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Single soft gluon emission at two loops

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Abstract

We study the single soft-gluon current at two loops with two energetic partons in massless perturbative QCD, which describes, for example, the soft limit of the two-loop amplitude for $gg \to Hg$. The results are presented as Laurent expansions in ϵ in $D = 4 - 2\epsilon$ spacetime dimension. We calculate the expansion to order ϵ^2 analytically, which is a necessary ingredient for Higgs production at hadron colliders at next-to-next-to-next-to-leading order in the soft-virtual approximation. We also give two-loop results of the single soft-gluon current in $\mathcal{N} = 4$ Super-Yang-Mills theory, and find that it has uniform transcendentality. By iteration relation of splitting amplitudes, our calculations can determine the three-loop single soft-gluon current to order ϵ^0 in $\mathcal{N} = 4$ Super-Yang-Mills theory in the limit of large N_c.

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I. Introduction

Amplitudes in gauge theory develop infrared divergences when one or multiple external partons become soft/collinear. Fortunately, in the soft/collinear limit, there exist universal factorization properties for such amplitudes, which are the foundation of higher-order perturbative-QCD computations. Extensive discussion on the factorization of gauge-theory amplitudes in the infrared region can be found, for example, in Refs. [1–24].

In QCD, the radiation of an arbitrary number of soft gluons off a tree-level amplitude can be obtained using the well-known Berends-Giele recursion relation [3]. Due to the long-range properties of soft gluon radiation, amplitudes in the soft limit have non-local color correlations. Compact expressions for tree-level amplitudes with two soft partons have been obtained in the color space formalism in Ref. [7]. Emission of a single soft gluon from a generic one-loop amplitude have also been studied by several groups [8, 12, 14, 15]. These results have been proven to be important in the program of next-to-next-to-leading order (NNLO) QCD computations for jet physics, see for example Refs. [25–31].

While the NNLO revolution is under way, there is strong motivation for going one order in α_s further. This is driven by both experimental and theoretical demands. On the experimental side, the discovery of Higgs boson marks one of the most important progress in particle physics in the last few decades [32, 33]. It's certainly important to give the most precise theoretical prediction for its production cross section. On the theory side, uncertainties estimated by scale variation for Higgs production is around $\pm 10\%$ at NNLO [34–37], and improved to $\pm 7\%$ by including soft-gluon resummation up to next-to-next-to-leading logarithmic accuracy [38]. Further decreasing the scale uncertainties to percent level requires the computation of next-to-next-to-next-to-leading order (NNNLO) QCD corrections.

In this paper we consider single soft-gluon radiation at two loops, which plays an important role in NNNLO QCD corrections, similar to the one-loop soft-gluon current does in NNLO computations, see for example, Refs. [25–27, 39]. To simplify the situation, we confine ourselves to the case that only two hard partons are present. This corresponds to the cases such as $e^+e^- \to$ dijet, deep-inelastic scattering, or Drell-Yan/Higgs production at hadron collider. Previously, such amplitudes have been derived [19] by taking the soft limit of collinear splitting amplitudes at two loops to order ϵ^0, using the two-loop aamplitudes for $\gamma^* \to q\bar{q}g$ [40, 41] and $H \to ggg$ [42]. However, for a NNNLO computation, one needs the Laurent expansion in ϵ through the ϵ^2 terms, which we have given for the first time in this paper. Our results for the two-loop soft amplitude agree with the soft limit of the two-loop splitting amplitudes [18, 19] through the ϵ^0 terms, serving as a strong check of our calculation.

As a by-product, we obtain the soft-gluon current in $\mathcal{N}=4$ Super-Yang-Mills theory to order ϵ^2, which coincides with the QCD result at leading transcendentality. We also derive the soft limit of splitting amplitudes at three loops through order ϵ^0 at leading order of $N_c \to \infty$, using the results of Refs. [43, 44].

The paper is organized as follows. In Sec. II, we review the general result on the factorization of the single soft-gluon current at tree level and one loop. In Sec. III we calculate the soft-gluon current to two loops. We conclude at Sec. IV. We present some details for the computation of one of the master integral in the appendix.

II. Review of the soft-gluon current

In this section we review the factorization of amplitudes in the soft limit, closely following the notation in Ref. [15]. It's well-known [2, 4] that tree-level QCD amplitudes with two hard partons and one soft gluon can be written as

$$|\mathcal{M}^{(0)}(q,p_1,p_2)|^2 \simeq 4g_s^2\mu^{2\epsilon}C_R S_{12}^{(0)}(q)|\mathcal{M}^{(0)}(p_1,p_2)|^2 \tag{1}$$

where $S_{12}^{(0)}(q) = \frac{p_1\cdot p_2}{2(q\cdot p_1)(q\cdot p_2)}$, and $\mathcal{M}^{(0)}(q,p_1,p_2)$ is the tree-level amplitude for 2 hard partons (massless quark or gluon) and one soft gluon, and $\mathcal{M}^{(0)}(p_1,p_2)$ is the corresponding amplitude with the soft gluon stripped off. Dependence of the amplitudes on the extra colorless particles in the process is left implicit. The symbol $\simeq$ means that we have neglected terms that are less singular than $1/q^2$. g_s is the strong coupling constant, μ is the mass scale introduced by continuing the space-time dimension to $D = 4-2\epsilon$ dimension. C_R is the quadratic Casimir invariant. $C_R = C_A$ if parton 1 is a gluon, $C_R = C_F$ if parton 1 is a quark, where $C_A = N_c$ and $C_F = \frac{N_c^2-1}{2N_c}$, with N_c being the number of color. Note that the functional dependence of the eikonal function $S_{12}^{(0)}(q)$ is uniquely determined by its invariance under the rescaling of p_1 and p_2, which is a simple consquence of the QCD Feynman rule in the eikonal limit. In our convention, all momenta are massless and have positive-definite energies. The generalization of Eq. (1) to processes with any number of hard partons can be found, for example, in Ref. [7].

