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$\mathcal{N} \geq 4$ Supergravity Amplitudes from Gauge Theory at One Loop

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Abstract

We expose simple and practical relations between the integrated four- and five-point one-loop amplitudes of $\mathcal{N} \geq 4$ supergravity and the corresponding (super-)Yang-Mills amplitudes. The link between the amplitudes is simply understood using the recently uncovered duality between color and kinematics that leads to a double-copy structure for gravity. These examples provide additional direct confirmations of the duality and double-copy properties at loop level for a sample of different theories.

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I. INTRODUCTION

One of the remarkable theoretical ideas emerging in the last decade is the notion that gravity theories are intimately tied with gauge theories. The most celebrated connection is the AdS/CFT correspondence [1] which relates maximally supersymmetric Yang-Mills gauge theory to string theory (and supergravity) in anti-de Sitter space. Another surprising link between the two theories is the conjecture that to all perturbative loop orders the kinematic numerators of diagrams describing gravity scattering amplitudes are double copies of the gauge-theory ones [2, 3]. This double-copy relation relies on a novel conjectured duality between color and kinematic diagrammatic numerators of gauge-theory scattering amplitudes. At tree level, the double-copy relation encodes the Kawai-Lewellen-Tye (KLT) relations between gravity and gauge-theory amplitudes [4].

The duality between color and kinematics offers a powerful tool for constructing both gauge and gravity loop-level scattering amplitudes, including nonplanar contributions [3, 5–7]. The double-copy property does not rely on supersymmetry and is conjectured to hold just as well in a wide variety of supersymmetric and non-supersymmetric theories. In recent years there has been enormous progress in constructing planar $\mathcal{N} = 4$ super-Yang-Mills amplitudes. For example, at four and five points, expressions for amplitudes of this theory—believed to be valid to all loop orders and nonperturbatively—have been constructed [8]. (For recent reviews, see refs. [5, 9].) Many of the new advances stem from identifying a new symmetry, called dual conformal symmetry, in the planar sector of $\mathcal{N} = 4$ super-Yang-Mills theory [10]. This symmetry greatly enhances the power of methods based on unitarity [11, 12] or on recursive constructions of integrands [13]. The nonplanar sector of the theory, however, does not appear to possess an analogous symmetry. Nevertheless, the duality between color and kinematics offers a promising means for carrying advances in the planar sector of $\mathcal{N} = 4$ super-Yang-Mills theory to the nonplanar sector and then to $\mathcal{N} = 8$ supergravity. In particular, the duality interlocks planar and nonplanar contributions into a rigid structure. For example, as shown in ref. [3], for the three-loop four-point amplitude, the maximal cut [14] of a single planar diagram is sufficient to determine the complete amplitude, including nonplanar contributions.

Here we will explore one-loop consequences of the duality between color and kinematics for supergravity theories with $4 \leq \mathcal{N} \leq 6$ supersymmetries. These cases are less well understood than the cases of maximal supersymmetry. (Some consequences for finite one-loop amplitudes in non-supersymmetric pure Yang-Mills theory have been studied recently [15].) Since the duality and its double-copy consequence remain a conjecture, it is an interesting question to see if the properties hold in the simplest nontrivial loop examples with less than maximal supersymmetry. In particular, we will explicitly study the four- and five-point amplitudes of these theories. These cases are especially straightforward to investigate because the required gauge theory and gravity amplitudes are known. Our task is then to find rearrangements that expose the desired properties. The necessary gauge-theory four-point amplitudes were first given in dimensional regularization near four dimensions in ref. [16], and later in a form valid to all orders in the dimensional regularization parameter [17]. At five points, the dimensionally regularized gauge-theory amplitudes near four dimensions were presented in ref. [18]. The four-graviton amplitudes in theories with $\mathcal{N} \leq 6$ supersymmetries were first given in ref. [19]. More recently, the MHV one-loop amplitudes of $\mathcal{N} = 6$

and $\mathcal{N} = 4$ supergravity were presented, up to rational terms in the latter theory [20].[1]

Here we point out that the double-copy relations can be straightforwardly exploited, allowing us to obtain complete integrated four- and five-point amplitudes of $\mathcal{N} \geq 4$ supergravity amplitudes as a simple linear combinations of corresponding gauge-theory amplitudes. Because these relations are valid in any number of dimensions, we can use previously obtained representations of QCD and $\mathcal{N} = 4$ super-Yang-Mills four-point amplitudes valid with D-dimensional momenta and states in the loop to obtain such representations for $\mathcal{N} \geq 4$ supergravity. These D-dimensional results are new, while our four-dimensional results reproduce ones found in refs. [19, 20]. At the level of infrared singularities, possible relations between $\mathcal{N} = 4$ super-Yang-Mills and $\mathcal{N} = 8$ supergravity four- and five-point amplitudes had been discussed previously in refs. [21, 22].

