu\rho}(X)\frac{\partial X^\nu}{\partial \xi^\beta}\frac{\partial X^\rho}{\partial \xi^\gamma}. \tag{2.18}$$

Choosing to work with co-moving co-ordinates $t,\ y^1,\ y^2$ in which the field strength is constant, the coupling becomes[1]

$$\frac{\mu_2(2\pi\alpha')}{2}\int dtd^2y \epsilon^{0ij} C_k(X)\frac{\partial X^k}{\partial t}B_{ij} = \frac{\mu_2(2\pi\alpha')HB}{2}\int dtd^2y \epsilon_{ij}X^i\partial_t X^j. \tag{2.19}$$

This term has an intuitive explanation: a single $D0$-brane (labeled by an index α) moving in a constant magnetic field will have a term in its Lagrangian proportional to

$$\sim \frac{H}{2}\epsilon_{ij}x^i_\alpha \partial_t x^j_\alpha. \tag{2.20}$$

Now in describing the many $D0$-brane system as a fluid, x^i_α are replaced by the fields $X^i(y)$ and the sum over all such particles, $\sum_\alpha$, by the integral $\int d^2yB$, where B is the density of the particles.

In terms of the displacement fields, this term becomes

$$-\frac{\mu_2(2\pi\alpha')H}{2}\int dtd^2y \Theta^{ij}\hat{A}_i\partial_t\hat{A}_j = \frac{\kappa}{4\pi}\int dtd^2y \epsilon^{ij}\hat{A}_i\partial_t\hat{A}_j, \tag{2.21}$$

up to a total time derivative. So it becomes an ordinary Chern-Simons coupling for the displacement fields. The level κ is given by

$$\kappa = 2\pi\mu_2(2\pi\alpha')HB^{-1} = 4\pi^2\alpha'\mu_2\mu_6\frac{K}{N} = \frac{K}{N} = \nu. \tag{2.22}$$

[1]In our conventions, $\epsilon^{012} = \epsilon^{12} = 1$.

Equation of constraint. Since we are working in the $A_0 = 0$ gauge, we must impose the A_0 equation of motion as a constraint. From (2.18), the part of the Lagrangian containing A_0 is given by

$$\frac{\mu_2(2\pi\alpha')H}{2}\epsilon^{ij}\epsilon_{kl}\partial_i X^k \partial_j X^l A_0 \tag{2.23}$$

and so the contribution to the equation of motion is given by

$$\frac{\mu_2(2\pi\alpha')H}{2}\epsilon^{ij}\epsilon_{kl}\partial_i X^k \partial_j X^l + ... = 0. \tag{2.24}$$

This term has the following origin. In the static gauge, the effect of the $D6$-branes at the center is to couple to the A field just like a uniform charge density given by [1]

$$J^0 = \frac{\mu_2(2\pi\alpha')H}{2}. \tag{2.25}$$

Then, the corresponding density in the comoving frame is given by

$$J^0\left(\frac{\partial X}{\partial y}\right) \tag{2.26}$$

where the last factor is the Jacobian of the transformation from the comoving frame to the "Eulerian frame" of the static gauge. Expanding to linear order in the displacement fields, we obtain

$$J^0 + \frac{\nu}{2\pi}\epsilon^{ij}\partial_i \hat{A}_j + ... = J^0 + \frac{\nu}{2\pi}\hat{F}_{12} + ... = 0. \tag{2.27}$$

Here

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i. \tag{2.28}$$

Including the non-linear terms in eq. (2.24) amounts to replacing $\hat{F}$ with

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i\{\hat{A}_i, \hat{A}_j\}, \tag{2.29}$$

i.e. with the non-commutative field strength.

The equations of motion, together with the constraint equation, can be derived by introducing a scalar potential $\hat{A}_0$ and varying the following action [2]

$$S = \int dtd^2y \left[-\hat{A}_0 J^0 - \frac{\nu}{4\pi}\epsilon^{\mu\nu\rho}\hat{A}_\mu \partial_\nu \hat{A}_\rho\right]. \tag{2.30}$$

[2]We have dropped total derivatives.

The first term is a chemical potential term for $\hat{A}_0$. It acts like an induced positive background charge density on the membrane. As we shall see in the next section, it can be cancelled by adding K string ends on the membrane of opposite charge:

$$\int J^0 = K. \tag{2.31}$$

The second term is a Chern-Simons coupling. We have kept terms to quadratic order in the displacement fields only. Both terms have been derived in [1] using Matrix Theory with the full non-commutative gauge symmetry manifest. As we shall see in the next section, however, the fluid of string ends has the effect of cancelling both terms including the Chern-Simons coupling.

What we have said above applies to the case of the spherical Quantum Hall soliton as well. In this case, the co-moving co-ordinates $y^1,\ y^2$ are two angles θ and ϕ and the fixed density of the $D0$-brane fluid is given by

$$B = B_{12} = \frac{N}{2}\sin\theta. \tag{2.32}$$

The symmetry of the problem is the group of area preserving diffeomorphisms of the sphere. Under an infinitesimal area preserving diffeomorphism, the density B transforms covariantly

$$B \to B' = \frac{N}{2}\sin\theta'. \tag{2.33}$$

The effect of the $D6$-branes at the center is to induce a chemical potential term and a Chern-Simons coupling for the displacement fields as we have argued above. The "charge density" is now given by

$$J^0 = \frac{K}{4\pi}\sin\theta. \tag{2.34}$$

Born-Infeld Dynamics. The Quantum Hall soliton lives in the near horizon geometry of a stack of K $D6$-branes. The background metric and dilaton fields are given by

$$ds^2 = \sqrt{\frac{\rho}{l_s}}(dt^2 - dy^a dy^a) - \sqrt{\frac{l_s}{\rho}}(d\rho^2 + \rho^2 d\Omega_2^2), \tag{2.35}$$

where ρ is a radial coordinate, and

$$g_s^2 e^{2\Phi} = \frac{4}{K^2}\left(\frac{\rho}{l_s}\right)^{\frac{3}{2}}. \tag{2.36}$$

The soliton can be stabilized at a co-ordinate distance given by [1]

$$\rho_* = \frac{(\pi N)^{\frac{2}{3}}}{2} l_s \tag{2.37}$$

for all N and K. The values of the induced metric $g_{\mu\nu}$ on the $D2$-brane and dilaton field at ρ_* are given by

$$ds^2_{ind} = \frac{(\pi N)^{\frac{1}{3}}}{\sqrt{2}} dt^2 - \frac{(\pi N)}{2\sqrt{2}} \alpha'(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.38}$$

and

$$g_s e^{\Phi}|_{\rho_*} = \frac{2}{K}\left(\frac{\rho_*}{l_s}\right)^{\frac{3}{4}} = 2^{\frac{1}{4}} \frac{\sqrt{\pi N}}{K} \tag{2.39}$$

independent of g_s at infinity.

The proper area of the stable soliton is given by

$$A = 4\pi\sqrt{\rho_*{}^3 l_s} = \sqrt{2}\pi^2 N l_s^2. \tag{2.40}$$

Thus there is a universal density of $D0$-branes which is of order one in string units for all N and K. This means that the fluid of $D0$-branes is incompressible. The separation of the $D0$-branes is the magnetic length and this is of order one in string units.