At the one-loop level, Eq. (1) receives quantum corrections, which can be written as

$$\begin{aligned}
&\mathcal{M}^{(0)}(q,p_1,p_2)\mathcal{M}^{(1)}(q,p_1,p_2)^* + \text{c.c.} \\
\simeq &\left(4(g_s\mu^\epsilon)^2 C_R S_{12}^{(0)}(q)\mathcal{M}^{(0)}(p_1,p_2)\mathcal{M}^{(1)}(p_1,p_2)^* + \text{c.c.}\right) \\
&+\left(4(g_s\mu^\epsilon)^2 C_R S_{12}^{(1)}(q)|\mathcal{M}^{(0)}(p_1,p_2)|^2 + \text{c.c.}\right),
\end{aligned} \tag{2}$$

where c.c. denotes complex conjugate. $\mathcal{M}^{(i)}$ is the ith order in α_s unrenormalized amplitudes in dimensional regularization, where UV and IR divergences are simutaneously regularized by the dimensional regularization parameter ϵ. The one-loop corrections to the eikonal function have been calculated to be [8, 12, 14, 15]

$$S_{12}^{(1)}(q) = -S_{12}^{(0)}(q)\frac{\alpha_s}{4\pi}C_A S_\epsilon \frac{e^{\epsilon\gamma_E}\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\epsilon^2\Gamma(1-2\epsilon)}, \tag{3}$$

where

$$S_\epsilon = \left(4\pi e^{-\gamma_E}e^{i\sigma_{12}\pi}\mu^2 S_{12}^{(0)}(q)\right)^\epsilon, \tag{4}$$

and $\sigma_{12} = -1$ if both p_1 and p_2 are incoming, otherwise $\sigma_{12} = 1$. Note that the one-loop eikonal function doesn't depend on C_R, which may be explained by the non-abelian exponentiation theorem [45, 46], if one replaces the polarization summation for the soft gluon by a cut propagator. Eq. (3), and its generalization to processes with any number of hard partons have been used, for example, in the calculation of soft-virtual approximation to Higgs production at NNLO [47–49], in the calculation of the two-loop soft function in soft-collinear-effective theory [50–53], and in the construction of subtraction term in general NNLO corrections [25–27, 39].

III. Calculation of the two-loop soft-gluon current

At two loops, the factorized soft-gluon current has the form

$$\begin{aligned}
&\mathcal{M}^{(0)}(q,p_1,p_2)\mathcal{M}^{(2)}(q,p_1,p_2)^* + \text{c.c.} \\
&\simeq 4(g_s\mu^\epsilon)^2 C_R\Big[\Big(S_{12}^{(0)}(q)\mathcal{M}^{(0)}(p_1,p_2)\mathcal{M}^{(2)}(p_1,p_2)^* + \text{c.c.}\Big) \\
&+\Big(S_{12}^{(1)}(q)\mathcal{M}^{(0)}(p_1,p_2)\mathcal{M}^{(1)}(p_1,p_2)^* + \text{c.c.}\Big) \\
&+\Big(S_{12}^{(2)}(q)|\mathcal{M}^{(0)}(p_1,p_2)|^2 + \text{c.c.}\Big)\Big].
\end{aligned} \tag{5}$$

The two-loop generalization is consistent with the soft limit of two-loop collinear splitting amplitudes [18, 19]. For the latter, it has been shown that similar factorization form holds to all orders in α_s [9]. The two-loop eikonal function, $S_{12}^{(2)}(q)$, is known through the order ϵ^0 terms by taking the soft limit of the two-loop collinear splitting amplitudes [18, 19] or the two-loop squared amplitudes for $\gamma^* \to q\bar{q}g$ and $H \to ggg$ [40–42]. However, for computation accurate to NNNLO, one also needs the order ϵ and order ϵ^2 terms. In this section, we calculate the Laurent expansions of $S_{12}^{(2)}(q)$ in ϵ through order ϵ^2, using a method different from Refs. [19].

In Ref. [15], the one-loop soft-gluon current is derived by taking the eikonal approximation of the integrand of the amplitudes before the loop integrals are carried out. This has the advantage that the one-loop eikonal function can be directly obtained without the subtraction of the product of the tree-level eikonal function and the one-loop squared amplitude, that is, the second line of Eq. (2). The same procedure can be used in the calculation of the two-loop eikonal function.