For cases with larger numbers of external legs, the loop momentum is expected to become entangled with the relations making them more intricate. Nevertheless, we expect that the duality should lead to simple structures at one loop for all multiplicity, and once understood these should lead to improved means for constructing gravity loop amplitudes. Indeed, the duality has already been enormously helpful for constructing four- and five-point multiloop amplitudes in $\mathcal{N} = 8$ supergravity [3, 5–7].

This paper is organized as follows. In section II we review some properties of scattering amplitudes, including the conjectured duality between color and kinematics and the gravity double-copy property. Then in section III, we give some one-loop implications, before turning to supergravity. We also make a few comments in this section on two-loop four-point amplitudes. We give our summary and outlook in section IV. Two appendices are included collecting gauge-theory amplitudes and explicit forms of the integrals used in our construction.

II. REVIEW

In this section we review some properties of gauge and gravity amplitudes pertinent to our construction of supergravity amplitudes. We first summarize the duality between color and kinematics which allows us to express gravity amplitudes in terms of gauge-theory ones. We then review decompositions of one-loop $\mathcal{N} = 4, 5, 6$ supergravity amplitudes in terms of contributions of matter multiplets, simplifying the construction of the amplitudes.

[1] While completing the present paper, version 2 of ref. [20] appeared, giving the missing rational terms of the $\mathcal{N} = 4$ supergravity five-point amplitudes.

A. Duality between color and kinematics

We can write any m-point L-loop-level gauge-theory amplitude where all particles are in the adjoint representation as

$$\frac{(-i)^L}{g^{m-2+2L}}\mathcal{A}_m^{\rm loop} = \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} \,. \tag{2.1}$$

The sum runs over the set of distinct m-point L-loop graphs, labeled by j, with only cubic vertices, corresponding to the diagrams of a ϕ^3 theory. The product in the denominator runs over all Feynman propagators of each cubic diagram. The integrals are over L independent D-dimensional loop momenta, with measure $d^{DL}p = \prod_{l=1}^{L} d^D p_l$. The c_i are the color factors obtained by dressing every three vertex with an $\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \text{Tr}\{[T^a, T^b]T^c\}$ structure constant, and the n_i are kinematic numerator factors depending on momenta, polarizations and spinors. For supersymmetric amplitudes expressed in superspace, there will also be Grassmann parameters in the numerators. The S_j are the internal symmetry factors of each diagram. The form in eq. (2.1) can be obtained in various ways; for example, starting from covariant Feynman diagrams, where the contact terms are absorbed into kinematic numerators using inverse propagators.

Any gauge-theory amplitude of the form (2.1) possesses an invariance under "generalized gauge transformations" [2, 3, 23–25] corresponding to all possible shifts, $n_i \to n_i + \Delta_i$, where the Δ_i are arbitrary kinematic functions (independent of color) constrained to satisfy

$$\sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} = 0 \,. \tag{2.2}$$

By construction this constraint ensures that the shifts by Δ_i do not alter the amplitude (2.1). The condition (2.2) can be satisfied either because of algebraic identities of the integrand (including identities obtained after trivial relabeling of loop momenta in diagrams) or because of nontrivial integration identities. Here we are interested in Δ_i that satisfy (2.2) because of the former reason, as the relations we will discuss below operate at the integrand level. We will refer to these kind of numerator shifts valid at the integrand level as point-by-point generalized gauge transformations. One way to express this freedom is by taking any function of the momenta and polarizations and multiplying by a sum of color factors that vanish by the color-group Jacobi identity, and then repackaging the functions into Δ_i's over propagators according to the color factor of each individual term. Some of the resulting freedom corresponds to gauge transformations in the traditional sense, while most does not. These generalized gauge transformations will play a key role, allowing us to choose different representations of gauge-theory amplitudes, aiding our construction of gravity amplitudes from gauge-theory ones.

The conjectured duality of refs. [2, 3] states that to all loop orders there exists a form of the amplitude where triplets of numerators satisfy equations in one-to-one correspondence with the Jacobi identities of the color factors,

$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k \,, \tag{2.3}$$

where the indices i, j, k schematically indicate the diagram to which the color factors and numerators belong to. Moreover, we demand that the numerator factors have the same antisymmetry property as color factors under interchange of two legs attaching to a cubic vertex,

$$c_i \to -c_i \;\Rightarrow\; n_i \to -n_i\,. \tag{2.4}$$

At tree level, explicit forms satisfying the duality have been given for an arbitrary number of external legs and any helicity configuration [26]. An interesting consequence of this duality is nontrivial relations between the color-ordered partial tree amplitudes of gauge theory [2] which have been proven in gauge theory [27] and in string theory [28]. Recently these relations played an important role in the impressive construction of the complete solution to all open string tree-level amplitudes [29]. The duality has also been studied from the vantage point of the heterotic string, which offers a parallel treatment of color and kinematics [23]. A partial Lagrangian understanding of the duality has also been given [24]. The duality (2.3) has also been expressed in terms of an alternative trace-based representation [30], emphasizing the underlying group-theoretic structure of the duality. Indeed, at least for self-dual field configurations and MHV amplitudes, the underlying infinite-dimensional Lie algebra has been very recently been identified as area preserving diffeomorphisms [31].