The action for the $D2$–brane in the background geometry is given as usual by the Dirac Born Infeld (DBI) action. In terms of co-moving co-ordinates in which the gauge field on the brane is non-fluctuating, the Born-Infeld part of the action takes the following form

$$S_{D2} = -\frac{1}{4\pi^2 g_s l_s^3} \int dt d^2y e^{-\Phi} det^{\frac{1}{2}}[h_{\mu\nu} + 2\pi\alpha' B_{\mu\nu}], \tag{2.41}$$

with $B_{0i} = 0$ and $B_{12} = N\sin\theta/2$. Here, $h_{\mu\nu}$ is the induced metric on the brane and it depends on the dynamical fields $X^i(y,t)$.

We may use the DBI action to obtain an effective action for small oscillations of the soliton about the spherical equilibrium configuration at ρ_*. To do so, we expand the action to quadratic order in the displacement fields $\hat{A}_i$ and the fluctuation of the radial field χ defined by

$$\rho(X) = \rho_* + 2\pi\alpha'\chi(X). \tag{2.42}$$

For later convenience, we take the radial field χ to be a function of the space-fixed co-ordinates X^i. Naturally χ transforms as a scalar under co-ordinate transformations [3]. We do not consider motions along the directions parallel to the $D6$-branes because there is translational symmetry in these directions and so they are uninteresting. The effective action governs the dynamics of the soliton at distance scales comparable to the size of the

[3] that is, $\chi_y(y) = \chi(X) = \chi(y) - \partial_i\chi(y)\Theta^{ij}\hat{A}_j$.

sphere. Alternatively, the effective action describes the dynamics of the $D0$-brane fluid at macroscopic length scales, distances bigger than the magnetic length.

Expanding the DBI action and writing down only the terms quadratic in the fields, we obtain

$$S_{gaugefield} = -\frac{1}{2g_{YM}^2}\int dtd^2y(detG)^{\frac{1}{2}}\left[G^{00}G^{ij}\partial_t\hat{A}_i\partial_t\hat{A}_j + \frac{1}{2}G^{ik}G^{jl}\hat{F}_{ij}\hat{F}_{kl}\right], \tag{2.43}$$

$$S_{scalar} = \frac{1}{2g_{YM}^2}\int dtd^2y(detG)^{\frac{1}{2}}|g_{\rho\rho}|\left[G^{00}(\partial_t\chi)^2 + G^{ij}\partial_i\chi\partial_j\chi - \frac{16\sqrt{2}}{9\pi N\alpha'}\chi^2\right] \tag{2.44}$$

and the interaction piece

$$S_{int} = \frac{8\nu}{9\pi}\int dtd^2y\chi\hat{F}_{12}. \tag{2.45}$$

The indices are contracted with the effective "open string" metric $G_{\mu\nu}$. In terms of the "closed string" metric at ρ_*, eq. (2.38), this is given by

$$G_{00} = g_{00}(\rho_*), \;\; G_{ij} = g_{ij}(\rho_*) - 2\pi\alpha' B_{ik}g^{kl}(\rho_*)B_{lj}. \tag{2.46}$$

Therefore,

$$ds^2_{open} = \frac{(\pi N)^{\frac{1}{3}}}{\sqrt{2}}dt^2 - \frac{9(\pi N)}{2\sqrt{2}}\alpha'(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.47}$$

The gauge coupling constant g_{YM} is given in terms of the effective "open string" coupling constant as follows

$$g_{YM}^2 = G_s(\rho_*)l_s^{-1}, \tag{2.48}$$

where

$$G_s(\rho_*) = g_s(\rho_*)\left(\frac{detG_{\mu\nu}}{det(g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu})}\right)^{\frac{1}{2}}. \tag{2.49}$$

Therefore,

$$g_{YM}^2 = 3\frac{2^{\frac{1}{4}}\sqrt{\pi N}}{K}l_s^{-1} = 3\frac{2^{\frac{1}{4}}\sqrt{\pi}}{\nu\sqrt{N}}l_s^{-1}. \tag{2.50}$$

The contribution to the constraint equation, eq. (2.27), from the Born-Infeld term is given by

$$\frac{1}{g_{YM}^2}\partial_i\left((detG)^{\frac{1}{2}}G^{00}G^{ij}\partial_t\hat{A}_j\right) + J^0 + \frac{\nu}{2\pi}\hat{F}_{12} + ... = 0. \tag{2.51}$$

The new term in the equation is simply the analogue of minus the divergence of the electric field in ordinary electrodynamics.

Finally, as in [4], let us do a conformal transformation and rescale the fields so that the metric takes the form

$$d\tilde{s}^2 = dt^2 - \frac{9(\pi N)^{2/3}}{2}\alpha'(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.52}$$

and the fields appear with canonically normalized kinetic terms in the action. To this end, we set

$$\tilde{A}_i^{\ 2} = \frac{1}{g_{YM}^2\sqrt{G_{00}}}\hat{A}_i^{\ 2}, \quad \tilde{\chi}^2 = \frac{|g_{\rho\rho}|\sqrt{G_{00}}}{g_{YM}^2}\chi^2 = \frac{1}{g_{YM}^2\sqrt{G_{00}}}\chi^2 \tag{2.53}$$

and express the action in terms of the rescaled fields. We obtain the following action

$$S = \int dtd^2y(det\tilde{G})^{\frac{1}{2}}\left(-\frac{1}{4}\tilde{F}_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\tilde{\chi})^2 - \frac{8}{9(\pi N)^{2/3}\alpha'}\tilde{\chi}^2\right) + \frac{8}{3(\pi N)^{1/3}l_s}\tilde{\chi}\tilde{F}_{12}. \tag{2.54}$$

In these units, the size of the sphere is of order $N^{1/3}l_s$. Accordingly, the magnetic length is of order $N^{-1/6}l_s$. The scalar field is massive with mass given by

$$m_{\tilde{\chi}} = \frac{4}{3(\pi N)^{1/3}l_s}. \tag{2.55}$$

So its Compton wavelength is comparable to the size of the sphere. Since the scalar field is massive, we conclude as in [1] that the soliton is stable under small spherically symmetric oscillations. The energy scale of these oscillations is set by the mass of the scalar field. As we shall see in the next section, the Chern-Simons term for the displacement fields is cancelled by the collective motions of the electron fluid and we do not include it in the action. We also note that the supersymmetry breaking scale for the $D2$-brane theory is set by the size of the sphere [4].

The rescaled fields couple to ordinary charges with coupling constant

$$g_{YM}G_{00}^{1/4}. \tag{2.56}$$

Therefore, one expects the strength of interactions between string ends to be of order

$$\sim \frac{1}{\nu N^{1/3}} \tag{2.57}$$

in string units.

Focusing on a small patch of the sphere bigger than the magnetic length so that we can approximate it as flat, the gauge kinetic term is of the standard form

$$-\frac{1}{4}\int d^3x\tilde{F}^2 \tag{2.58}$$

and so the speed of sound waves of the $D0$-brane fluid is one

$$c = 1. \tag{2.59}$$

Thus the $D0$-brane fluid is stiff and difficult to compress.

The interaction term between the scalar and the gauge fields is also noteworthy. Consider a non-spherically symmetric deformation of the soliton as in [6]. Then the $D0$-branes would tend to accumulate in the regions farther away from the $D6$-branes where the repulsion is less strong. In other words, where χ is positive ($D2$-brane farther from the $D6$-branes), the energy is lowered by F_{12} (excess density of $D0$-branes) being positive, so that the term in the Hamiltonian will be $-\chi F_{12}$ and thus $+\chi F_{12}$ in the Lagrangian.