Specifically, we generate the integrand corresponding to the interference of tree-level and two-loop amplitudes, the first line of Eq. (5). For this purpose, we consider the process $\gamma^* \to q(p_1)\bar{q}(p_2)g(q)$, keeping in mind that the eikonal function is independent of the colorless particles in the process. Summation of polarization for the external gluon is done in Feynman gauge. We then take the eikonal approximation of the integrand, assumming that the energy of the internal and external gluons are parametrically smaller then p_1^0 and p_2^0. The integrand after the eikonal approximation is taken can also be generated by treating $q(p_1)$ and $q(p_2)$ as two out-going Wilson lines, whose directions are given by p_1^μ/p_1^0 and p_2^μ/p_2^0. We have checked that this indeed gives the same integrand [1]. We note that after the eikonal approximation is taken on the right hand side of Eq. (5), the second and the third lines of it vanish. The reason is that $\mathcal{M}^{(1)}(p_1,p_2)$ and $\mathcal{M}^{(2)}(p_1,p_2)$ become scaleless integrals and vanish identically in dimensional regularization. Therefore, The two-loop eikonal function can be obtained by evaluating the resulting integrand, without the need of subtraction.

A. Warm up: one-loop soft-gluon current

For the convenience of reader, we reproduce the one-loop results in this section. At one loop, there is only one non-zero diagram (from now on eikonal approximation is assumed for the integrand), which is depicted in Fig. 1. All the remaining diagrams are zero in dimensional regularization, because their loop integrals are scaleless. One example of such

[1] We use QGRAF [54] extensively in generating various diagrams.

a vanishing diagram is depicted in Fig. 2. We note that the external soft-gluon momentum only enters the loop integral through $q \cdot p_1$. However, the invariance of the integral under the rescaling of p_1 and p_2 demands that a factor of $\left(\frac{\mu^2(p_1 \cdot p_2)}{(q \cdot p_1)(q \cdot p_2)}\right)^\epsilon$ must be generated per loop. This is impossible for this diagram, leading to the conclusion that it must vanish.

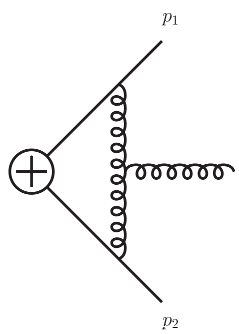


FIG. 1: Non-vanishing diagram for soft gluon emission at one-loop. Solid line are quark/anti-quark lines in the high energy limit.

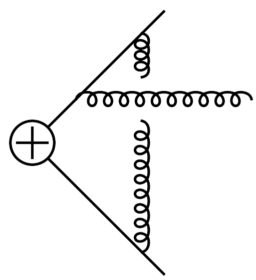


FIG. 2: Diagram which vanishes in dimensional regularization.

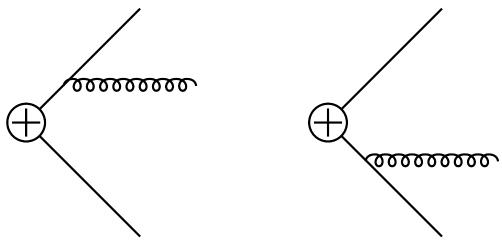


FIG. 3: Tree-level diagrams for single soft gluon emission.

We calculate the interference between the one-loop non-zero diagram, Fig. 1, and the tree-level diagrams in Fig. 3. The one-loop eikonal function can then be extracted from the one-loop integral in the interference term, which, after some simplification, reads

$$S_{12}^{(1)}(q) = i4g_s^2 C_A (p_1 \cdot p_2) \mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} \frac{1}{[2k \cdot p_1][2(q-k) \cdot p_2][k^2][(k-q)^2]}, \tag{6}$$

where the Feynman prescription $i0^+$ is implicitly understood for all propagators in square brackets, for example, $[k^2] \equiv k^2 + i0^+$. Carrying out the loop integral, we reproduce the one-loop eikonal function in Eq. (3).

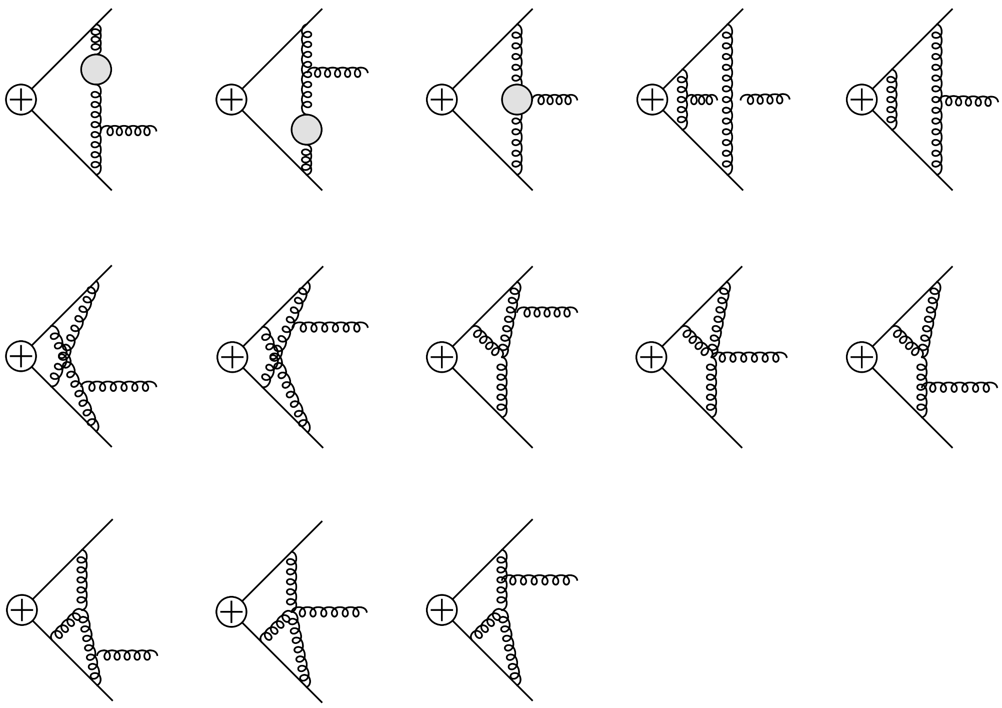
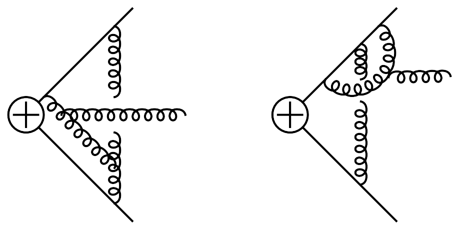