At loop level, less is known though some nontrivial tests have been performed. In particular, the duality has been confirmed to hold for the one-, two- and three-loop four-point amplitudes of $\mathcal{N}=4$ super-Yang-Mills theory [3]. It is also known to hold for the one- and two-loop four-point identical helicity amplitudes of pure Yang-Mills theory [3]. Very recently it has also been shown to hold for the four-loop four-point amplitude of $\mathcal{N}=4$ super-Yang-Mills theory [6], and for the five-point one-, two- and three-loop amplitudes of the same theory [7].

B. Gravity as a double copy of gauge theory

Perhaps more surprising than the gauge-theory aspects of the duality between color and kinematics is a directly related conjecture for the detailed structure of gravity amplitudes. Once the gauge-theory amplitudes are arranged into a form satisfying the duality (2.3), corresponding gravity amplitudes can be obtained simply by taking a double copy of gauge-theory numerator factors [2, 3],

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_m^{\rm loop} = \sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p^2_{\alpha_j}}\,, \tag{2.5}$$

where $\mathcal{M}_m^{\rm loop}$ are m-point L-loop gravity amplitudes. The $\tilde{n}_i$ represent numerator factors of a second gauge-theory amplitude, the sum runs over the same set of diagrams as in eq. (2.1). At least one family of numerators (n_j or $\tilde{n}_j$) for gravity must be constrained to satisfy the duality (2.3) [3, 24]. This is expected to hold in a large class of gravity theories, including all theories that are low-energy limits of string theories. We obtain different gravity theories by taking the n_i and $\tilde{n}_i$ to be numerators of amplitudes from different gauge theories. Here we are interested in $\mathcal{N} \geq 4$ supergravity amplitudes in $D=4$. For example, we obtain the pure supergravity theories as products of $D = 4$ Yang-Mills theories as,

$$\begin{aligned}
\mathcal{N} = 8 \text{ supergravity} &: (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 4 \text{ sYM})\,, \\
\mathcal{N} = 6 \text{ supergravity} &: (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 2 \text{ sYM})\,, \\
\mathcal{N} = 5 \text{ supergravity} &: (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 1 \text{ sYM})\,, \\
\mathcal{N} = 4 \text{ supergravity} &: (\mathcal{N} = 4 \text{ sYM}) \times (\mathcal{N} = 0 \text{ sYM})\,,
\end{aligned} \tag{2.6}$$

where $\mathcal{N} = 0$ super-Yang-Mills is ordinary non-supersymmetric Yang-Mills theory, consisting purely of gluons. ($\mathcal{N} = 7$ supergravity is equivalent to $\mathcal{N} = 8$ supergravity, so we do not list it.)

Since the duality requires the numerators and color factors to share the same algebraic properties (2.3) and (2.4), eq. (2.2) implies that

$$\sum_j \int \frac{d^{DL}p}{(2\pi)^{DL}} \frac{1}{S_j} \frac{\Delta_j \tilde{n}_j}{\prod_{\alpha_j} p^2_{\alpha_j}} = 0\,, \tag{2.7}$$

so that the gravity amplitude (2.5) is invariant under the same point-by-point generalized gauge transformation $n_j \to n_j + \Delta_j$ as in gauge theory.

At tree level, the double-copy property encodes the KLT [4] relations between gravity and gauge theory [2]. The double-copy formula (2.5) has been proven at tree level for pure gravity and for $\mathcal{N} = 8$ supergravity, when the duality (2.3) holds in the corresponding gauge theories [24]. At loop level a simple argument based on the unitarity cuts strongly suggests that the double-copy property should hold if the duality holds in gauge theory [3, 24]. In any case, the nontrivial part of the loop-level conjecture is the assumption of the existence of a gauge-theory loop amplitude representation that satisfies the duality between color and kinematics. The double-copy property (2.5) has been explicitly confirmed in $\mathcal{N} = 8$ supergravity through four loops for the four-point amplitudes [3, 6] and through two loops for the five-point amplitudes [7]. (The three- and four-loop $\mathcal{N} = 4$ super-Yang-Mills and $\mathcal{N} = 8$ supergravity four-point amplitudes had been given earlier, but in a form where the duality and double copy are not manifest [32–35].)

