

# Reissner-Nordström Black Holes in the Inverse Electrodynamics Model

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We study electric and magnetic monopoles in static, spherically symmetric and constant curvature geometries in the context of the inverse electrodynamics model. We prove that this U(1) invariant Lagrangian density is able to support the standard metric of a Reissner-Nordström Black Hole, but with more complex thermodynamical properties than in the standard case. By employing the Euclidean Action approach we perform a complete analysis of its phase space depending on the sign and singularities of the heat capacity and the Helmholtz free energy.

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## I. INTRODUCTION

In General Relativity coupled with the usual U(1) invariant Electrodynamics theory, the Reissner-Nordström black-hole (BH) solution arises, corresponding to a massive, charged, non-rotating and spherically symmetric body [1, 2]. This kind of solution has been widely studied in the last decades (*c.f.* Refs. [3–5]). Nevertheless the divergence of self-energy of point charges (like electrons) in the standard Electrodynamics theory has suggested that modified Electrodynamics theories might be required in order to circumvent this shortcoming. Non-linear models have also been studied from the point of view of effective Lagrangians which attempt to describe Quantum Electrodynamics [6]. Some important examples of these kinds of theories are the Born-Infeld [7, 8] and the Euler-Heisenberg models [9–14]. Following this line of reasoning, in the last years, different works have studied modified Electrodynamics models coupled with gravity [15]. In particular models providing static and spherically symmetric solutions for electrostatic spherically symmetric fields have drawn remarkable attention (*c.f.* [16] and references therein).

On the other hand, the study of the thermodynamics properties of BH solutions began in the 1970's with the attainment of the four laws of BHs dynamics [17]. These mechanics laws seem very similar to the four laws of Thermodynamics, where the BH mass, the area of the horizon and the surface gravity play analogous roles to the energy, the entropy and the temperature, respectively. One approach in order to compute the thermodynamical properties of a BH solution is the Euclidean Action Method [18, 19]. The Euclidean approach exhibits some difficulties when is applied to General Relativity.

Except in special cases it is generally impossible to represent an analytic spacetime as a Lorentzian section of a four-complex-dimensional manifold with a complex metric which possesses a Euclidean section. Therefore there is not a general prescription for analytically *continuing* Lorentzian signature metrics to Riemannian metrics. However, in static metrics on which we shall focus, the aforementioned *continuation* procedure can be done. Nevertheless, even if possible to be performed, there are not any theorems guaranteeing the analyticity of the obtained quantities (for further details, *c.f.* Ref. [20], [21]).

The paper is organized as follows: in Section II we introduce the Inverse Electrodynamics Model (IEM) and the static, spherically symmetric solutions supported therein by electric and magnetic monopoles. In Section III we apply the Euclidean Method in order to distinguish the different thermodynamics phases of the solutions, defined in terms of their stability, and we shall compare the phase diagrams with the standard electrodynamics model counterparts. The appearance of a new thermodynamical phase, absent in the standard case, shall be extensively discussed. In Section IV we then perform a classification of the BH configurations depending on the phase transitions that they present. Finally, in Section V we summarize the main results and conclusions.

Unless otherwise specified, Planck units, ( $G = c = k_B = \hbar = 4\pi\epsilon_0 = 1$ ) will be used throughout this paper, Greek indices run from 0 to 3. The symbol  $\nabla$  denotes the standard covariant derivative and the signature  $+, -, -, -$  is used.

## II. INVERSE ELECTRODYNAMICS MODEL

In this section, we shall show the static and spherically symmetric solutions for the IEM in General Relativity. Thus the action is given by

$$S = S_g + S_{U(1)}, \quad (1)$$

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where  $S_g$  and  $S_{U(1)}$  denote the gravitational and matter terms of the action, respectively. The usual gravitational action term takes the form

$$S_g = \frac{1}{16\pi} \int d^4x \sqrt{|g|} (R - 2\Lambda), \quad (2)$$

being  $g$  the determinant of the metric  $g_{\mu\nu}$ ,  $R$  the scalar of curvature and  $\Lambda$  a cosmological constant.

On the other hand, we assume that the matter term of the action,  $S_{U(1)}$ , is given by the IEM Lagrangian density  $\mathcal{L}(X, Y)$ , namely,

$$\mathcal{L}(X, Y) = -\frac{1}{8\pi} X + \frac{\eta}{8\pi} \frac{Y^2}{X}, \quad (3)$$

which is a function of the Maxwell invariants  $X$  and  $Y$ , defined as

$$X \equiv -\frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad Y \equiv -\frac{1}{2} F_{\mu\nu} F^{*\mu\nu}, \quad (4)$$

being  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  the usual electromagnetic tensor and  $F_{\mu\nu}^* \equiv \frac{1}{2} \sqrt{|g|} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$ , with  $\epsilon_{\mu\nu\alpha\beta}$  the Levi-Civita symbol. In terms of the Lagrangian density  $\mathcal{L}(X, Y)$ , the matter term of the action (1) takes the form

$$S_{U(1)} = - \int d^4x \sqrt{|g|} \mathcal{L}(X, Y). \quad (5)$$

This action is parity-invariant and can be interpreted as a perturbation of the standard Electrodynamics theory ( $\mathcal{L}(X, Y) \sim X$ ) for a small enough value of the parameter  $\eta$ . Moreover, provided  $F_{\mu\nu}$  represents an electric monopole with a null magnetic field, the standard Lagrangian and the standard point-like solutions are recovered as one might expect. Another interesting property of the IEM is its conformal invariance. In fact, the trace of the associated energy-momentum tensor vanishes as in standard Electrodynamics:

$$T \equiv T^\mu{}_\mu = g^{\mu\nu} T_{\mu\nu} = -\frac{2g^{\mu\nu}}{\sqrt{|g|}} \frac{\delta S_{U(1)}}{\delta g^{\mu\nu}} = 0. \quad (6)$$

In this paper we restrict ourselves to the study of static and spherically symmetric solutions. Hence, for the metric tensor let us consider the most general ansatz for static and spherically symmetric scenarios,

$$ds^2 = \lambda(r) dt^2 - \frac{1}{\mu(r)} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (7)$$

where the functions  $\lambda(r)$  and  $\mu(r)$  depend solely on  $r$  in order to ensure staticity and spherical symmetry. Besides, with this metric (7) we consider an ansatz for the electromagnetic tensor

$$F_{01} = -F_{10} = E(r), \quad F_{23} = -F_{32} = -B(r)r^2 \sin\theta, \quad (8)$$

being identically null the other components, and  $E(r)$  and  $B(r)$  functions on  $r$ . In Minkowski spacetime, where

$\lambda(r)$  and  $\mu(r)$  equal to 1, (8) is the electromagnetic tensor for radial electric and magnetic fields  $E(r)$  and  $B(r)$ , respectively [22]. For this reason, we shall refer to these functions as ‘‘electric’’ and ‘‘magnetic’’ fields.