FIG. 4: Two-loop non-vanishing diagrams for single soft gluon emission.



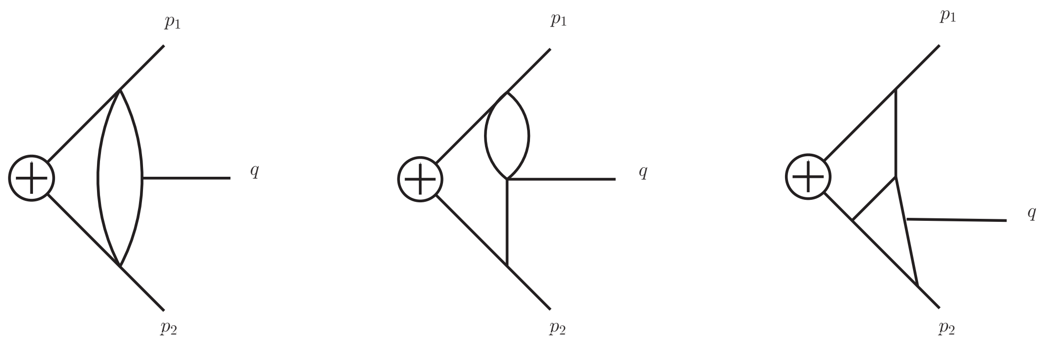
(a) (b)

FIG. 5: Examples of diagrams which vanish identically.

B. Two-loop soft-gluon current

As explained above, the two-loop eikonal function $S_{12}^{(2)}(q)$ can be extracted from the calculation of the non-vanishing diagrams at two-loop level, as depicted in Fig. 4. The grey blobs represent all possible two-point and three point insertions, where no eikonal approximation is made. We include N_f flavour of massless fermions and N_s flavour of massless scalar in the blob, besides the gluon. In QCD, $N_f = 5$, $N_s = 0$. Before describing the calculation of these diagrams, we comment on the diagrams that vanish identically. There are two classes of vanishing diagrams. The first class vanishes due to color or Lorentz algebra. An example of it is depicted in Fig. 5a. The second class vanishes because the corresponding loop integral is scaleless, as in Fig. 5b. Because of the vanishing of these two classes of diagrams, the actual number of diagrams that need to be evaluated is significantly reduced.

We now come to the actual evaluation of the non-zero diagrams in Fig. 4. We calculate the interference terms between the tree-level diagrams in Fig. 3 and the two-loop



(a) I_1 (b) I_2 (c) I_3

FIG. 6: Master integrals encountered in the computation. Eikonal approximations are taken on the directions p_1 and p_2.

diagrams in Fig. 4. After the evaluation of color factor and kinematical factor, the resulting loop integrals are reduced to three master integrals in Fig. 6. To that end, we use the techniques of Integration-By-Parts (IBP) [55, 56], implemented in the MATHEMATICA package FIRE [57] using the Laporta algorithm [58]. The reduction to master integrals has also been cross checked using a different MATHEMATICA package LiteRed [59]. The results after the IBP reduction procedure can be written as

$$
\begin{aligned}
S_{12}^{(2)}(q) = g_s^4 \frac{p_1\cdot p_2}{(q\cdot p_1)(q\cdot p_2)} \times \Bigg\{ & C_A N_f \left[\frac{2(-7+2D)(12-6D+D^2)}{(-6+D)(-3+D)(-2+D)(-1+D)} I_1 \right. \\
& \left. - \frac{6(-4+D)^2}{(-6+D)(-2+D)(-1+D)} I_2 \right] + C_A N_s \left[- \frac{(-7+2D)(-4-4D+D^2)}{2(-6+D)(-2+D)(-1+D)} I_1 \right. \\
& \left. + \frac{3(-4+D)^2}{(-6+D)(-2+D)(-1+D)} I_2 \right] + C_A^2 \left[+ \frac{8}{3} I_3 \right. \\
& - \frac{(2(-156+D(72+D(11+(-9+D)D))) - 3(-4+D)^3 D_s)}{(-6+D)(-4+D)(-2+D)(-1+D)} I_2 \\
& + \left(\frac{(-7+2D)(504-1308D+874D^2-213D^3+17D^4)}{3(-6+D)(-4+D)(-3+D)(-2+D)(-1+D)} \right. \\
& \left. \left. - \frac{(-7+2D)(-4-4D+D^2)D_s}{2(-6+D)(-2+D)(-1+D)} \right) I_1 \right] \Bigg\}, \qquad (7)
\end{aligned}
$$

The parameter D_s selects the particular variant of dimensional regularization. For $D_s = 4-2\epsilon$ the scheme is the conventional dimensional regularization scheme, while for $D_s = 4$ it is the four-dimensional helicity scheme (FDH) [60, 61].