C. Decomposing one-loop $\mathcal{N} \geq 4$ supergravity amplitudes.

To simplify the analysis, we consider amplitudes with only gravitons on the external legs. (One can, of course, use an on-shell superspace as described in ref. [36] to include other cases as well.) At one loop it is well known that the graviton scattering amplitudes of various supersymmetric theories satisfy simple linear relations dictated by the counting of states in each theory. In table I we give the particle content of relevant supergravity multiplets. (The $\mathcal{N} = 5$ matter multiplet is the same as the $\mathcal{N} = 6$ matter one, hence, it is not explicitly listed. Similarly, the $\mathcal{N} = 8$ supergravity multiplet is equivalent to the $\mathcal{N} = 7$ one.) Looking at this table, we can easily assemble some simple relations between the contributions from

	scalars	spin 1/2	spin 1	spin 3/2	spin 2
$\mathcal{N}=8$	70	56	28	8	1
$\mathcal{N}=6$ gravity	30	26	16	6	1
$\mathcal{N}=5$ gravity	10	11	10	5	1
$\mathcal{N}=4$ gravity	2	4	6	4	1
$\mathcal{N}=6$ matter	20	15	6	1	
$\mathcal{N}=4$ matter	6	4	1		

TABLE I: Particle content of relevant supergravity multiplets. The scalars are taken to be real for counts in this table.

different multiplets circulating in the loop,

$$\begin{aligned}
\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6}(1,2,\ldots,m) &= \mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8}(1,2,\ldots,m) - 2\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{mat.}}(1,2,\ldots,m)\,,\\
\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=5}(1,2,\ldots,m) &= \mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8}(1,2,\ldots,m) - 3\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{mat.}}(1,2,\ldots,m)\,,\\
\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4}(1,2,\ldots,m) &= \mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8}(1,2,\ldots,m) - 4\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{mat.}}(1,2,\ldots,m)\\
&\qquad + 2\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{mat.}}(1,2,\ldots,m)\,,
\end{aligned} \tag{2.8}$$

where the subscript "mat" denotes a matter multiplet contribution. Thus, in the rest of the paper, we will consider only one-loop amplitudes with the two types of matter going around the loop in addition to the $\mathcal{N}=8$ amplitudes. The remaining $\mathcal{N}\geq 4$ amplitudes (with generic amounts of $\mathcal{N}\geq 4$ matter) can be assembled by linear combination of these three types.

III. IMPLICATIONS OF THE DUALITY AT ONE LOOP

In this section we first present a few general one-loop implications of the duality between color and kinematics. Our initial considerations are general and apply as well to non-supersymmetric theories. We will then specialize to $\mathcal{N}\geq 4$ supergravity four- and five-point amplitudes, taking advantage of special properties of $\mathcal{N}=4$ super-Yang-Mills theory.

A. Implications for generic one-loop amplitudes

As shown in ref. [37] all color factors appearing in a one-loop amplitude can be obtained from the color factors of "ring diagrams", that is the $(m-1)!/2$ one-particle-irreducible (1PI) diagrams in the shape of a ring, as illustrated in fig. 1 for the cyclic ordering $1,2,\ldots,m$. We will denote the color and kinematic numerator factors of such a diagram with external leg ordering $1,2,\ldots,m$ by $c_{123\cdots m}$ and $n_{123\cdots m}(p)$. Its color factor is given by the adjoint trace,

$$c_{123\ldots m} = \text{Tr}_A[\tilde{f}^{a_1}\tilde{f}^{a_2}\tilde{f}^{a_3}\cdots\tilde{f}^{a_m}]\,, \tag{3.1}$$

where $(\tilde{f}^{a_i})^{bc} = \tilde{f}^{ba_ic}$.

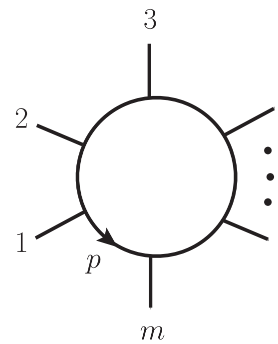

FIG. 1: the one-loop m-gon master diagram for the cyclic ordering $1, 2, \ldots, m$.

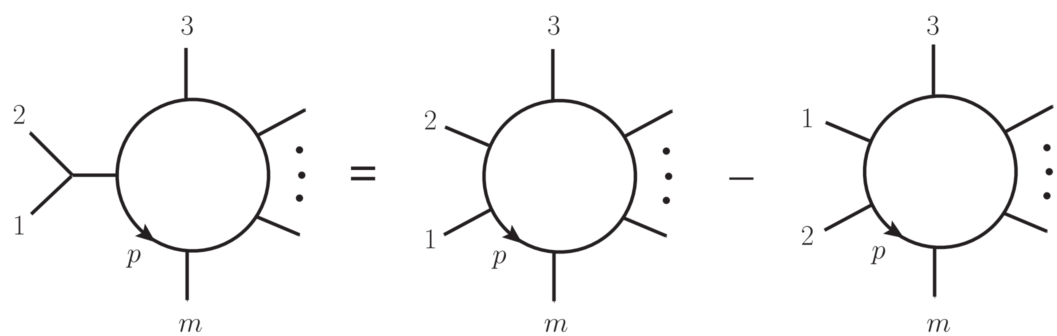

FIG. 2: The basic Jacobi relation between three one-loop graphs that can be used to express any color factor or kinematic numerator factor for any one-loop graph in terms of the parent m-gons.