With the metric (7), we can raise or lower indices in (8), and then the gauge invariants (4) can be rewritten in terms of the electric and magnetic fields as follows

$$X = \frac{\mu(r)}{\lambda(r)} E(r)^2 - B(r)^2, \quad Y = 2\sqrt{\frac{\mu(r)}{\lambda(r)}} E(r) \cdot B(r). \quad (9)$$

Furthermore, with (7) we get the scalar curvature  $R$  as a function on the coefficients  $\lambda(r)$  and  $\mu(r)$ :

$$R(r) = \frac{1}{2\lambda(r)^2 r^2} [\lambda'(r)\mu'(r)\lambda(r)r^2 + 2\mu(r)\lambda''(r)\lambda(r)r^2 - \lambda'(r)^2\mu r^2 + 4r\mu(r)\lambda'(r)\lambda(r) + 4r\mu'(r)\lambda(r)^2 - 4\lambda(r)^2 + 4\lambda(r)^2\mu(r)], \quad (10)$$

where prime denotes derivative with respect to  $r$ .

By performing variations of the total action (1) with respect to the metric tensor, we achieve the Einstein field equations in metric formalism,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} + 8\pi T_{\mu\nu} = 0, \quad (11)$$

where  $R_{\mu\nu}$  holds for the Ricci Tensor. Then, by taking the trace in the previous expression, we reach  $R = 4\Lambda$ , due to the traceless character of the Electromagnetic energy-momentum tensor.

Furthermore, by replacing the Lagrangian density (3), and the metric and electromagnetic tensors (7) and (8) in the energy-momentum definition (6), we obtain the non zero components of the energy-momentum tensor

$$T_0^0(r) = T_1^1(r) = -T_2^2(r) = -T_3^3(r) = -\frac{1}{8\pi} \left( 1 + 2\frac{B^2}{X} \right) \left( X + \eta \frac{Y^2}{X} \right), \quad (12)$$

With the components of the energy-momentum tensor above and the metric tensor (7), one defines the quantity  $\zeta(r) \equiv \lambda(r)/\mu(r)$ . Then by subtracting the field equations (11) corresponding to  $\mu = \nu = t$  and  $\mu = \nu = r$ , one reaches the condition

$$\zeta'(r) = 0, \quad (13)$$

i.e., the quantity  $\lambda(r)/\mu(r)$  is a constant, which can be fixed to one by performing a time reparametrization. In others words, equation (13) is equivalent to

$$\lambda(r) = \mu(r). \quad (14)$$

With this expression, we could simplify the gauge invariants (9), which now read as

$$X = E(r)^2 - B(r)^2, \quad Y = 2E(r) \cdot B(r), \quad (15)$$

i.e., the usual gauge invariants in standard Electrodynamics are recovered. Moreover, we can replace (14) in the rest of Euler field equations (11), achieving the expressions

$$-r\lambda'(r) - \lambda(r) + 1 + 8\pi T_0^0(r)r^2 = 0, \quad (16)$$

$$-16\pi T_2^2(r)r + 2\lambda'(r) + r\lambda''(r) = 0, \quad (17)$$

where (16) is obtained from the equation (11) with  $\mu = \nu = r$  and, on the other hand, equation (17) is proportional to (11) with  $\mu = \nu = \theta$  or  $\mu = \nu = \phi$  (both equations are in fact equivalents). Finally, replacing (14) in (12), we reach that the non-vanishing components of the energy-momentum tensor can be rewritten as

$$\begin{aligned} T_0^0(r) &= T_1^1(r) = -T_2^2(r) = -T_3^3(r) \\ &= -\frac{E^2 + B^2}{8\pi} \left[ 1 + 4\eta \frac{E^2 B^2}{(E^2 - B^2)^2} \right]. \end{aligned} \quad (18)$$

The general solution of the field equations system (16)-(17) reads

$$\lambda(r) = 1 - \frac{2M}{r} - \frac{8\pi}{r} \int_r^\infty x^2 T_0^0(x) dx + \frac{1}{3} \Lambda r^2, \quad (19)$$

where  $M$  is an integration constant, that can be identified as the BH mass. The metric function (19) can be rewritten in terms of an ‘‘external energy function’’, which is defined as

$$\varepsilon_{ex}(r) = -4\pi \int_r^\infty x^2 T_0^0(x) dx. \quad (20)$$

This external energy represents the energy provided by the U(1) fields  $E(r)$  and  $B(r)$  outside a sphere of radius  $r$  (see Ref. [16]).

Considering now  $\mathcal{L}(X, Y)$  and its derivatives, the associated Euler field equations, together with the Bianchi identities for the electromagnetic field, take the form

$$\nabla_\mu (\mathcal{L}_X F^{\mu\nu} + \mathcal{L}_Y F^{*\mu\nu}) = 0, \quad \nabla_\mu F^{*\mu\nu} = 0. \quad (21)$$

These generalized Maxwell equations can be expressed for static and spherically symmetric solutions of the IEM with the electromagnetic tensor (8) as

$$r^2 B(r) = Q_t, \quad (22)$$

$$\begin{aligned} r^2 \left[ 1 + 4\eta \left( \frac{E(r)B(r)}{E(r)^2 - B(r)^2} \right)^2 \right] E(r) \\ = 4\eta \frac{E(r)B(r)}{E(r)^2 - B(r)^2} Q_t + Q_c, \end{aligned} \quad (23)$$

with  $Q_c$  and  $Q_t$  the sources of the generalized Maxwell equations (21), i.e., the current and the topological charges respectively. It is easy to see that equation (23) possesses solutions for electric fields that decrease as  $r^{-2}$ .