There are three master integrals encountered in this computation. They are defined as

$$
\begin{aligned}
I_1 &= \mu^{4\epsilon}\int\frac{d^Dk_1d^Dk_2}{(2\pi)^{2D}}\frac{p_1\cdot p_2}{[2k_1\cdot p_1][2(q-k_1)\cdot p_2][k_2^2][(k_2-q)^2][(k_1-k_2)^2]},\\
I_2 &= \mu^{4\epsilon}\int\frac{d^Dk_1d^Dk_2}{(2\pi)^{2D}}\frac{p_1\cdot p_2}{[2k_1\cdot p_1][2(q-k_1)\cdot p_2][k_1^2][k_2^2][(k_1+k_2-q)^2]},\\
I_3 &= \mu^{4\epsilon}\int\frac{d^Dk_1d^Dk_2}{(2\pi)^{2D}}\frac{(q\cdot p_1)(q\cdot p_2)^2}{[2k_1\cdot p_1][2(q-k_1)\cdot p_2][2(k_2+q)\cdot p_2][k_2^2][(k_1+k_2)^2][(k_2+q)^2]},
\end{aligned}
\tag{8}
$$

where $i0^+$ dependences in the propagators are understood. The first two masters are calculated to all orders in ϵ. For the last master integral, we give the Laurent expansion of it to order ϵ^2, which is the order relevant for NNNLO computation. The details of the computation of the last integral are presented in the appendix. Here we only list the results for the three master integrals:

$$
\begin{aligned}
I_1 &= -\frac{1}{(16\pi^2)^2}S_\epsilon^2\frac{e^{2\epsilon\gamma_E}\Gamma^2(1-2\epsilon)\Gamma^2(1-\epsilon)\Gamma^2(1+2\epsilon)}{8\epsilon^3(1-4\epsilon)\Gamma(1-4\epsilon)},\\
I_2 &= -\frac{1}{(16\pi^2)^2}S_\epsilon^2\frac{e^{2\epsilon\gamma_E}\Gamma(1-2\epsilon)\Gamma^3(1-\epsilon)\Gamma^2(1+2\epsilon)}{8\epsilon^3(1-2\epsilon)\Gamma(1-3\epsilon)},\\
I_3 &= -\frac{1}{(16\pi^2)^2}S_\epsilon^2\Bigg[-\frac{1}{8\epsilon^4}-\frac{5\zeta_2}{16\epsilon^2}+\frac{25\zeta_3}{48\epsilon}-\frac{17\zeta_4}{16}+\epsilon\left(\frac{67\zeta_2\zeta_3}{48}+\frac{319\zeta_5}{80}\right)\\
&\quad+\epsilon^2\left(\frac{101\zeta_3^2}{36}+\frac{1723\zeta_6}{256}\right)+\mathcal{O}(\epsilon^3)\Bigg],
\end{aligned}
\tag{9}
$$

where ζ_s is the Riemann zeta vaule, $\zeta_s=\sum_{n=1}^\infty\frac{1}{n^s}$. It's interesting to note that I_3 coincides with the soft limit of the corresponding master integral in full QCD, where no eikonal approximation is taken in the denominator. The latter was calculated in Ref. [62] to order ϵ^0. This is probably due to the fact that the divergences in I_3 have only infrared origin. While we have only presented the Laurent expansions of I_3 to order ϵ^2 analytically, the higher-order terms can easily be obtained numerically, using its two-fold Mellin-Barnes integral representation derived in the appendix, and the `MBintegrate` routine of Czakon [63]. For example, the next three terms in the ϵ expansion of I_3 are given by

$$
(82.1443689\pm0.0000007)\epsilon^3+(198.904248\pm0.000002)\epsilon^4+(726.325910\pm0.000007)\epsilon^5 \tag{10}
$$

However, it's difficult to convert them into Riemann zeta values due to lack of significant digits.

Substituting the master integral into Eq. (7) and setting $N_s=0$, we obtain the two-loop eikonal function in QCD in the conventional dimensional regularization scheme ($D=D_s=$