The color factors of the one-particle-reducible diagrams are simply given by antisymmetrizations of ring-diagram ones as dictated by the Jacobi relations (2.3). For example, the color factor of the diagram with a single vertex external to the loop shown in fig. 2 is

$$c_{[12]3\cdots m} \equiv c_{123\cdots m} - c_{213\cdots m} \,. \tag{3.2}$$

If we have a form of the amplitude where the duality holds, then the numerator of this diagram is

$$n_{[12]3\cdots m}(p) \equiv n_{123\cdots m}(p) - n_{213\cdots m}(p) \,. \tag{3.3}$$

The color factors of other diagrams, with multiple vertices external to the loop, can similarly be obtained with further antisymmetrizations such as $c_{[[12]3]\cdots m} = c_{[12]3\cdots m} - c_{3[12]\cdots m}$. In this way all color factors and numerators can be expressed in terms of the ones of the ring diagram, so it serves as our "master" diagram.

It is also useful to consider representations where the dual Jacobi relations do not hold. For any m-point one-loop amplitude, we can use the color-group Jacobi identity to eliminate all color factors except those of the master diagram and its relabelings. Indeed, this is how one arrives at the adjoint-representation color basis [37]. In this color basis we express the one-loop amplitude in terms of a sum over permutations of a planar integrand,

$$\mathcal{A}^{\text{1-loop}}(1, 2, \ldots, m) = g^m \sum_{S_m/(Z_m \times Z_2)} \int \frac{d^D p}{(2\pi)^D} \, c_{123\ldots m} \, \mathscr{A}(1, 2, \ldots, m; p) \,, \tag{3.4}$$

where $\mathscr{A}(1,2,\ldots,m;p)$ is the complete integrand of the color-ordered amplitude, $A^{\text{1-loop}}(1,2,\ldots,m)$. The sum runs over all permutations of external legs (S_n), but with the cyclic (Z_m) and reflection (Z_2) permutations modded out. In this representation all numerator factors except for the m-gon ones are effectively set to zero, since their color factors no longer appear in the amplitude. This is equivalent to a generalized gauge transformation applied to the numerators[2]

$$\begin{aligned} n_{123\cdots m}(p) \;&\rightarrow\; n_{123\cdots m}(p) + \Delta_{123\cdots m}(p) = \mathscr{A}(1,2,3,\ldots,m;p)\prod_{\alpha=1}^{m} p_\alpha^2\,, \\ n_i \;&\rightarrow\; n_i + \Delta_i = 0, \qquad \text{for 1PR graphs } i\,, \end{aligned} \tag{3.5}$$

where the product $\prod p_\alpha^2$ runs over the inverse propagators of the m-gon master diagram. In this representation the m-gon numerators are in general nonlocal to account for propagators carrying external momenta present in the one-particle reducible (1PR) diagrams but not in master diagrams. In general, the new numerators in eq. (3.5) will not satisfy the duality relations (2.3).

Recall that generalized gauge invariance implies that only one of the two copies of numerators needs to satisfy the duality in order for the double-copy property to work. For the first copy we use the duality-violating representation (3.5) where all one-particle reducible numerator factors are eliminated in favor of nonlocal m-gon master numerator factors. For the second copy we use the duality-satisfying numerators, $\tilde{n}_{12\ldots m}$. Then according to the double-copy formula (2.5), by making the substitution $c_i \rightarrow \tilde{n}_i$ in eq. (3.4), we obtain a valid gravity amplitude. We then have

$$\mathcal{M}^{\text{1-loop}}(1,2,\ldots,m) = \left(\frac{\kappa}{2}\right)^m \sum_{S_m/(Z_m\times Z_2)} \int \frac{d^Dp}{(2\pi)^D}\, \tilde{n}_{123\ldots m}(p)\, \mathscr{A}(1,2,\ldots,m;p)\,, \tag{3.6}$$

where $\tilde{n}_{12\ldots m}(p)$ is the m-gon master numerator with the indicated ordering of legs and we have replaced the gauge-theory coupling constant with the gravity one.

At first sight, it may seem surprising that only the m-gon numerators are needed, but as noted above, these master numerators contain all the nontrivial information in the amplitudes. The nontrivial step in this construction is to find at least one copy of m-gon numerators $\tilde{n}_i$ such that the duality relations (2.3) hold manifestly.