2.2 The Electron Fluid

As we saw in [1], K fundamental strings must stretch from the $D6$-branes to the $D2$-brane due to the Hanany-Witten effect [7]. The strings will tend to distribute themselves homogeneously so as to cancel the background charge density induced by the $D6$-branes on the membrane. In what follows, we consider a model in which the strings remain in their ground state apart from the motion of their ends on the $D6$- and $D2$-branes. As was argued in [1], the strong gauge dynamics on the $D6$-brane side will tend to bind the K string ends into a "baryon". The properties of the "baryon" wavefunction under interchange of two strings would then determine the effective statistics of the string ends on the $D2$-brane side. We ignore the intrinsic statistics of the string ends here.

When K is large, and at distances larger than the magnetic length, we can treat the string ends on the $D2$-brane as a fluid of non-relativistic charged particles. We will take the effective mass of the particles to be of order the mass of a radially stretched string at rest

$$m_{string} = \frac{\rho_*}{2\pi l_s^2} = \frac{(\pi N)^{\frac{2}{3}}}{4\pi} l_s^{-1}. \tag{2.60}$$

More precisely, the effective string mass above is computed by assuming that the string remains straight as its ends move around on the branes. As discussed in [1], the energy scale of long string oscillations is of the same order as the typical energy scale controlling the dynamics of the soliton, much lower than the mass of a stretched string at rest, and we do not expect them to modify the effective mass of the electrons. In the large N limit the non-relativistic approximation a good one.

Non-commutative charged particles. Consider a $D0$-brane fluid element fixed at the origin in its rest frame (the y-frame) and denote the relative position of the string end with respect to it by ${y_s}^i(t)$. Then, the position of the string end in space is given by

$$ {x_s}^i(t) = {y_s}^i(t) - \Theta^{ij} \hat{A}_j(y_s, t). \tag{2.61} $$

Similarly, the particle velocity is given by

$$ \frac{d}{dt} {x_s}^i = {\dot{x_s}}^i = {\dot{y_s}}^i - \Theta^{ij} \dot{\hat{A}}_j = {\dot{y_s}}^i - \Theta^{ij} \partial_t \hat{A}_j - {\dot{y_s}}^k \Theta^{ij} \partial_k \hat{A}_j. \tag{2.62} $$

In the frame comoving with the $D0$-brane fluid, the particle interacts with a fixed background magnetic field and so

$$ L_{gaugeinter} = \frac{eB}{2} \epsilon_{ij} {y_s}^i {\dot{y_s}}^j. \tag{2.63} $$

We have chosen the background potential to be given by $-B\epsilon_{ij} y^j/2$. Using equations (2.61) and (2.62), we can write this in terms of the space fixed variables, x_s and $\dot{x_s}$. Keeping terms to quadratic order in the fluctuations, the Lagrangian (2.63) becomes

$$ L_{gaugeinter} = \frac{eB}{2} \epsilon_{ij} {x_s}^i {\dot{x_s}}^j - e {\dot{x_s}}^i \hat{A}_i(x_s) - \frac{e}{2} \Theta^{ij} \hat{A}_i(x_s) \partial_t \hat{A}_j(x_s). \tag{2.64} $$

We see that the particle couples to the displacement field as it couples to the ordinary commutative gauge field but it also feels a potential.

Under the infinitesimal area preserving diffeomorphisms, eq. (2.9), the Lagrangian (2.63) changes by a total time derivative

$$ \delta L = -\frac{e}{2} \frac{d}{dt} (\partial_i \lambda y_s^i). \tag{2.65} $$

The action is invariant under such co-ordinate transformations and so under the non-commutative gauge symmetry of the problem.

The above result has also an intuitive explanation. Suppose we can ignore the kinetic term. Then the full Lagrangian is given by (2.63) and the equations of motion imply that

$$ {y_s}^i = constant. \tag{2.66} $$

That is, the particle is fixed with respect to any $D0$-brane fluid element and simply follows the fluid. This can also be understood as follows. Let v^i denote the velocity of the particle in fixed space. Since the particle is massless, the Lorentz force on the particle must be set to zero

$$ E_i(x) + B(x) \epsilon_{ij} v^j = 0. \tag{2.67} $$

It follows that v^i is equal to $-\epsilon^{ij}E_j(x)/B(x)$ which according to (2.3) is simply the velocity field of the $D0$-brane fluid evaluated at the position of the particle. Thus, the particle follows the fluid.

The rest of the Lagrangian involves the non-relativistic kinetic energy term

$$L_{KE} = -\frac{1}{2}mg^{00}g_{ij}\dot{x_s}^i\dot{x_s}^j \tag{2.68}$$

and the interaction with the scalar field χ

$$L_{scalarinter} = -\chi(x_s). \tag{2.69}$$

The last term arises since the rest mass of a string is proportional to its length.

Fluid description. We describe the many string ends as a fluid. To this end, we use co-ordinates t and z^1, z^2 comoving with the electron fluid[4]. In this frame, the electron number density is fixed and given by

$$\eta_0 = \frac{K}{4\pi}\sin\theta. \tag{2.70}$$

To pass to the fluid description, we replace $x_{s_\alpha}^i$ by the fields $X_s^i(z,t)$ and the sum $\sum_\alpha$ by the integral $\int d^2z\eta_0$. We obtain the following effective action

$$S_{strings} = \int dtd^2z\frac{-\eta_0 m}{2}g^{00}g_{ij}\partial_t X_s^i\partial_t X_s^j + \frac{e\eta_0 B}{2}\epsilon_{ij}{X_s}^i\partial_t {X_s}^j + V, \tag{2.71}$$

where

$$V = -e\eta_0\partial_t {X_s}^i\hat{A}_i + \frac{e\nu}{4\pi}\epsilon^{ij}\hat{A}_i\partial_t\hat{A}_j + \eta_0\chi\partial_i(X_s^i - z^i) + h.o. \tag{2.72}$$

In the potential, we have kept terms up to quadratic order in the displacement field $\hat{A}$ and spatial and temporal derivatives of the field X_s. Since we are expanding about an extremum of the potential, any linear terms will cancel once we add the two fluid actions together. Thus we ignore them. As we already mentioned in the previous section, the string-end fluid Lagrangian contains a Chern-Simons coupling for the the displacements fields $\hat{A}$ that cancels the Chern-Simons coupling, eq. (2.21), if e is negative. So we can omit it.

The fluid action we have obtained has also an exact gauge symmetry which consists of time independent area preserving diffeomorphisms

$$z'^i = z^i + \frac{\eta_0}{2\pi}\epsilon^{ij}\partial_j\lambda. \tag{2.73}$$

[4] z^1 and z^2 are two angles θ and ϕ

Under such transformations the fields X_s^i transform as scalars. The fields describing the $D0$-brane fluid appear as functionals of X_s in this action and so they transform as scalars. Under such transformations the electron number density transforms covariantly.