Thus, provided that we impose  $E(r) = Q_e/r^2$ , and using equation (22), we achieve a equation for this parameter  $Q_e$

$$Q_e \left[ 1 + 4\eta \left( \frac{Q_e Q_t}{Q_e^2 - Q_t^2} \right)^2 \right] = 4\eta \frac{Q_e Q_t^2}{Q_e^2 - Q_t^2} + Q_c. \quad (24)$$

From this equation, one can obtain the parameter  $Q_e$  as a function of  $\eta$  and the charges  $Q_c$  and  $Q_t$ , and seeing that  $Q_e$  coincides with the current charge  $Q_c$  in standard Electrodynamics ( $\eta = 0$ ). The analytic expression of this parameter is not trivial, but for a small enough  $\eta$ , could be expressed as

$$Q_e = Q_c - 4\eta \frac{Q_c Q_t^4}{(Q_c^2 - Q_t^2)^2} + \mathcal{O}(\eta^2), \quad (25)$$

whereas if the topological charge is smaller than the current one, the expression reads

$$Q_e = Q_c \left[ 1 - 4\eta \left( \frac{Q_t}{Q_c} \right)^4 + \mathcal{O} \left( \frac{Q_t}{Q_c} \right)^6 \right]. \quad (26)$$

In the following, instead of using as charges  $\{Q_c, Q_t\}$  we choose  $\{Q_e, Q_m\}$  (being  $Q_m \equiv Q_t$ ), denoted as ‘‘electric’’ and ‘‘magnetic’’ charges. This election has the important advantage that the electric and magnetic fields read directly as  $E = Q_e/r^2$  and  $B = Q_m/r^2$  and therefore the interpretation of the following results.

In terms of these charges, the non-null components of the electromagnetic tensor (18) read

$$\begin{aligned} T_0^0(r) &= T_1^1(r) = -T_2^2(r) = -T_3^3(r) \\ &= -\frac{1}{8\pi} \frac{Q_e^2 + Q_m^2}{r^4} \left[ 1 + 4\eta \left( \frac{Q_e Q_m}{Q_e^2 - Q_m^2} \right)^2 \right] \end{aligned} \quad (27)$$

Then we can replace these components in the expression (20), and rewrite the external energy in the form

$$\varepsilon_{ex}(r) = \frac{Q_e^2 + Q_m^2}{2r} \left[ 1 + 4\eta \left( \frac{Q_e Q_m}{Q_e^2 - Q_m^2} \right)^2 \right]. \quad (28)$$

One can realize that, as also occurs in the standard case ( $\eta = 0$ ) the external energy diverges at the origin, i.e., the total energy from the U(1) fields is divergent. Furthermore, we can replace (28) in the expression (19), so the metric parameter  $\lambda(r)$  can be rewritten as

$$\lambda(r) = 1 - \frac{2M}{r} + \frac{k_1}{r^2} + \frac{1}{3} \Lambda r^2, \quad (29)$$

where  $k_1$  is defined as

$$k_1 \equiv (Q_e^2 + Q_m^2) \left[ 1 + 4\eta \left( \frac{Q_e Q_m}{Q_e^2 - Q_m^2} \right)^2 \right]. \quad (30)$$

The obtained metric corresponds to a Reissner-Nordström-like with a scalar curvature  $R = 4\Lambda$ , and a

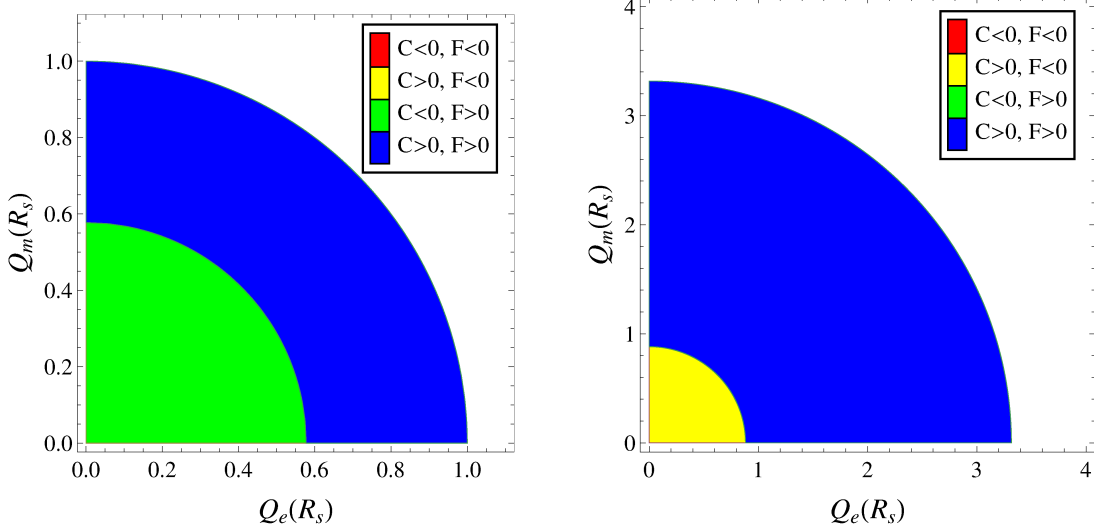


Figure 1: Phase diagrams of BH solutions with  $\eta = 0$  (usual Electrodynamics Lagrangian) corresponding to  $r_h = R_s$ , in flat spacetime ( $\Lambda = 0$ ) (left panel) and AdS spacetime with  $\Lambda = 10R_s^{-2}$  (right panel), being  $R_s$  the Schwarzschild radius of an object with a solar mass,  $R_s \simeq 10^{38} l_p$ . The electric and magnetic charges are also expressed in  $R_s$  Planck charges. In the flat spacetime case, two different phases exist: in blue both  $C$  and  $F$  are positive, while in green  $C < 0$  and  $F > 0$ . For AdS scenario  $\Lambda = 10R_s^{-2}$  there is a phase with  $C > 0$  and  $F < 0$  (yellow) and again a phase with both quantities positive. The phase with both  $C$  and  $F$  negative does not hold on for the usual Electrodynamics theory. Regions in white correspond to either negative temperature  $T$ , entropy  $S$  or mass parameter  $M$ . The diagram is represented solely for positive values of the charges; however, under the change  $Q_e \rightarrow -Q_e$  or  $Q_m \rightarrow -Q_m$  the diagram would be completely symmetric.