So far these considerations have been general. An important simplification occurs if the numerators of one of the gauge-theory copies are independent of the loop momenta, $\tilde{n}_{123\ldots m}(p) = \tilde{n}_{123\ldots m}$. We can then pull these numerators out of the integral in eq. (3.6) giving relations between *integrated* gravity and gauge theory amplitudes. Below we will identify two cases where this is indeed true: the four- and five- point one-loop amplitudes of $\mathcal{N}=4$ super-Yang-Mills theory [7, 38]. Taking one copy to be the $\mathcal{N}=4$ super-Yang-Mills amplitude and the other to be a gauge-theory amplitude with fewer supersymmetries, we then get a remarkably simple relation between integrated one-loop ($\mathcal{N}+4$) supergravity and

[2] Here we have absorbed a phase factor i into the numerator definition, $i\,n_j \rightarrow n_j$, compared to eq. (2.1), as is convenient for one-loop amplitudes. For the remaining part of the paper we will use this convention.

super-Yang-Mills amplitudes with $\mathcal{N}$ supersymmetries,

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}+4\text{ susy}}(1,2,\ldots,m) = \left(\frac{\kappa}{2}\right)^m \sum_{S_m/(Z_m\times Z_2)} \tilde{n}_{123\ldots m}\, A^{\text{1-loop}}_{\mathcal{N}\text{ susy}}(1,2,\ldots,m)\,, \tag{3.7}$$

valid for $m = 4, 5$. This construction makes manifest the remarkably good power counting noted in refs. [20, 39]. We do not expect higher points to be quite this simple, but we do anticipate strong constraints between generic one-loop amplitudes of gravity theories and those of gauge theory.

B. Four-point one-loop $\mathcal{N} \geq 4$ supergravity amplitudes

We now specialize the above general considerations to four-point supergravity amplitude. There is only one independent four-graviton amplitude, $\mathcal{M}^{\text{1-loop}}_{\mathcal{N}\text{susy}}(1^-,2^-,3^+,4^+)$, as the others either vanish or are trivially related by relabelings. As a warmup exercise, we start with $\mathcal{N}=8$ supergravity and we reevaluate this supergravity amplitude using the above considerations. Our starting point is the $\mathcal{N}=4$ super-Yang-Mills one-loop four-point amplitude [38, 41],

$$\mathcal{A}^{\text{1-loop}}_{\mathcal{N}=4}(1,2,3,4) = istg^4 A^{\text{tree}}(1,2,3,4)\Big(c_{1234} I_4^{1234} + c_{1243} I_4^{1243} + c_{1423} I_4^{1423}\Big)\,, \tag{3.8}$$

where $s=(k_1+k_2)^2$ and $t=(k_2+k_3)^2$ are the usual Mandelstam invariants, and the tree amplitude is

$$A^{\text{tree}}(1^-,2^-,3^+,4^+) = \frac{i\,\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle}\,, \tag{3.9}$$

where the angle brackets $\langle i\,j\rangle$ (also $[i\,j]$ below) denotes spinor products. (See *e.g.* ref. [42].) The function I_4^{1234} is the massless scalar box integral defined in eqs. (B6) and (B7) of appendix B. The other box integrals are just relabelings of this one. The expression in eq. (3.8) in terms of the box integral (B6) is valid in dimensions $D < 10$.

The first color factor in eq. (3.8) is given by

$$c_{1234} = \tilde{f}^{ba_1c}\,\tilde{f}^{ca_2d}\,\tilde{f}^{da_3e}\,\tilde{f}^{ea_4b}\,, \tag{3.10}$$

and the others are just relabelings of this one. The kinematic numerator in each case is

$$n_{1234} = n_{1243} = n_{1423} = istA^{\text{tree}}(1,2,3,4)\,. \tag{3.11}$$

These numerators happen to have full crossing symmetry, but that is a special feature of the four-point amplitude in $\mathcal{N}=4$ super-Yang-Mills theory. Because the triangle and bubble diagrams vanish, eq. (3.11) is equivalent to the duality relations (2.3). Thus, this representation of the amplitude trivially satisfies the duality.

Using eq. (3.6), by replacing color factors with numerators and compensating for the coupling change, we then immediately have the four-point $\mathcal{N}=8$ supergravity amplitude,

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8}(1,2,3,4) = -\left(\frac{\kappa}{2}\right)^4 [stA^{\text{tree}}(1,2,3,4)]^2\Big(I_4^{1234} + I_4^{1243} + I_4^{1423}\Big)\,, \tag{3.12}$$

which matches the known amplitude [38, 43].

We now generalize to supergravity amplitudes with fewer supersymmetries. Specifically, consider the one-loop four-graviton amplitudes with the $\mathcal{N}=6$ and $\mathcal{N}=4$ matter multiplets in the loop. These multiplets can be expressed as products of two gauge-theory multiplets:

$$\begin{aligned}\mathcal{N}=6 \text{ matter} &: (\mathcal{N}=4 \text{ sYM}) \times (\mathcal{N}=1 \text{ sYM})_{\text{mat.}}\,,\\ \mathcal{N}=4 \text{ matter} &: (\mathcal{N}=4 \text{ sYM}) \times (\text{scalar})\,,\end{aligned} \tag{3.13}$$

where the $\mathcal{N}=1$ Yang-Mills matter multiplet consists of a Weyl fermion with two real scalars (this combination actually has two-fold supersymmetry so it can also be thought of as a $\mathcal{N}=2$ matter multiplet), and on the second line "(scalar)" denotes a single real scalar.