$4-2\epsilon$),

$$
\begin{aligned}
S_{12}^{(2)}(q) = S_{12}^{(0)}(q)\left(\frac{\alpha_s}{4\pi}\right)^2 S_\epsilon^2 \Bigg\{ & C_A N_f \left[\frac{1}{6\epsilon^3}+\frac{5}{18\epsilon^2}+\frac{19}{54\epsilon}+\frac{\zeta_2}{6\epsilon}+\frac{65}{162}+\frac{5\zeta_2}{18}-\frac{31\zeta_3}{9}\right.\\
&+\epsilon\left(-\frac{35\zeta_2}{54}-\frac{155\zeta_3}{27}-\frac{185\zeta_4}{24}+\frac{211}{486}\right)\\
&\left.+\epsilon^2\left(-\frac{31}{9}\zeta_3\zeta_2-\frac{367\zeta_2}{162}-\frac{994\zeta_3}{81}-\frac{925\zeta_4}{72}-\frac{511\zeta_5}{15}+\frac{665}{1458}\right)\right]\\
&+C_A^2\left[\frac{1}{2\epsilon^4}-\frac{11}{12\epsilon^3}+\frac{-\frac{67}{36}+\zeta_2}{\epsilon^2}+\frac{-\frac{193}{54}-\frac{11\zeta_2}{12}-\frac{11\zeta_3}{6}}{\epsilon}-\frac{571}{81}-\frac{67\zeta_2}{36}+\frac{341\zeta_3}{18}\right.\\
&+\frac{7\zeta_4}{8}+\epsilon\left(-\frac{7}{6}\zeta_3\zeta_2-\frac{139\zeta_2}{54}+\frac{2077\zeta_3}{54}+\frac{2035\zeta_4}{48}-\frac{247\zeta_5}{10}-\frac{3410}{243}\right)\\
&+\epsilon^2\left(-\frac{205\zeta_3^2}{18}+\frac{341\zeta_2\zeta_3}{18}+\frac{6388\zeta_3}{81}-\frac{436\zeta_2}{81}+\frac{12395\zeta_4}{144}+\frac{5621\zeta_5}{30}\right.\\
&\left.\left.-\frac{3307\zeta_6}{48}-\frac{20428}{729}\right)\right]+\mathcal{O}(\epsilon^3)\Bigg\}.
\end{aligned} \tag{11}
$$

Eq. (11) is the main result of this paper. We remind the reader that this result is for the unrenormalized amplitudes. To obtain the renormalized ones, one only needs to perform a renormalization on the strong coupling α_s. We have checked Eq. (11) against the two-loop splitting amplitudes in the soft limit calculated in Refs. [18, 19], and found full agreement to order ϵ^0. To the best of our knowledge, the order ϵ and ϵ^2 terms presented in this paper are new.

C. Single soft-gluon current in $\mathcal{N}=4$ Super-Yang-Mills theory

Using the generic results presented above, it's straightforward to obtain the single soft-gluon current in $\mathcal{N}=4$ Super-Yang-Mills theory, by setting $N_f=4C_A$, $N_s=6C_A$, and $D_s=4$ (corresponding to FDH scheme [60, 61]) in Eq. (7):

$$
S_{12,\mathcal{N}=4}^{(2)}(q) = g_s^4 C_A^2 \frac{p_1\cdot p_2}{(q\cdot p_1)(q\cdot p_2)}\left[-\frac{1-4\epsilon}{3\epsilon}I_1+\frac{1-2\epsilon}{\epsilon}I_2+\frac{8}{3}I_3\right]. \tag{12}
$$

This result is remarkably simple. It becomes obvious that the result in $\mathcal{N}=4$ Super-Yang-Mills theory has uniform transcendentality, as long as I_3 does. Substituting the explicit form of the master integrals into Eq. (12), we obtain

$$
\begin{aligned}
&S_{12,\mathcal{N}=4}^{(2)}(q) = S_{12}^{(0)}(q)\left(\frac{\alpha_s}{4\pi}\right)^2 S_\epsilon^2 C_A^2\\
&\times\left[\frac{1}{2\epsilon^4}+\frac{\zeta_2}{\epsilon^2}-\frac{11\zeta_3}{6\epsilon}+\frac{7\zeta_4}{8}+\epsilon\left(-\frac{7\zeta_2\zeta_3}{6}-\frac{247\zeta_5}{10}\right)+\epsilon^2\left(-\frac{205\zeta_3^2}{18}-\frac{3307\zeta_6}{48}\right)+\mathcal{O}(\epsilon^3)\right].
\end{aligned} \tag{13}
$$

We note that at leading transcendentality, the eikonal soft function in $\mathcal{N}=4$ Super-Yang-Mills theory coincides with the one in QCD through ϵ^2, as also happens in some other context [64].

It's also interesting to notice that Eq. (13) can be reorganized as [65] [2]

$$S^{(2)}_{12,\mathcal{N}=4}(q) \equiv 4S^{(0)}_{12}(q)\left(\frac{\alpha_s}{4\pi}\right)^2 S_\epsilon^2 C_A^2 r_S^{(2)}(\epsilon)$$
$$= 4S^{(0)}_{12}(q)\left(\frac{\alpha_s}{4\pi}\right)^2 S_\epsilon^2 C_A^2 \left(\frac{1}{2}\left(r_S^{(1)}(\epsilon)\right)^2 + f(\epsilon) r_S^{(1)}(2\epsilon)\right) + \mathcal{O}(\epsilon), \quad (14)$$

where $r_S^{(1)}(\epsilon) = -e^{\epsilon\gamma_E}\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{2\epsilon^2\Gamma(1-2\epsilon)}$ is the soft limit of the one-loop collinear splitting amplitudes in $\mathcal{N}=4$ Super-Yang-Mills theory (up to an overall z-dependent factor, same below), and $f(\epsilon) = -\sum_{i=1}^{\infty}\zeta_{i+1}\epsilon^{i-1}$ [65]. Eq. (14) makes explicit the iterative structure of $\mathcal{N}=4$ splitting amplitudes and eikonal function [43]. Eq. (14) also determines the soft limit of two-loop splitting amplitudes beyond order ϵ^0,

$$r_S^{(2)}(\epsilon) = \frac{1}{8\epsilon^4} + \frac{\zeta_2}{4\epsilon^2} - \frac{11\zeta_3}{24\epsilon} + \frac{7\zeta_4}{32} + \epsilon\left(-\frac{7\zeta_2\zeta_3}{24} - \frac{247\zeta_5}{40}\right)$$
$$+\epsilon^2\left(-\frac{205\zeta_3^2}{72} - \frac{3307\zeta_6}{192}\right) + \mathcal{O}(\epsilon^3). \quad (15)$$