modified charge term equal to  $k_1$  which in the standard case ( $\eta = 0$ ) provides the well-known sum of squares of charges  $Q_e^2 + Q_m^2$ . Once the metric parameter  $\lambda(r)$  for the Reissner-Nordström-like solution has been obtained, the horizons structure for this solution can be determined. In order to obtain the radii of the horizons, one has to calculate the roots of  $\lambda(r)$  or, equivalently, satisfying the condition

$$M - \frac{r_h}{2} - \frac{1}{6}\Lambda r_h^2 = \varepsilon_{ex}(r_h), \quad (31)$$

whose solutions may provide in general one external (event) horizon and one internal horizon. Note that the external energy could be positive or negative depending on the sign of  $k_1$ . We are just interested in the anti-de Sitter (AdS) case  $\Lambda > 0$ , since otherwise ( $\Lambda < 0$ ) some problems of normalization of the temporal Killing  $\partial_t$  arise [24]. Thus, the value of the external horizon yields [23]

$$r_h = \frac{1}{2} \left( \sqrt{x} + \sqrt{-\frac{6}{\Lambda} - x + \frac{12M}{\Lambda\sqrt{x}}} \right), \quad (32)$$

with

$$x = \left( \frac{1 + 4\Lambda k_1}{\Lambda} \right) \sqrt[3]{\frac{2}{y}} + \frac{3}{\Lambda} \sqrt[3]{\frac{y}{32}} - \frac{2}{\Lambda}, \quad (33)$$

and

$$y = 2 + 36\Lambda M^2 - 24\Lambda k_1 + \sqrt{(2 + 36\Lambda M^2 - 24\Lambda k_1)^2 - 4(1 + 4\Lambda k_1)^3}. \quad (34)$$

Then, using (31) we can write the BH mass as a function of the external horizon radius  $r_h$ , the charge term  $k_1$  and the cosmological constant  $\Lambda$ , provided that at least one horizon is present, as

$$M(r_h) = \frac{r_h}{2} \left( 1 + \frac{k_1}{r_h^2} + \frac{1}{3}\Lambda r_h^2 \right). \quad (35)$$

If we assume both  $k_1$  and  $\Lambda$  positive (as in the standard AdS case), the function  $M(r_h)$  has a minimum at  $r_{h \min} = \sqrt{\Lambda} \left( \sqrt{1 + 4k_1/\Lambda^2} - 1 \right) / 2$ . This means that provided the mass of the configuration is small enough, no horizon appears and then such configuration would not constitute a proper BH configuration. However, if  $k_1$  takes a negative value and  $\Lambda$  is non negative, the ranges of values of  $M(r_h)$  covers entirely the interval  $[0, \infty)$ . Then, for any positive mass value, it is possible to have a BH solution with at least one horizon.

### III. THERMODYNAMICS ANALYSIS IN AdS SPACE

In this section, we shall apply the so-called Euclidean Action method [18] in order to obtain a thermodynamics analysis of the Reissner-Nordström-like solution corresponding to the IEM model defined by the Lagrangian density (3). We shall focus on the AdS space case ( $\Lambda > 0$ ), in order to avoid the normalization problem mentioned above. With this method, we shall obtain the