For small oscillations of the charged fluid, it is convenient to introduce the displacement fields defined by

$$X_s^i = z^i - \frac{1}{2\pi\eta_0}\epsilon^{ij}D_j. \tag{2.74}$$

In analogy with the displacement fields describing small oscillations of the $D0$-brane fluid $\hat{A}$, the displacement fields D are also non-commutative gauge fields. At the linearized level, they transform as ordinary gauge fields

$$D_i \to D_i + \partial_i\lambda. \tag{2.75}$$

In terms of the displacement fields, the string fluid action becomes

$$S_{strings} = \int dtd^2z\frac{-1}{2g_e^2}(detG_e)^{\frac{1}{2}}G_e^{00}G_e^{ij}\partial_t D_i\partial_t D_j - \frac{e}{4\pi\nu}\epsilon^{ij}D_i\partial_t D_j - \frac{e}{2\pi}\epsilon^{ij}\hat{A}_i\partial_t D_j - \frac{1}{2\pi}\chi\epsilon^{ij}\partial_i D_j. \tag{2.76}$$

Notice the similarity with the $D0$-brane fluid action. The first term is a conventional Maxwell term. The effective metric appearing in the Maxwell term is given by

$$G_e^{00} = g^{00},\ \ G_e^{ij} = -\frac{1}{(2\pi\eta_0)^2(2\pi\alpha')^2}\epsilon^{ik}g_{kl}\epsilon^{lj} \sim \frac{1}{\nu^2 N\alpha'} \tag{2.77}$$

and the coupling constant by

$$g_e^2 = \frac{1}{(2\pi\alpha'^2)(2\pi\eta_0)m}(detG)^{\frac{1}{2}} \sim \frac{\nu}{\sqrt{N}l_s}. \tag{2.78}$$

The formulae are analogous to the formulae for the effective open string metric and coupling constant in the zero slope limit [5]! Here, the non-commutativity parameter is set by the number density of the electrons

$$\Theta_e = \frac{1}{2\pi\eta_0} \tag{2.79}$$

and the Maxwell coupling constant is inversely proportional to the mass of the strings as in the case of the $D0$-brane fluid, where the coupling constant is inversely proportional to the $D0$-brane mass. Unlike the Maxwell term describing the $D0$-brane fluid however, the magnetic part of the Maxwell term is absent. This means that unlike the $D0$-brane fluid case, it does not cost much energy to compress the electron fluid. Physically, what is going on is that the electrons only interact with each other via the electromagnetic field since we are ignoring their intrinsic statistics, so the density of electrons only comes into the action via the coupling to the A field.

The second term is a Chern-Simons term for the gauge field D. As in [2], it arises in the fluid description of charged particles in a magnetic field. The presence of the Chern-Simons term means that the gauge field D is massive with mass given by

$$m_D \sim \frac{g_e^2}{\sqrt{G_{00}}4\pi\nu} \sim \frac{\eta_0}{\sqrt{g_{11}}m\nu\alpha'} \tag{2.80}$$

in the units we are working with. With the mass of the string given by eq. (2.60), we find that

$$m_D \sim N^{-\frac{1}{3}} l_s^{-1}. \tag{2.81}$$

In fact, this mass is really the cyclotron frequency setting the energy scale of higher Landau levels. As in [1], we see a single energy scale describing the low energy dynamics of the Quantum Hall soliton.

The last two terms involve the interactions of the electron fluid gauge field with the $D0$-brane fluid gauge field and the scalar field. The interaction between the two gauge fields is a Chern-Simons coupling consistent with the two gauge symmetries of the problem. The origin of the interaction with the scalar field is also easy to understand. Consider a non-spherically symmetric perturbation of the soliton. Then the strings would tend to concentrate in the region closer to the $D6$-branes since their mass is smaller there. The sign of this coupling is opposite to that between the scalar field and $\hat{A}$. We will comment more on the interaction terms in the next two sections.

The contribution of the electron fluid to the constraint equation, eq. (2.51), can easily be obtained. The electron density in the space fixed frame is given by

$$\eta = \eta_0 \left(\frac{\partial z}{\partial X} \right) = \eta_0 + \frac{1}{2\pi} \epsilon^{ij} \partial_i D_j. \tag{2.82}$$

The density in the frame comoving with the $D0$-brane fluid is

$$\eta \left(\frac{\partial X}{\partial y} \right) = \eta_0 + \frac{1}{2\pi} \epsilon^{ij} \partial_i D_j + \frac{\nu}{2\pi} \epsilon^{ij} \partial_i \hat{A}_j. \tag{2.83}$$

So the contribution to the constraint equation is simply

$$e\eta_0 + \frac{e}{2\pi} \epsilon^{ij} \partial_i D_j + \frac{e\nu}{2\pi} \epsilon^{ij} \partial_i \hat{A}_j. \tag{2.84}$$

With $e = -1$, the full constraint equation becomes

$$\frac{1}{g_{YM}^2} \partial_i \left((detG)^{\frac{1}{2}} G^{00} G^{ij} \partial_t \hat{A}_j \right) + \frac{e}{2\pi} \epsilon^{ij} \partial_i D_j = 0. \tag{2.85}$$

We see that both the background charge density J_0 and the contribution from the Chern-Simons term, eq. (2.27), cancel.

Conserved Quantities. Under the transformation eq. (2.75) the Lagrangian of (2.76) changes by a total time derivative

$$\frac{e}{4\pi\nu}\partial_t(\epsilon^{ij}\partial_i\lambda D_j). \tag{2.86}$$

Therefore, a conserved quantity exists and is given by

$$\partial_i(\Pi^i_D - \frac{e}{4\pi\nu}\epsilon^{ij}D_j), \tag{2.87}$$

where Π^i_D is the momentum conjugate to D_i. The conserved quantity is therefore given by

$$-\frac{1}{g_e^2}\partial_i\left((detG_e)^{\frac{1}{2}}G_e^{00}G_e^{ij}\partial_t D_j\right) - \frac{e}{2\pi\nu}\epsilon^{ij}\partial_i D_j + \frac{e}{2\pi}\epsilon^{ij}\partial_i\hat{A}_j. \tag{2.88}$$

In the absence of vortices we may consistently impose

$$-\frac{1}{g_e^2}\partial_i\left((detG_e)^{\frac{1}{2}}G_e^{00}G_e^{ij}\partial_t D_j\right) - \frac{e}{2\pi\nu}\epsilon^{ij}\partial_i D_j + \frac{e}{2\pi}\epsilon^{ij}\partial_i\hat{A}_j = 0 \tag{2.89}$$

as a second constraint equation. Thus, changes in the $D0$-brane fluid density source an electric field for the gauge field D. In this way, $D0$-branes may be thought of as charges under the electron fluid gauge field! Adding a single $D0$-brane corresponds to adding a unit of magnetic flux. It follows then from the above equation that it creates a fluctuation in the electron density of total charge ν. As in [1], additional $D0$-branes can be thought of as the Laughlin quasiparticles [10].

3 Stability Analysis

In this section, we use the two-fluid action we have obtained to show that the Quantum Hall soliton is stable under a variety of perturbations. There are reasons to think that the system might be unstable. The velocity independent part of the potential is given by

$$V \sim \tilde{\chi}\epsilon^{ij}\partial_i(D_j - b\tilde{A}_j), \tag{3.1}$$

with b of order one, and it is not positive definite. The physical origin of the potential instability can be described as follows [6]. Consider a configuration in which the $D6$-branes are displaced away from the center of the spherical membrane. The $D0$-branes are repelled by the $D6$-branes, while the electrons are attracted to the $D6$-branes by the stretched strings. It is conceivable then that with the electrons moving closer and the $D0$-branes farther away from the $D6$-branes the energy of the configuration gets lowered. This argument on the other hand assumes that the electrons and the $D0$-branes move independently of each other, neglecting the velocity dependent forces in the potential. It is well known that charged particles moving in a magnetic field tend to follow magnetic flux lines, and so there is also a tendency for the electrons and the $D0$-branes to stick together. The velocity dependent terms in the potential are crucial for the stability of the soliton.