At three loops, the soft limit of splitting amplitudes at leading color is predicted to be [43]

$$r_S^{(3)}(\epsilon) = -\frac{1}{3}\left(r_S^{(1)}(\epsilon)\right)^3 + r_S^{(1)}(\epsilon) r_S^{(2)}(\epsilon) + f^{(3)}(\epsilon) r_S^{(1)}(3\epsilon) + \mathcal{O}(\epsilon), \quad (16)$$

where $f^{(3)}$ has been calculated through order ϵ^2 [44],

$$f^{(3)}(\epsilon) = \frac{11\zeta_4}{2} + (5\zeta_2\zeta_3 + 6\zeta_5)\epsilon + a\epsilon^2 + \mathcal{O}(\epsilon^3), \quad (17)$$

with $a = 85.263 \pm 0.004$. Using the above results, we obtain

$$r_S^{(3)}(\epsilon) = -\frac{1}{48\epsilon^6} - \frac{3\zeta_2}{32\epsilon^4} + \frac{\zeta_3}{12\epsilon^3} - \frac{1487\zeta_4}{2304\epsilon^2} - \frac{13\zeta_2\zeta_3}{144\epsilon} + \frac{71\zeta_5}{30\epsilon} + \frac{11005\zeta_6}{2048} + \frac{167\zeta_3^2}{96} - \frac{a}{18} + \mathcal{O}(\epsilon). \quad (18)$$

For completeness, the $\mathcal{N}=4$ eikonal function at leading color is then given by

$$S^{(3)}_{12,\mathcal{N}=4}(q) = 8S^{(0)}_{12}(q)\left(\frac{\alpha_s}{4\pi}\right)^3 S_\epsilon^3 C_A^3 r_S^{(3)}(\epsilon). \quad (19)$$

We note that Eq. (18) is actually exact through order ϵ^{-1} for finite N_c. There are potential $\frac{1}{N_c}$ corrections, starting from order ϵ^0. Unlike the one-loop and two-loop cases, these corrections would depend explicitly on the color representation of the hard partons, through the product of fourth order invariant tensor, $d_R^{ijkl}d_A^{ijkl}$. An explicit calculation of these corrections would be necessary in obtaining them.

Using the iterative predictions for the $\mathcal{N}=4$ splitting amplitudes [43] and the cusp anomalous dimension at leading color [66], the results above can further determine the leading-color $\mathcal{N}=4$ eikonal function at four loops through order ϵ^{-2}.

[2] We are grateful to Lance Dixon for pointing us to the discussion in the rest of this section.

IV. Conclusion

In this paper we have computed the single soft-gluon current to two-loop order. we have compared our results with those in Refs. [18, 19], and found full agreement to order ϵ^0. The order ϵ and order ϵ^2 terms presented in this paper are new. As a by-product, we have also given the soft-gluon current in $\mathcal{N} = 4$ Super-Yang-Mills theory to order ϵ^2, which in turn enables us to derive the splitting amplitudes in the soft limit, or the single soft-gluon current, at three loops and large N_c, using the results of Refs. [43, 44]. We observe uniform transcendentality for the single soft-gluon current in $\mathcal{N} = 4$ Super-Yang-Mills theory, and confirm that the leading transcendentality terms for the eikonal function are the same in QCD and $\mathcal{N} = 4$ Super-Yang-Mills theory at two loops, up to order ϵ^2.

The main purpose of the computation done in this paper is to provide the necessary ingredient for a calculation of Higgs production cross section at hadron collider at NNNLO. A lot of progress have been made recently in this direction [67–72]. A useful step towards the full NNNLO QCD corrections is the soft-virtual approximation at NNNLO. Using the results presented in this paper, the cross section for Higgs + one gluon emission can be computed by trivially integrating over the soft-gluon phase space. The cross section for Higgs + 3 partons production in the soft limit has also been calculated recently in an impressive paper [70]. The only missing piece is the cross section for Higgs + 2 partons production at one loop in the soft limit. It's reasonable to expect that the soft-virtual approximation for Higgs production at hadron collider at NNNLO will be available in the foreseeable future.

Besides Higgs physics, the soft gluon current at two loops is also useful in soft-collinear-effective theory [50–53]. For example, the two-loop soft gluon current can be used to calculate soft function at NNNLO. Finally, we have only computed this soft gluon current with two energetic partons, or two Wilson lines. It's certainly interesting to extend our results to processes with an arbitrary number of Wilson lines. This will be relevant to jet physics at NNNLO. It will also be useful in understanding the structure of infrared divergences for multiple Wilson lines at three loops, see for example, Ref. [73].