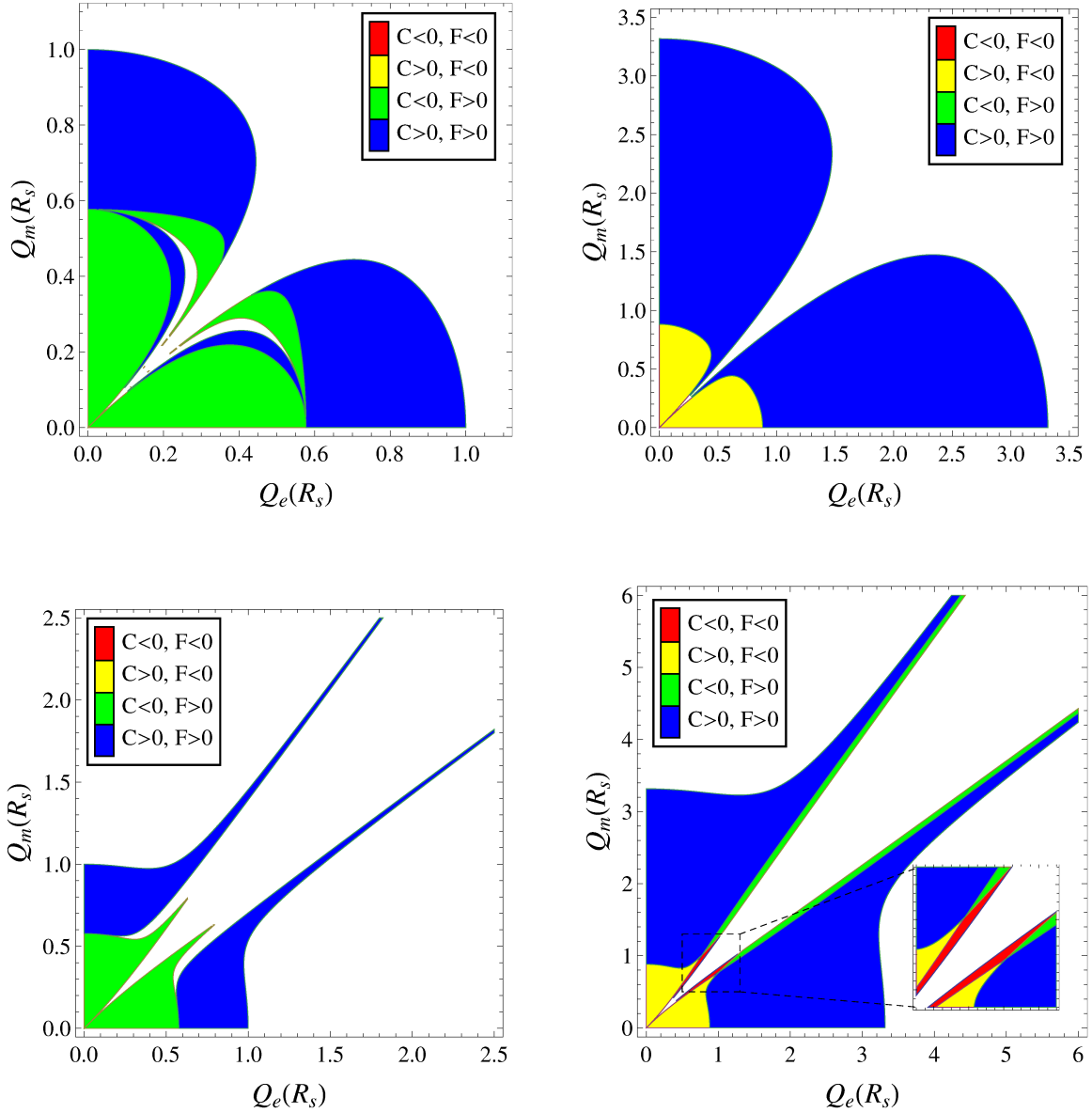


Figure 2: Phase diagrams of BH solutions in the IEM model: The horizon radius was taken  $r_h = R_s$  and different values of  $\eta$  and  $\Lambda$ . Regions with either negative temperature, entropy or mass were again avoided. Phase diagrams for  $\eta = 0.1$  in flat spacetime ( $\Lambda = 0$ ) (upper left panel) and AdS spacetime with  $\Lambda = 10 R_s^{-2}$  with the same value of  $\eta$  (right upper panel). One can appreciate that for positive  $\eta$  the phase diagrams are just deformed with respect to the Standard Electrodynamics, but they do not hold any new phase. In the lower left and lower right panels, the phase diagrams of the solutions for  $\eta = -0.1$ , in flat spacetime and in AdS spacetime - with  $\Lambda = 10 R_s^{-2}$  - are respectively plotted. Here, we see that for curved (AdS) spacetime and negative  $\eta$  a new phase, where both  $C$  and  $F$  are negative, appears. Such a phase is not realized in the standard case.

thermodynamics properties of the BH solutions. Consequently the BH configurations stability shall thus be studied.

First of all, the BH temperature can be defined in terms of the horizon gravity  $\kappa$  as [25]

$$T = \frac{\kappa}{4\pi}, \quad (36)$$

where the horizon gravity is defined as

$$\kappa = \lim_{r \rightarrow r_h} \frac{\partial_r g_{tt}}{|g_{tt} g_{rr}|}. \quad (37)$$

This expression can be simplified by replacing (29) in the temperature definition, yielding

$$T = \frac{1}{4\pi r_h} \left( 1 - \frac{k_1}{r_h^2} + \Lambda r_h^2 \right). \quad (38)$$

For large BHs with  $r_h \rightarrow \infty$  the temperature goes to infinity, whereas near  $r_h \sim 0$  the temperature diverges with its sign opposite to the sign of  $k_1$ . Let remind that, by imposing the positivity of the temperature (38), we

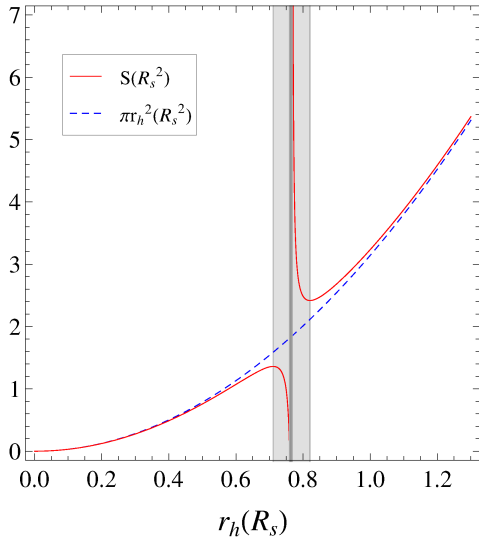


Figure 3: BH entropy in the standard and inverse electrodynamics models versus horizon radius  $r_h$ : Entropy for  $\Lambda = 0$ ,  $\eta = 0.1$ ,  $Q_e = 0.4R_s$  and  $Q_m = 0.15R_s$  as a function of the horizon radius is represented with the red solid line. The dashed blue line corresponds to the usual result  $S = \pi r_h^2$ . According to expression (49), for the choice of parameters and for a determined horizon radius interval, the entropy turns out to be a decreasing function of  $r_h$ , and so of the horizon area. The grey area shows the region where the entropy decreases with the horizon radius; whereas the black area is forbidden since the entropy would become negative.

achieve the condition

$$k_1 < r_h^2 (1 + \Lambda r_h^2). \quad (39)$$

This condition will later be employed to discard some sets of parameters in the IEM model.