Following eq. (3.6), we get the gravity amplitude by taking the first copy of the gauge-theory amplitude and replacing the color factors with the kinematic numerator of the second copy, constrained to satisfy the duality (2.3), and switching the coupling to the gravitational one. Because the duality satisfying $\mathcal{N}=4$ super-Yang-Mills kinematic factors at four points (3.11) are independent of the loop momentum, they simply come out of the integral as in eq. (3.7) and behave essentially the same way as color factors. Thus, we have a remarkably simple general formula at four points,

$$\begin{aligned}\mathcal{M}^{\text{1-loop}}_{\mathcal{N}+4\text{ susy}}(1,2,3,4) \;=\; \left(\frac{\kappa}{2}\right)^4 istA^{\text{tree}}(1,2,3,4)\Big(A^{\text{1-loop}}_{\mathcal{N}\text{ susy}}(1,2,3,4) + A^{\text{1-loop}}_{\mathcal{N}\text{ susy}}(1,2,4,3)\\ + A^{\text{1-loop}}_{\mathcal{N}\text{ susy}}(1,4,2,3)\Big)\,,\end{aligned} \tag{3.14}$$

where $A^{\text{1-loop}}_{\mathcal{N}\text{ susy}}$ are one-loop color- and coupling-stripped gauge-theory amplitudes for a theory with $\mathcal{N}$ (including zero) supersymmetries. We were able pull out an overall $stA^{\text{tree}}(1,2,3,4)$ because of the crossing symmetry apparent in eq. (3.11).

Using eq. (3.14) we can straightforwardly write down the four-graviton supergravity amplitude $\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{mat.}}(1^-,2^-,3^+,4^+)$ with the $\mathcal{N}=6$ matter multiplet in the loop. We use the $\mathcal{N}=1$ one-loop amplitude representation[3] from ref. [17] which is valid to all order in the dimensional regularization parameter ϵ:

$$\begin{aligned}A^{\text{1-loop}}_{\mathcal{N}=1,\text{mat.}}(1^-,2^-,3^+,4^+) \;&=\; ig^4A^{\text{tree}}(1^-,2^-,3^+,4^+)\Big(tJ_4(s,t)-I_2(t)\Big)\,,\\ A^{\text{1-loop}}_{\mathcal{N}=1,\text{mat.}}(1^-,2^-,4^+,3^+) \;&=\; ig^4A^{\text{tree}}(1^-,2^-,3^+,4^+)\Big(tJ_4(s,u)-\frac{t}{u}I_2(u)\Big)\,,\\ A^{\text{1-loop}}_{\mathcal{N}=1,\text{mat.}}(1^-,4^+,2^-,3^+) \;&=\; ig^4A^{\text{tree}}(1^-,2^-,3^+,4^+)\Big(I_2(t)+\frac{t}{u}I_2(u)\\ &\qquad - tJ_4(t,u) - tI_4^{D=6-2\epsilon}(t,u)\Big)\,,\end{aligned} \tag{3.15}$$

where the integrals I_2, J_4 and $I_4^{D=6-2\epsilon}$ are defined in appendix B. Using eq. (3.14) we can see that the bubble integrals cancel and we have the amplitude in a form valid to all orders

[3] Here we removed the factor of $i(-1)^{m+1}(4\pi)^{2-\epsilon}$ present in the integrals of ref. [17], where m is 2 for the bubble, 3 for the triangle and 4 for the box. (Compare eq. (B1) with eq. (A.13) of ref. [17].)

in ϵ. Also using the relation $J_4 = -\epsilon I_4^{D=6-2\epsilon}$, we get

$$\begin{aligned}\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{mat.}}(1^-,2^-,3^+,4^+) &= \left(\frac{\kappa}{2}\right)^4 \frac{1}{s}[stA^{\text{tree}}(1^-,2^-,3^+,4^+)]^2 \\ &\quad\times \Big[I_4^{D=6-2\epsilon}(t,u) + \epsilon\Big(-I_4^{D=6-2\epsilon}(t,u) + I_4^{D=6-2\epsilon}(s,t) \\ &\qquad + I_4^{D=6-2\epsilon}(s,u)\Big)\Big]\,.\end{aligned} \tag{3.16}$$

Using the explicit value of $I_4^{D=6-2\epsilon}$ given in eq. (B14), we get the remarkably simple result to order ϵ^0,

$$\begin{aligned}\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{mat.}}(1^-,2^-,3^+,4^+) &= \frac{ic_\Gamma}{2}\left(\frac{\kappa}{2}\right)^4 [stA^{\text{tree}}(1^-,2^-,3^+,4^+)]^2\,\frac{1}{s^2}\left[\ln^2\left(\frac{-t}{-u}\right)+\pi^2\right] + \mathcal{O}(\epsilon) \\ &= -\frac{ic_\Gamma}{2}\left(\frac{\kappa}{2}\right)^4 \frac{\langle 1\,2\rangle^4\,[3\,4]^4}{s^2}\left[\ln^2\left(\frac{-t}{-u}\right)+\pi^2\right] + \mathcal{O}(\epsilon)\,,\end{aligned} \tag{3.17}$$

where the constant c_Γ is defined in eq. (B3). On the last line we plugged in the value of the tree amplitude, $stA^{\text{tree}}(1^-,2^-,3^+,4^+) = -i\,\langle 1\,2\rangle^2\,[3\,4]^2$. Indeed, this reproduces the known result from ref. [19].