In addition, the system might be unstable to forming small ripples, with the $D0$-branes becoming concentrated on the peaks of the ripples where they are farther from the $D6$-branes and the electrons becoming concentrated in the troughs where the strings attached to them can be shorter.

Let us illustrate how the potential instability gets removed with a simple example. Consider the case of a charged particle in an upside-down harmonic oscillator potential moving in the presence of a uniform magnetic field along the z direction. The Lagrangian is given by

$$L = \frac{1}{2} m \dot{\vec{X}}^2 + \frac{1}{2} k \vec{X}^2 + \frac{1}{2} eB \epsilon_{ij} X^i \dot{X}^j. \tag{3.2}$$

We will see that in spite of the tachyonic potential, the system is stable for some values of the parameters. In terms of $Z = X^1 + iX^2$, and ignoring the X^3 direction which is uninteresting, the Lagrangian becomes

$$L = \frac{1}{2} m \dot{Z} \dot{Z}^\star + \frac{1}{2} k Z Z^\star + \frac{i}{4} eB (\dot{Z} Z^\star - Z \dot{Z}^\star). \tag{3.3}$$

The equation of motion is

$$m\ddot{Z} + kZ + ieB\dot{Z} = 0 \tag{3.4}$$

This is the equation of motion of a damped harmonic oscillator. Looking at the characteristic frequencies shows that if

$$(eB)^2 > 4km, \tag{3.5}$$

then the system oscillates; otherwise it exhibits exponential growth or decay. Physically this means that if the B field is large enough, then the particle has closed orbits; otherwise it rolls down the potential hill. Therefore, for a certain range of parameters the extremum of the oscillator potential is stable. The stability condition (3.5) can also be written as follows

$$\omega_{cycl} > \omega_{oscil}, \tag{3.6}$$

where the cyclotron frequency is given by $|eB|/m$ and the characteristic harmonic oscillator frequency by $(k/m)^{1/2}$. We conclude that with all other quantities fixed, the oscillator is stable if the mass m is not too large.

This example demonstrates that the stability of the Quantum Hall Soliton depends on the relative size of the velocity dependent terms to the velocity independent terms in the potential. The strength of the velocity dependent terms is set by the cyclotron frequency or the mass of the gauge field D

$$\omega_{cycl} \sim N^{-1/3} l_s. \tag{3.7}$$

The strength of velocity independent terms is set by the product of the gauge coupling constants of each fluid

$$\sqrt{g_e g_{YM}} G_{00}^{1/4} \sim N^{-1/3} l_s. \tag{3.8}$$

Since these are comparable for all N and K, a careful analysis of the stability of the configuration is needed.

We first worry about perturbations on the scale of the whole sphere. Thus we do a multipole expansion of the fields and look at the dipole term. Then we focus on a patch of the sphere small enough so that it can be approximated as flat and examine the fluctuations in complete generality. We find the results of these two calculations reassuring enough that we do not compute the stability under higher multipole perturbations of the sphere.

For the purposes of this calculation, we write out the Lagrangian in gory detail and eliminate all hats and tildes:

$$\begin{aligned} L = \int d\theta d\phi \sin\theta \{ \frac{1}{2}(\dot{A_\theta}^2 + \dot{A_\phi}^2) - \frac{1}{2}c\left[\frac{1}{\sin\theta}(\partial_\theta(A_\phi \sin\theta) - \partial_\phi A_\theta)\right]^2 + \frac{1}{2}c^{-1}(\dot{D}_\theta^2 + \dot{D}_\phi^2) \\ +12\sqrt{2}c^{-\frac{1}{2}}\dot{D}_\theta D_\phi + \frac{1}{2}c^{-1}\dot{\chi}^2 - \frac{1}{2}\left[(\partial_\theta\chi)^2 + (\frac{\partial_\phi\chi}{\sin\theta})^2\right] - 4\chi^2 \\ +3\sqrt{6}\epsilon^{ij}A_i\dot{D}_j - 3\sqrt{6}\chi\frac{1}{\sin\theta}\left[\partial_\theta(\sin\theta D_\phi) - \partial_\phi D_\theta\right] + 4\sqrt{2}c^{\frac{1}{2}}\chi\frac{1}{\sin\theta}\left[\partial_\theta(\sin\theta A_\phi) - \partial_\phi A_\theta\right]\} \end{aligned} \tag{3.9}$$

where

$$c = \frac{2}{9(\pi N)^{\frac{2}{3}} l_s^2}. \tag{3.10}$$

For later convenience, we have set $\tilde{A}_\phi = A_\phi \sin\theta$ and similarly for D_ϕ.

Since the system has rotational symmetry, there is no harm in taking the dipole to lie along the z axis. Then the perturbation has azimuthal symmetry so we drop ϕ derivatives in the equations of motion:

$$
\begin{aligned}
-\ddot{A}_\theta + 3\sqrt{6}c^{-1}\dot{D}_\phi &= 0 \\
c\partial_\theta\left(\frac{\partial_\theta(A_\phi\sin\theta)}{\sin\theta}\right) - \ddot{A}_\phi - 4\sqrt{2}c^{\frac{1}{2}}\partial_\theta\chi - 3\sqrt{6}\dot{D}_\theta &= 0 \\
-c^{-1}\ddot{D}_\theta - 12\sqrt{2}c^{-\frac{1}{2}}\dot{D}_\phi + 3\sqrt{6}\frac{\dot{A}_\phi}{\sin\theta} &= 0 \\
3\sqrt{6}\partial_\theta\chi - c^{-1}\ddot{D}_\phi + 12\sqrt{2}c^{\frac{1}{2}}\dot{D}_\theta - 3\sqrt{6}A_\theta &= 0 \\
-c^{-1}\ddot{\chi} + \frac{\partial_\theta(\sin\theta\partial_\theta\chi)}{\sin\theta} - 8\chi + 4\sqrt{2}c^{\frac{1}{2}}\frac{\partial_\theta(\sin\theta A_\phi)}{\sin\theta} - 3\sqrt{6}\frac{\partial_\theta(\sin\theta D_\phi)}{\sin\theta} &= 0
\end{aligned}
\tag{3.11}
$$