Once we have obtained the temperature, we can compute the other thermodynamics quantities. First we use the Euclidean quantum gravity definition [26] introducing the Euclidean time  $t \rightarrow i\tau$ . When in the total action (1), we replace the time coordinate by the Euclidean time, the action becomes Euclidean and the metric becomes periodical with a period  $\beta$  which coincides with the inverse of the temperature (38). The change to the imaginary time implies that, due to the fact that the magnetic field is a pseudo-vector, the magnetic charge becomes imaginary ( $B \rightarrow iB$ ) in the action, i.e.

$$X = E^2 - B^2 \rightarrow \tilde{X} = E^2 + B^2 = \frac{Q_e^2 + Q_m^2}{r^4}, \quad (40)$$

$$Y = 2E \cdot B \rightarrow \tilde{Y} = 2iE \cdot B = 2i \frac{Q_e Q_m}{r^4}. \quad (41)$$

So after performing the corresponding changes, the Euclidean action reads as

$$\Delta S_E = -\frac{1}{16\pi} \int d^4x \sqrt{|g|} \left[ R - 2\Lambda - 16\pi\mathcal{L}(\tilde{X}, \tilde{Y}) \right]. \quad (42)$$

Then, we just need to evaluate the integral in the difference of four-volume of two metrics: the first volume, when there is solely an AdS metric ( $M = 0$ ,  $Q_e = 0$  and  $Q_m = 0$ ); and second one, when there is our metric solution (29) (see Ref. [27]). The computation of the difference leads to the expression for the Euclidean action as follows

$$\Delta S_E = \beta \left[ -\frac{\Lambda}{12} \left( r_h^3 - \frac{3}{\Lambda} r_h \right) + \frac{k_1 + 2k_2}{4r_h} \right], \quad (43)$$

where we have defined the parameter  $k_2$  as

$$k_2 \equiv (Q_e^2 + Q_m^2) \left[ 1 + 4\eta \left( \frac{Q_e Q_m}{Q_e^2 + Q_m^2} \right)^2 \right]. \quad (44)$$

From (43) we can obtain the different thermodynamics quantities. The Helmholtz free energy is just the quotient between the Euclidean action and the inverse of temperature:  $F = \Delta S_E / \beta$ . Therefore,

$$F = -\frac{\Lambda}{12} \left( r_h^3 - \frac{3}{\Lambda} r_h \right) + \frac{k_1 + 2k_2}{4r_h}. \quad (45)$$

On the other hand, the total energy is defined as

$$E = \frac{\partial \Delta S_e}{\partial \beta} = -T^2 \frac{\partial \Delta S_e}{\partial r_h}, \quad (46)$$

$T$  being the temperature of the BH solution (38). Then the total energy is given by the expression

$$E = \frac{\Lambda^2 r_h^8 + 2\Lambda r_h^6 - 3r_h^4(1 - 2\Lambda k_2) + 3k_1(2r_h^2 + k_1 + 2k_2)}{6r_h(\Lambda r_h^4 - r_h^2 + 3k_1)}. \quad (47)$$

Additionally, the entropy of the BH is defined as the difference

$$S = \beta E - \beta F, \quad (48)$$

so we can express the entropy as

$$S = \frac{\Lambda r_h^4 - r_h^2 + k_1 + 2k_2}{\Lambda r_h^4 - r_h^2 + 3k_1} \pi r_h^2. \quad (49)$$

In general, the BH entropy is not proportional to the horizon area  $A = 4\pi r_h^2$ , unlike the well-known result in standard Electrodynamics. For small  $\eta$ , the entropy can be approximated as

$$S = \frac{1}{4}A - \frac{128\pi^2 A}{\Lambda A^2 - 4\pi A + 48\pi^2(Q_e^2 + Q_m^2)} \frac{Q_e^4 Q_m^4}{(Q_e^2 + Q_m^2)(Q_e^2 - Q_m^2)^2} \eta + \mathcal{O}(\eta^2), \quad (50)$$

where we can see the the standard result  $S = A/4$  is recovered in the limit  $\eta \rightarrow 0$ . However, the previous result proves that general relation between entropy and horizon area turns out to be cumbersome. In fact, for the IEM model under study (3) the entropy might decrease with the BH horizon area depending upon the charges values and the parameter  $\eta$  characterizing the model. Without entering into details at this stage, let us mention that this behavior affects the fulfillment of the Second Law of the BH dynamics and may have relevant consequences in the so-called Focusing theorem where the entropy usually encodes the degrees of freedom inside the BH which in the standard case are proportional to the area [28]. For scenarios where magnetic charges are smaller than the

electric charges, the entropy (49) becomes

$$S = \frac{1}{4}A \left[ 1 - \frac{2^9 \eta \pi^2 Q_e^2}{\Lambda A^2 - 4\pi A + 48\pi^2 Q_e^2} \left( \frac{Q_m}{Q_e} \right)^4 + \mathcal{O} \left( \frac{Q_m}{Q_e} \right)^6 \right] \quad (51)$$

i.e., for small magnetic charges compared to electric charges, the entropy behaves very similarly to the standard case since the first correction is of order  $(Q_m/Q_e)^4$ . Finally, the heat capacity  $C$  can be defined as  $C = T \frac{\partial S}{\partial T}$ , so we can replace expressions (38) and (49) in this definition, yielding

$$C = 2\pi r_h^2 \frac{(\Lambda r_h^4 + r_h^2 - k_1) [\Lambda^2 r_h^8 - 2\Lambda r_h^6 + r_h^4 - 6k_1 r_h^2 - 2\Lambda k_2 r_h^4 + 8\Lambda k_1 r_h^4 + 3k_1(k_1 + 2)]}{(\Lambda r_h^4 - r_h^2 + 3k_1)^3}. \quad (52)$$

Once the relevant thermodynamics quantities are obtained, it is possible to discuss the BH stability regions in terms of the sign of the heat capacity (52) and the Helmholtz free energy (45) [29]. BH configurations with  $F > 0$  are more energetic than pure radiation, so they eventually decay to radiation by tunneling; whereas BH solutions with  $F < 0$  will not decay to radiation since they are less energetic. Furthermore, if the solution has  $C < 0$  it is unstable under acquiring mass, on the contrary to solutions with  $C > 0$  [29]. In the following, we discuss the stability regions for the IEM model as well as we compare the results with the standard Electrodynamics theory which are briefly revised below.

#### Standard case: $\eta = 0$

For illustrative purposes let us consider the case  $\eta = 0$  in the IEM Lagrangian density (3). In this case the defined quantities  $k_1$  and  $k_2$  coincide, being their expression

$$k_1 = k_2 = (Q_e^2 + Q_m^2). \quad (53)$$

Using this result, we can simplify the free Helmholtz energy (45) and the heat capacity (52) of the BH solution

are given by

$$F = -\frac{\Lambda}{12} \left( r_h^3 - \frac{3}{\Lambda} r_h \right) + \frac{3}{4} \frac{Q_e^2 + Q_m^2}{r_h}, \quad (54)$$

$$C = 2\pi r_h^2 \frac{\Lambda r_h^4 + r_h^2 - (Q_e^2 + Q_m^2)}{\Lambda r_h^4 - r_h^2 + 3(Q_e^2 + Q_m^2)}. \quad (55)$$

In Figure 1, we represent the phase diagrams of a BH solutions in the Standard Electrodynamics theory in flat and AdS spacetimes. One can see that the phase with both  $C$  and  $F$  negative does not appear in the standard Electrodynamics theory.