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Appendix: Evaluation of the master integral I_3 to order ϵ^2

In this appendix, we briefly explain the evaluation of the most difficult master integral, I_3. In fact, I_1 and I_2 can be obtained by deleting two propagators from I_3. We proceed by first performing the k_2 sub-loop integral by Feynman's trick,

$$\begin{aligned} I_3' &= \int \frac{d^D k_2}{i\pi^{D/2}} \frac{1}{[2(k_2+q)\cdot p_2][k_2^2][(k_1+k_2)^2][(k_2+q)^2]} \\ &= (-1)^{D/2}\Gamma\left(4-\frac{D}{2}\right)\int_0^\infty dy_2 \int_0^1 dx_1\, dx_2\, dx_3 \frac{\delta(1-x_1-x_2-x_3)}{\Delta^{4-D/2}}, \end{aligned} \tag{A.1}$$

where

$$\Delta = x_1 x_2 [k_1^2] + x_1 x_3 [(q+y_2 p_2)^2] + x_2 x_3 [(k_1 - q - y_2 p_2)^2]. \tag{A.2}$$

The resulting Feynman parameter integral over dx_i can be factorized by introducing a two-fold Mellin-Barnes integral,

$$\begin{aligned} \frac{1}{\Delta^{4-D/2}} &= \int_{-i\infty}^{+i\infty} \frac{dz_1\, dz_2}{(2\pi i)^2} \Gamma(-z_1)\Gamma(-z_2) \frac{\Gamma(4-D/2+z_1+z_2)}{\Gamma(4-D/2)} \\ &\times \left(x_1 x_2 [k_1^2]\right)^{z_1} \left(x_1 x_3 [(q+y_2p_2)^2]\right)^{z_2} \left(x_2 x_3 [(k_1-q-y_2p_2)^2]\right)^{-z_1-z_2-4+D/2}, \end{aligned} \tag{A.3}$$

where the contour for z_i separates the poles of $\Gamma(\cdots + z_i)$ from those of $\Gamma(\cdots - z_i)$. After this step, the Feynman parameter integral over dx_i can be done in closed form in terms of Γ functions. The remaining k_1 sub-loop integral has the form

$$\int \frac{d^D k_1}{i\pi^{D/2}} \frac{1}{[k_1^2]^{1-z_1}[(k_1-q-y_2p_2)^2]^{4-D/2+z_1+z_2}[2k_1\cdot p_1][2(q-k_1)\cdot p_2]}, \tag{A.4}$$

which can be straightforwardly done. We then arrive at a two-fold Mellins-Barnes integral representation for I_3,

$$\begin{aligned} I_3 &= \frac{1}{8(16\pi^2)^2} S_\epsilon^2 e^{2\epsilon\gamma_E}\Gamma(5-D) \\ &\times \int_{-i\infty}^{+i\infty} \frac{dz_1\, dz_2}{(2\pi i)^2} \Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma\left(\frac{D}{2}-z_1-2\right)\Gamma\left(\frac{D}{2}+z_1-2\right) \\ &\times \frac{\Gamma\left(\frac{D}{2}-z_2-3\right)\Gamma\left(-D+z_2+6\right)\Gamma\left(1+z_1+z_2\right)\Gamma(D-z_1-z_2-5)}{\Gamma\left(1-z_1\right)\Gamma\left(2+z_2\right)\Gamma\left(\frac{3D}{2}-z_2-7\right)}. \end{aligned} \tag{A.5}$$

We were not able to find an all order in ϵ solution of this integral. Instead, we calculate the Laurent expansion of the Mellin-Barnes integral to order ϵ^2, which is relevant for NNNLO phenomenology. To that end, we make use of the MATHEMATICA pachages MB [63] and BARNESROUTINES of D. Kosower to resolve the singularity, to expand the integrand in ϵ, and to apply the Barnes lemma in an automatic way, which results in a series of one-fold and two-fold Mellin-Barnes integrals. The one-fold integral can easily be done numerically using MATHEMATICA's `NIntegrate` routine, and the results can be converted into Riemann zeta values using the PSLQ algorithm [75, 76]. The only remaining two-fold Mellin-Barnes integral is

$$\begin{aligned} &\int_{-i\infty}^{+i\infty} \frac{dz_1\, dz_2}{(2\pi i)^2} \frac{\Gamma(-z_1)^2\Gamma(z_1)\Gamma(-z_2)\Gamma(1+z_2)\Gamma(-1-z_1-z_2)\Gamma(1+z_1+z_2)}{\Gamma(1-z_1)} \\ &\times \left(\psi(-1-z_2)+\psi(2+z_2)\right)\left(2\psi(-1-z_1-z_2)+\psi(-z_1)+\psi(z_1)\right), \end{aligned} \tag{A.6}$$

where $\psi(x)$ is the logarithmic derivative of Γ function, and the integration contours are straight vertical lines defined by

$$\mathrm{Re}(z_1) = -\frac{1091}{1641}, \qquad \mathrm{Re}(z_2) = -\frac{554}{1671}. \tag{A.7}$$

The integral can be performed by closing the contour to the left or right, and summing up the residues at the poles. The results are double sums of the form

$$\sum_{m,n=1}^{\infty} \frac{S_{\vec{i}_1}(m)}{m^{j_1}} \frac{S_{\vec{i}_2}(n)}{n^{j_2}} \frac{S_{\vec{i}_3}(m+n)}{(m+n)^{j_3}}, \tag{A.8}$$

where $S_{\vec{i}}(k)$ are nested harmonic sums defined in Ref. [77]. The summation can be conveniently done using XSummer [78]. The final result for this master integral is checked numerically using the package FIESTA [79] and the author's personal tool, based on the method of sector decomposition [80].

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