For the scalar and vector dipole perturbations we are investigating, the solutions take the form

$$
A_i(x^\mu) = A_i^o e^{i\omega t}\sin\theta;\;\; D_i(x^\mu) = D_i^o e^{i\omega t}\sin\theta;\;\; \chi(x^\mu) = \chi^o e^{i\omega t}\cos\theta; \tag{3.12}
$$

where A_i^o is a constant and i runs over θ and ϕ. These are the most general azimuthally symmetric scalar and vector dipole harmonics on the sphere. Plugging in this ansatz gives:

$$
\begin{aligned}
\omega^2 A_\theta^o + 3\sqrt{6}c^{-1}i\omega D_\phi^o &= 0 \\
(\omega^2 - 2c)A_\phi^o - 3\sqrt{6}i\omega D_\theta^o + 4\sqrt{2}c^{\frac{1}{2}}\chi^o &= 0 \\
3\sqrt{6}i\omega A_\phi^o + c^{-1}\omega^2 D_\theta^o - 12\sqrt{2}c^{-\frac{1}{2}}i\omega D_\phi^o &= 0 \\
-3\sqrt{6}i\omega A_\theta^o + 12\sqrt{2}c^{-\frac{1}{2}}D_\theta^o + c^{-1}\omega^2 D_\phi^o - 3\sqrt{6}\chi^o &= 0 \\
8\sqrt{2}c^{\frac{1}{2}}A_\phi^o - 6\sqrt{6}D_\phi^o + (c^{-1}\omega^2 - 10)\chi^o &= 0
\end{aligned}
\tag{3.13}
$$

The resulting system of equations can be put as usual in matrix form. In order for there to be nonzero solutions to these equations, the determinant of the resulting 5×5 matrix must be zero. Using this equation to solve for ω gives the following solutions

$$
\omega^2 = \frac{1}{N^{\frac{2}{3}}l_s^2}\{0, 0, 40.32, 1.37, 0.58\} \tag{3.14}
$$

The nonzero modes have positive energy and so the system is stable under such perturbations. There is no tachyonic mode. The energy scale of the resulting oscillations is again of the order of the characteristic energy scale identified in the previous sections. The $\omega = 0$ modes are a bit more complicated. We will discuss them in a moment since their complications have nothing to do with being on a sphere.

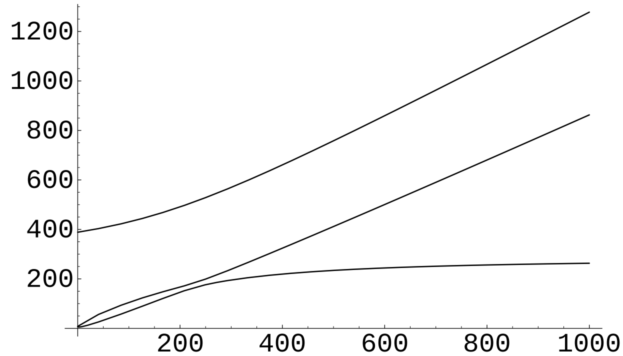

Figure 1: The three normal modes of the system; here we plot $c^{-1}\omega^2$ as a function of p^2

Next we examine perturbations with wavelengths smaller than the size of the sphere. Rather than doing a higher multipole expansion, we focus on a small patch of the sphere and approximate it as flat. The Lagrangian becomes

$$L = \int d^2z[\frac{1}{2}\dot{\vec{A}}^2 - \frac{1}{2}c(\nabla \times \vec{A})^2 + \frac{1}{2}c^{-1}\dot{\vec{D}}^2 + 12\sqrt{2}c^{-\frac{1}{2}}\dot{D}_1 D_2 + \frac{1}{2}c^{-1}\dot{\chi}^2$$
$$-\frac{1}{2}(\nabla\chi)^2 - 4\chi^2 + 3\sqrt{6}\epsilon^{ij}A_i\dot{D}_j - 3\sqrt{6}\chi(\nabla \times \vec{D}) + 4\sqrt{2}c^{\frac{1}{2}}\chi(\nabla \times \vec{A})] \quad (3.15)$$

Here the coordinates z_i are dimensionless. The dimensions can be restored by multiplying by the radius of the sphere $R = (\pi N)^{2/3}l_s/2$. Therefore the analysis is valid for dimensionless momenta $p \gg 1$. We guess a plane wave solution, $A_i = A_i^o e^{i\omega t + ipz_1}$, with similar expressions for the other fields. Since ω will be a function only of p^2, we lose nothing by assuming the fields are independent of z_2. Then the equations of motion become

$$\begin{pmatrix} \omega^2 & 0 & 0 & 3\sqrt{6}i\omega & 0 \\ 0 & \omega^2 - cp^2 & -3\sqrt{6}i\omega & 0 & -4\sqrt{2}c^{\frac{1}{2}}ip \\ 0 & 3\sqrt{6}i\omega & c^{-1}\omega^2 & -12\sqrt{2}c^{-\frac{1}{2}}i\omega & 0 \\ -3\sqrt{6}i\omega & 0 & 12\sqrt{2}c^{-\frac{1}{2}}i\omega & c^{-1}\omega^2 & 3\sqrt{6}ip \\ 0 & 4\sqrt{2}c^{\frac{1}{2}}ip & 0 & -3\sqrt{6}ip & c^{-1}\omega^2 - p^2 - 8 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ D_1 \\ D_2 \\ \chi \end{pmatrix} = 0 \quad (3.16)$$

The dispersion relations for the normal modes are shown in figures 1 and 2.

Since the system is quadratic, when written in terms of the normal modes it is a set of uncoupled oscillators whose energies are quantized as $\omega(n + \frac{1}{2})$. However, the values for ω given above are not the whole story because although $\omega = 0$ appears four times as a solution there are only two corresponding eigenvectors. These are pure gauge and are physically equivalent to $A_i = D_i = \chi = 0$. The fact that there is not a complete set of

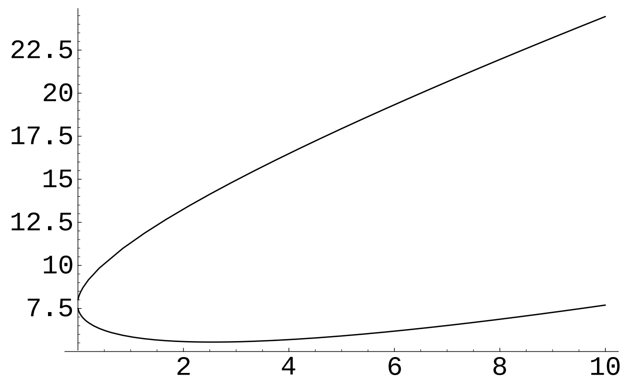

Figure 2: A close-up of the two lower modes near $p = 0$; again we plot $c^{-1}\omega^2(p^2)$

eigenvectors means that we must also consider solutions which are linear in time[5]. A simple example where such solutions are necessary is solving $\ddot{x} = 0$. If we guess a solution of the form $x = x_o e^{i\omega t}$ as we did above, then we find $\omega = 0$ is a double root of the characteristic equation. But there is only one corresponding eigenvector since it is a one-dimensional space, so we must supplement our solution with a solution which is linear in time. Getting back to our problem, one can picture the solutions this way: at each momentum there is a five-dimensional configuration space; a point in this space represents a state of the system. There is a potential on the space which is a harmonic oscillator potential in three of the directions and flat in the other two directions. The only subtlety is that the displacement of the system along the flat directions can be gauged away; all that matters is the velocity in these directions. Also, the constraint equation coming from the A_0 equation of motion prevents the system from moving along one of the flat directions. Thus there is only a one-parameter family of physically distinct non-oscillatory solutions.

Quasiparticles. We will now analyze the non-oscillatory solutions. These solutions are linear in time for the potentials but all physical quantities are constant in time. For this reason, we find it more convenient to work in terms of the physical fields. The genuine electric and magnetic fields we denote E_A and B_A, while the analogous quantities for the electron fluid we call E_D and B_D. These quantities have straightforward physical interpretations: B_D is proportional to the excess density of electrons, while $\epsilon_{ij} E_D^j$ is proportional to the velocity of the electrons.