#### General case

In the IEM, depending on the parameter  $\eta$  and the cosmological constant  $\Lambda$  the Thermodynamics phase corresponding to  $\{C < 0, F < 0\}$ , which is absent in the standard Electrodynamics theory, may exist. In order to illustrate this scenario, we represent in Figure 2 the phase diagrams for different signs of the parameter  $\eta$  in flat and AdS spacetimes. For the example  $\eta = 0.1$  (positive  $\eta$ ) we see that the phases are deformed with respect to the standard case but new phases do not appear. For the example  $\eta = -0.1$  (negative  $\eta$ ) we see that no new phases exist in the flat case, whereas for the Ads configuration

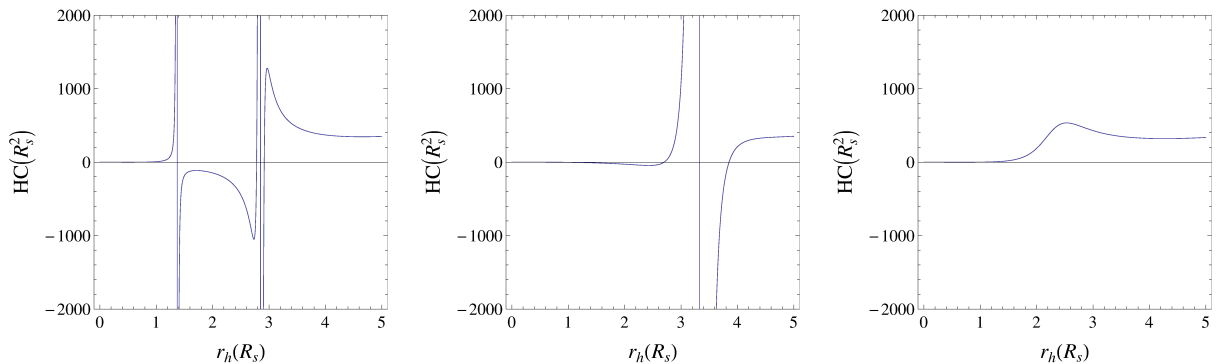


Figure 4: From left to right panels, behavior of the heat capacity (denoted HC) for *slow*, *inverse* and *fast* black holes, respectively. We can see that *slow* BHs present two phase transitions (for two horizon radii the heat capacity diverges). *Inverse* BHs present a unique phase transition and *fast* BHs doesn't have any phase transitions. The actual values for each panel were: Left:  $\eta = -0.1$ ,  $Q_e = 0.7R_s$ ,  $Q_m = 0.2R_s$ ,  $\Lambda = 0.1R_s^{-2}$ ; Center:  $\eta = -0.1$ ,  $Q_e = 0.5R_s$ ,  $Q_m = 0.4R_s$ ,  $\Lambda = 0.1R_s^{-2}$ ; Right:  $\eta = -0.1$ ,  $Q_e = 1R_s$ ,  $Q_m = 0.1R_s$ ,  $\Lambda = 0.1R_s^{-2}$ .

all the four possible phases could exist, including the corresponding to  $\{C < 0, F < 0\}$ . This means that the BH solutions in the IEM when the space-time is curved (AdS) host a different stability phenomenology from the standard Electrodynamics model.

Finally, we illustrate the found modified behavior of the entropy. In Figure 3, we depict the entropy for the flat case  $\Lambda = 0$  with  $\eta = 1$ ,  $Q_e = 0.4R_s$  and  $Q_m = 0.15R_s$ . As can be seen, the entropy (49) is not proportional to the horizon area ( $A = 4\pi r_h^2$ ). In fact, for the flat spacetime, the entropy decreases with the area, as seen in Figure 3 in the flat case  $\Lambda = 0$  with  $\eta = 1$ ,  $Q_e = 0.4R_s$  and  $Q_m = 0.15R_s$ . Thus, the BH entropy could decrease in some physical process violating the so-called second law of the BH dynamics. This violation is prevented provided that the corrections of the IEM model to the standard Electromagnetism are small, since the dominant term in (50) is still  $A/4$ .

#### IV. CLASSIFICATION OF BH SOLUTIONS IN TERMS OF THE NUMBER OF PHASE TRANSITIONS

In this section we shall perform a classification of BH solutions based on the number of phase transitions that they present. These phase transitions occur at a set of values of  $\Lambda$ ,  $\Theta$ ,  $Q_m$  and  $M$  for which the denominator of the heat capacity (52) goes to zero, i.e., the heat capacity goes through an infinite discontinuity [30]. Thus we have to obtain the parameters for which the derivative of the temperature (38) with respect to the external horizon radius is null,  $\left. \frac{\partial T}{\partial r_h} \right|_{\Lambda, \theta, Q_m} = 0$ , or equivalently, find the parameters for which the relation

$$r_h^2 = \frac{1}{2\Lambda} \left( 1 \pm \sqrt{1 - 12k_1\Lambda} \right), \quad (56)$$

is satisfied. Trying to solve this equation, we can distinguish three different classes of BH solutions:

- *Fast* BHs. If  $k_1 > \frac{1}{12\Lambda}$ , the radicand in (56) is negative, consequently expression (56) is not satisfied for any  $r_h$  and therefore phase transitions are absent for these BH configurations. We shall refer to these kinds of solutions as *fast* BHs. In flat spacetime,  $\Lambda = 0$ , this kind of solution is not allowed.
- *Slow* BHs. For  $0 < k_1 < \frac{1}{12\Lambda}$ , equation (56) can be satisfied for both plus and minus signs, since for both possibilities  $r_h^2 > 0$ . It means that for these BH configurations there are two horizon radii for which a phase transition occurs, i.e., there are two different phase transitions. We shall refer to these kinds of solutions as *slow* BHs.
- *Inverse* BHs. Provided  $k_1 < 0$ , equation (56) can be satisfied for the plus sign but not for the minus sign. In this case, there is solely one phase transition and we shall refer to these kinds of solutions as *inverse* BHs, since they appear in the IEM but not in the standard Electrodynamics theory.

In Figure 4, the heat capacity for different classes of BHs is represented. One can distinguish that *slow*, *inverse* and *fast* BHs present two, one or none phase transitions respectively. On the other hand, in Figure 5, we depicted the domain of each class in the case  $\eta = \pm 0.1$ ,  $r_h = R_s$  and  $\Lambda = 10^{-1}R_s^{-2}$ . For a negative parameter  $\eta$  all the three classes of BHs are present. However, for positive  $\eta$ , the *inverse* type does not appear. This is due to the fact that for positive  $\eta$ , the charge term given by (30) is always positive.



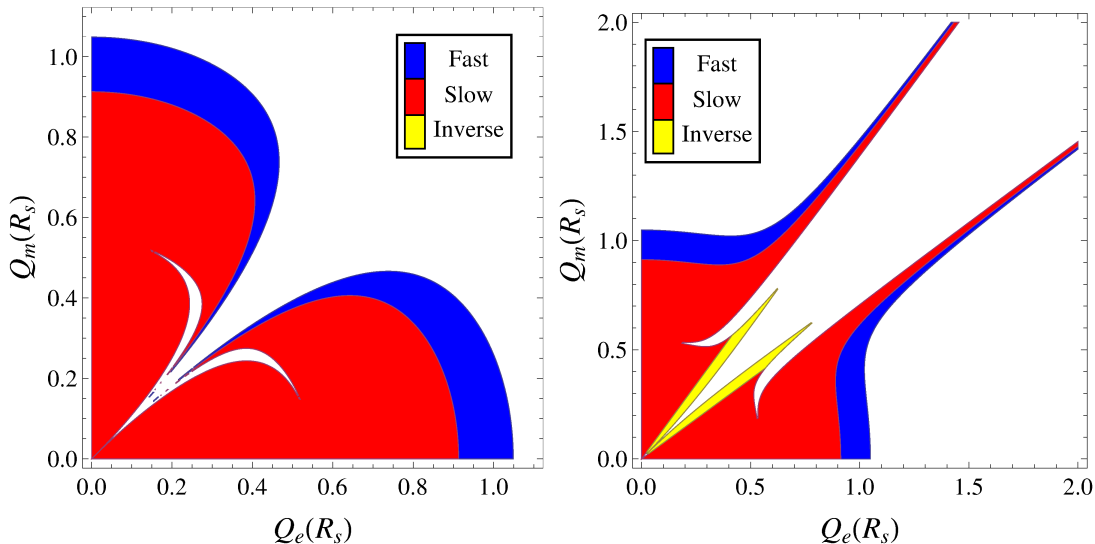


Figure 5: Classification of BH solutions as a function of the charges  $Q_e$  and  $Q_m$ : *Fast* BHs (blue), *slow* BHs in (red) and *inverse* BHs (yellow) are depicted. In the left panel, the case with  $\eta = 0.1$ ,  $r_h = 1 R_s$  and  $\Lambda = 10^{-1} R_s^{-2}$  is represented, and we can see that the *inverse* class does not appear. This is an expected result since for  $\eta > 0$ , the charge term  $k_1$  is always positive and the *inverse* class is not allowed. In the right panel, regions corresponding to the parameters  $\eta = -0.1$ ,  $r_h = R_s$  and  $\Lambda = 10^{-1} R_s^{-2}$  are depicted. We can see that the *inverse* BH does appear. In this case, all the classes described in Section IV are realized.

## V. CONCLUSIONS

In this paper we have examined gravitational solutions associated to the Inverse Electrodynamics Model as defined in expression (3). This model, which constitutes a straightforward extension of the usual Electrodynamics theory, is parity and gauge invariant and respects conformal invariance. In fact, it can be interpreted as a perturbation of the standard theory provided that the parameter  $\eta$  is sufficiently small. First, we have shown that for static and spherically symmetric U(1) fields, this model is able to support Reissner-Nordström-like black-hole solutions. After having obtained the metric tensor, we have performed a thermodynamics analysis of the solutions using the Euclidean Action approach. We have thus explored the phase diagrams of the different black holes types, i.e., enabling electrical and/or magnetic charges to exist. The stability of those configurations is fully characterized by the signs of the heat capacity and the free Helmholtz energy. We have found that for some sets of values of the Inverse Electrodynamics Model parameters, a new black-hole stability phase, namely a phase where both heat capacity and free energy are negative, which does not appear in the standard Electrodynamics theory, arises. This phase would imply that the black hole would possess a free energy smaller than pure radiation (null free energy) and consequently pure radiation will tend to tunnel or to collapse into the black-hole configuration. The fact that the heat capacity is negative means that, analogously to the Schwarzschild black holes, the more energy (mass) the black hole acquires the lower its temperature will be. To summarize this configuration would

never be in equilibrium with thermal radiation. This fact opens the possibility of further study in other extended electromagnetic theories in order to determine whether this behavior is shared by other non-linear theories.

Another novel result concerns the black hole entropy, that unlike the standard case, turns out to be in general not proportional to the horizon area. This quantity presents in the aforementioned model a cumbersome dependence with the horizon area as seen in expression (49). In fact, the entropy may decrease with the horizon area for a pathological choice of electric and magnetic charges.

Finally, we have classified the black-hole solutions in terms of the existing number of phase transitions, i.e., number of heat capacity divergences as a function of the horizon radius. Namely we have described the phenomenology of *fast*, *slow* and *inverse* black holes with none, two and one phase transitions respectively. This analysis shows explicitly a new difference with respect to the standard Electrodynamics since, whilst in the standard case *fast* and *slow* black holes are the only existing scenarios, in the Inverse Electrodynamics Model there may also exist a third configuration, the *inverse* black hole with a sole phase transition.

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