Writing the equations of motion in terms of the fields and dropping time derivatives

[5]Thanks to Boaz Nash for pointing this out and providing the example which follows.

of the fields (since we're looking for static solutions) gives the following momentum space equations:

$$\begin{aligned} i\vec{p}\cdot\vec{E_A} + 3\sqrt{6}B_D = 0 \\ -ic^{\frac{1}{2}}\vec{p}B_A + 4\sqrt{2}i\vec{p}\chi - 3\sqrt{6}c^{-\frac{1}{2}}\vec{E_D} = 0 \\ 3\sqrt{6}i\vec{p}\chi - 12\sqrt{2}c^{-\frac{1}{2}}\vec{E_D} + 3\sqrt{6}\vec{E_A} = 0 \\ -(\vec{p}^2+8)\chi + 4\sqrt{2}c(\frac{1}{2})B_A - 3\sqrt{6}B_D = 0 \end{aligned} \quad (3.17)$$

The first of these equations is the constraint from the A_0 equation of motion, eq. (2.85).

As in [2], we are interested in static solutions with the properties of Laughlin's quasiparticles. In our picture, quasiparticles can be thought of as vortices frozen in the electron fluid. The conserved quantity (2.88) is therefore non-zero but equal to a time-independent function appropriate for a localized vortex-type solution. The static solutions will be quantized. In terms of the fields, the conserved quantity (in momentum space) is proportional to

$$\widetilde{CQ}(\vec{p}) = -12\sqrt{2}B_D + 3\sqrt{6}c^{\frac{1}{2}}B_A + ic^{-\frac{1}{2}}\vec{p}\cdot\vec{E_D}. \quad (3.18)$$

Note that the value of this quantity at each point in space and not just its integral is conserved.

To get a sensible quantization condition, we follow [2]. From the Lagrangian (2.71), the canonical momentum density conjugate to X_s^i is

$$\Pi_i = \delta L/\delta\dot{X}_s^i = -\eta_0 m g^{00} g_{ij}\partial_t X_s^j - \frac{e\eta_0 B}{2}\epsilon_{ij}X_s^j + e\eta_0\hat{A}_i \quad (3.19)$$

Then the momentum per particle P_i is the momentum density Π_i divided by the density of electrons η_0. In order to quantize, we impose the condition

$$\oint P_i dX^i = 2\pi(k+\frac{1}{2}) \quad (3.20)$$

along any closed path in space. In terms of the canonically normalized D field, the positions of the electrons are given by

$$X_s^i = z^i + \frac{12\sqrt{2}\pi^{2/3}l_s^{1/2}}{N^{1/3}K^{1/2}}\epsilon^{ij}D_j. \quad (3.21)$$

Then, to first order, the quantization condition becomes

$$B\times area + \int dz_1 dz_2 \frac{\pi^{2/3}N^{2/3}l_s^{1/2}}{2K^{1/2}}CQ(z) = 2\pi(k+\frac{1}{2}) \quad (3.22)$$

where k is an integer, and we have turned the line integral into a volume integral. The zeroth order term gives the quantization of the magnetic flux; the first order term gives the quantization of the quasiparticle excitations.

The quantization condition for the quasiparticles thus becomes

$$\int dz_1 dz_2 CQ(z) = \frac{4\pi^{1/3}K^{1/2}}{N^{2/3}l_s^{1/2}}k \tag{3.23}$$

As in [2], we interpret this equation to mean that the introduction of quasiparticles can only change k by an integer, regardless of the region of integration. Thus we choose for our quasiparticle a delta function of quantized strength. This will lead in general to to field configurations that are singular at the position of the vortex. As noted in [2], the divergence is smoothed out if terms higher order in the fields are included, leading to non-linear terms in the equations; we do not find the divergence annoying enough to smooth it out. The smallest size quasiparticle sitting at the origin is given by $CQ(z) = (4\pi^{1/3}K^{1/2}/N^{2/3}l_s^{1/2})\delta^2(z)$. In momentum space, this condition becomes

$$\widetilde{CQ}(\vec{p}) = -12\sqrt{2}B_D + 3\sqrt{6}c^{\frac{1}{2}}B_A + ic^{-\frac{1}{2}}\vec{p}\cdot\vec{E_D} = \frac{4\pi^{1/3}K^{1/2}}{N^{2/3}l_s^{1/2}} \tag{3.24}$$

Solving the equations of motion in momentum space, we find the solution

$$\begin{aligned}
\vec{E_A} &= \frac{4\pi^{1/3}K^{1/2}}{N^{2/3}l_s^{1/2}}\frac{i\sqrt{3}(6-p^2)\vec{p}}{2(162+15p^2+2p^4)} \\
\vec{E_D} &= \frac{4\pi^{1/3}K^{1/2}c^{\frac{1}{2}}}{N^{2/3}l_s^{1/2}}\frac{i9\vec{p}}{162+15p^2+2p^4} \\
B_A &= \frac{4\pi^{1/3}K^{1/2}c^{-\frac{1}{2}}}{N^{2/3}l_s^{1/2}}\frac{(9+2p^2)\sqrt{6}}{162+15p^2+2p^4} \\
B_D &= \frac{4\pi^{1/3}K^{1/2}}{N^{2/3}l_s^{1/2}}\frac{6p^2-p^4}{6\sqrt{2}(162+15p^2+2p^4)} \\
\chi &= \frac{4\pi^{1/3}K^{1/2}}{N^{2/3}l_s^{1/2}}\frac{(18+p^2)\sqrt{3}}{2(162+15p^2+2p^4)}
\end{aligned} \tag{3.25}$$

Note that such a solution is not really allowed on the sphere because the total number of electrons is not conserved; our real interest is forming a particle/hole pair. To understand the above solution, note that B_D is the most badly behaved at large p and thus at small z; the singular part is proportional to a delta function in position space. Thus there is a finite charge at a point in z-space. As discussed in [2], this point actually takes up a finite area in real space. What is going on in the case of a hole is that the electron fluid is getting moved radially outward to make the hole. We find that the quasiparticle has charge ν as expected. To see this, note that the change in the density of electrons in real space is given by

$$\delta\eta(z) = \frac{3\sqrt{2}\pi^{-1/3}l_s^{1/2}K^{1/2}}{N^{1/3}}B_D(z) = -\nu\delta^2(z) + ... \tag{3.26}$$

Thus there is a hole of electrons at $z = 0$ with charge ν. The only other field that blows up is E_A, but we expect the electric field to blow up near a point-like charge. E_D is radial, which by (2.3) means that the electrons are moving in the angular direction. The magnetic force arising due to the motion of the electrons balances the Coulomb force and the force due to the scalar field. Thus the quasiparticle is a vortex in the electron fluid.

We would like to compute the energy of a single quasiparticle. The Hamiltonian in momentum space is

$$\int \frac{d^2p}{(2\pi)^2}[\frac{1}{2}|E_A|^2+\frac{1}{2}c|B_A|^2+\frac{1}{2}c^{-1}|E_D|^2+\frac{1}{2}(p^2+4)|\chi|^2+3\sqrt{6}\frac{\chi^*B_D+\chi B_D^*}{2}+4\sqrt{2}\frac{\chi^*B_A+\chi B_A^*}{2}] \tag{3.27}$$

which for this solution magically simplifies to

$$\int \frac{d^2p}{(2\pi)^2}\frac{16K\pi^{2/3}}{N^{4/3}l_s}\frac{-9}{2(162+15p^2+2p^4)} \tag{3.28}$$

Note that this is clearly negative, and also that we don't need to impose an ultraviolet cutoff; although various fields blow up at large p, they cancel in such a way that the energy is finite. Physically, this is because the forces between electrons at short distances become small due to cancellations between the scalar and the electromagnetic field [4].

The result of integrating gives

$$E_{qp} = -\frac{1.26K}{\pi^{1/3}N^{4/3}l_s} = -\frac{1.26\nu}{(\pi N)^{1/3}l_s} \tag{3.29}$$

This energy is of the scale predicted in [1] but it is negative!

Because the equations of motion are linear, if we can have one quasiparticle we can have as many as we want at any locations we want. The quasiparticles do not move in this solution. In order to have a quasiparticle/quasihole pair, one at the origin and one at x_0, we multiply the right side of the equation for the conserved quantity in momentum space by $(1 - e^{i\vec{p}\cdot\vec{x}_0})$. Clearly a particle-hole pair at zero separation has zero energy, and a pair at large separation has twice the energy of one quasiparticle, so the energy is lowered by creating particle-hole pairs and separating them. However, we note again that there are static solutions with the particle and hole at any separation.

This seems like an instability. However, note that these are all viable static solutions and have different values of the conserved quantitity. The issue is whether adding higher-order corrections to the Lagrangian will allow the system to move to a state with more quasiparticles, lowering its energy. We speculate about whether this happens in the conclusions.

4 Conclusions

We have shown unambiguously that the Quantum Hall Soliton is stable to small perturbations and that it contains fractionally charged excitations. The puzzling question is whether the negative energy associated with a quasiparticle indicates an instability.

One possibility is that the symmetry which leads to conservation of quasiparticles will remain good when we add higher terms to the Lagrangian. The gauge symmetry which leads to the conserved quantity is the symmetry of the electron fluid under area-preserving diffeomorphisms. When discretized, this symmetry becomes a $U(K)$ symmetry. It was conjectured in [2] that electrons in a magnetic field have an exact $U(K)$ symmetry when the kinetic term is dropped. We suspect that here because the electrons are connected to strings, they only truly have the permutation symmetry and not the full $U(K)$ symmetry. What could happen for example is that the strings could become excited while the brane goes into a lower- energy configuration with more quasiparticles.

We do not believe that this is actually an instability for the following reason. Let us return to our example of a particle subject to an upside-down oscillator potential and a magnetic field and continue to ignore the kinetic term. A solution exists for the particle orbiting at any radius, and the larger the radius the more negative the energy. Now consider weakly coupling the particle to an oscillator with higher frequency by the coupling $\epsilon\vec{x}_{part} \cdot \vec{x}_{osc}$. This coupling destroys angular momentum conservation and one might think it would allow the system to slide down to lower and lower energy states, giving energy to the oscillator. But the only effect of the coupling is to slightly mix the oscillator mode with the cyclotron mode, INCREASING the frequency of the cyclotron mode. We believe that this example is strongly analagous to the situation of a stable brane with negative energy states coupled to some strings, and for this reason we believe the effect of adding string oscillations will be to slightly change the frequencies of the normal modes we have found. In particular, we expect it to give our static solutions a small positive ω^2.

We briefly mention other possible ways the $D2$-brane could give up its energy. The $D6$-brane is infinite in 6 directions and thus has a large number of low-frequency modes. These basically act as a frictional force on the electrons and will eventually steal the energy. As was discussed in [1], we could compactify the directions parallel to the $D6$-brane to remove these modes. Finally, the $D2$-brane could emit closed string gravitons. Again, in the large N limit we are working in, the "closed string" coupling constant is small and the time scale for this process is large.

Unfortunately, the limitations of our method prevent us from making these arguments more rigorous.

5 Appendix

In this Appendix, we review the relation between the "Eulerian" and "Lagrangian" descriptions of the $D0$-brane fluid following [8][9].

Let us begin with the "Eulerian" description. We choose to work in the $A_0 = 0$ gauge and split the gauge field into its background value and the fluctuations

$$A_i(X) = -\frac{1}{2} B_{ij} X^j + A_i^f. \tag{5.1}$$

The field strength is given by

$$F_{0i} = \partial_t A_i = \partial_t A_i^f \tag{5.2}$$

and

$$F_{ij} = B_{ij} + F_{ij}^f. \tag{5.3}$$

Next we do a coordinate transformation of the form

$$X^0 = t, \;\; X^i = y^i + \Theta^{ij} \hat{A}_j(y,t) \tag{5.4}$$

so that

$$\tilde{F}_{0i} = F_{0j} \frac{\partial X^j}{\partial y^i} + F_{kl} \frac{\partial X^k}{\partial t} \frac{\partial X^l}{\partial y^i} = 0 \tag{5.5}$$

and

$$\tilde{F}_{ij} = F_{kl} \frac{\partial X^k}{\partial y^i} \frac{\partial X^l}{\partial y^j} = B_{ij}. \tag{5.6}$$

Eq. (6.6) can be satisfied to quadratic order in the fluctuations if we set

$$\hat{A}_i = A_i^f + \frac{1}{2} \Theta^{kl} (2 A_l^f \partial_k A_i^f + A_k^f \partial_i A_l^f) + h.o. \tag{5.7}$$

This is the Seiberg-Witten map between ordinary and non-commutative gauge fields to leading order in Θ. In this way the coordinate frame t, y^1, y^2 is determined up to infinitesimal area preserving diffeomorphisms of the form

$$y'^i = y^i + \Theta^{ij}\partial_j \hat{\lambda}(t, y). \tag{5.8}$$

Under such a tranformation the non-commutative gauge field transforms as in eq. (2.15) but $\tilde{F}_{ij} = B_{ij}$ remains unchanged. To satisfy equation (6.5) then, we do a particular time-dependent area preserving diffeomorphism. To satisfy $\tilde{F}_{0i} = 0$ to quadratic order in the fluctuations, we must set

$$\hat{A}_i \to \hat{A}_i + \partial_i \hat{\lambda} \tag{5.9}$$

with $\hat{\lambda}$ satisfying

$$\partial_t \hat{\lambda} = \frac{1}{2}\Theta^{ij} A_i^f \partial_t A_j^f + h.o. \tag{5.10}$$

Still, there is left over coordinate freedom consisting of time independent area preserving diffeomorphisms.

Alternatively, we may leave A_0 to be non-zero in the Eulerian description. Then we can satisfy eq. (6.5) for $\hat{A}_i$ given by eq. (6.7) if we set

$$A_0 = -\frac{1}{2}\Theta^{ij} A_i^f \partial_t A_j^f + h.o. \tag{5.11}$$

By doing then a time dependent commutative gauge transformation, we can reach again the $A_0 = 0$ gauge. Thus we must set

$$\partial_t \lambda = \frac{1}{2}\Theta^{ij} A_i^f \partial_t A_j^f + h.o. \tag{5.12}$$

The two gauge parameters are related to each other as in the Seiberg-Witten map

$$\hat{\lambda} = \lambda + O(\Theta). \tag{5.13}$$

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