Now consider the four-graviton amplitude with an $\mathcal{N}=4$ supergravity matter multiplet going around the loop. We take the four-gluon amplitudes with a scalar in the loop from ref. [17]. These are

$$\begin{aligned}A^{\text{1-loop}}_{\text{scalar}}(1^-,2^-,3^+,4^+) &= -ig^4 A^{\text{tree}}(1^-,2^-,3^+,4^+)\Big(\frac{1}{t}I_2^{D=6-2\epsilon}(t) + \frac{1}{s}J_2(t) - \frac{t}{s}K_4(s,t)\Big)\,,\\ A^{\text{1-loop}}_{\text{scalar}}(1^-,2^-,4^+,3^+) &= -ig^4 A^{\text{tree}}(1^-,2^-,3^+,4^+)\Big(\frac{t}{u^2}I_2^{D=6-2\epsilon}(u) + \frac{t}{su}J_2(u) - \frac{t}{s}K_4(s,u)\Big)\,,\\ A^{\text{1-loop}}_{\text{scalar}}(1^-,4^+,2^-,3^+) &= -ig^4 A^{\text{tree}}(1^-,2^-,3^+,4^+)\Big(-\frac{t(t-u)}{s^2}J_3(u) - \frac{t(u-t)}{s^2}J_3(t) - \frac{t^2}{s^2}I_2(u) \\ &\quad - \frac{tu}{s^2}I_2(t) - \frac{t}{u^2}I_2^{D=6-2\epsilon}(u) - \frac{1}{t}I_2^{D=6-2\epsilon}(t) - \frac{t}{su}J_2(u) - \frac{1}{s}J_2(t) \\ &\quad + \frac{t}{s}I_3^{D=6-2\epsilon}(u) + \frac{t}{s}I_3^{D=6-2\epsilon}(t) + \frac{t^2u}{s^2}I_4^{D=6-2\epsilon}(t,u) - \frac{t}{s}K_4(t,u)\Big)\,,\end{aligned} \tag{3.18}$$

where the integral functions are given in appendix B. Using eq. (3.14), we immediately have a form for the contributions of an $\mathcal{N}=4$ supergravity matter multiplet valid to all orders in ϵ,

$$\begin{aligned}\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{mat.}}(1^-,2^-,3^+,4^+) &= \left(\frac{\kappa}{2}\right)^4 [stA^{\text{tree}}(1^-,2^-,3^+,4^+)]^2\Big(-\frac{(t-u)}{s^3}J_3(u) - \frac{(u-t)}{s^3}J_3(t) \\ &\quad - \frac{t}{s^3}I_2(u) - \frac{u}{s^3}I_2(t) + \frac{1}{s^2}I_3^{D=6-2\epsilon}(u) + \frac{1}{s^2}I_3^{D=6-2\epsilon}(t) \\ &\quad + \frac{tu}{s^3}I_4^{D=6-2\epsilon}(t,u) - \frac{1}{s^2}K_4(t,u) - \frac{1}{s^2}K_4(s,t) - \frac{1}{s^2}K_4(s,u)\Big)\,.\end{aligned} \tag{3.19}$$

Expanding this through order ϵ^0 and using integral identities from refs. [17, 44] (see also appendix B) to reexpress everything in terms of six-dimensional boxes, bubbles and rational

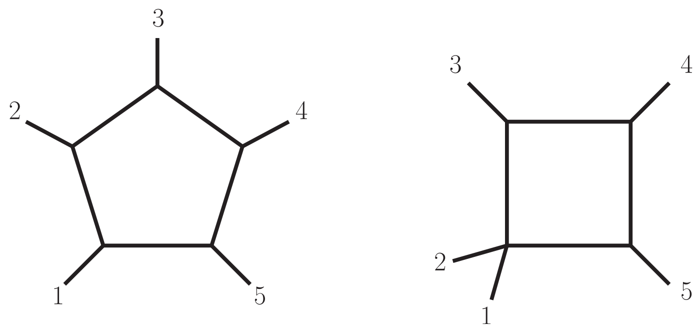

FIG. 3: Pentagon and box integrals appearing in the $\mathcal{N}=4$ super-Yang-Mills five-point one-loop amplitudes. The complete set of such integrals is generated by permuting external legs and removing overcounts.

terms, we obtain

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{mat.}}(1^-,2^-,3^+,4^+) \;=\; \frac{1}{2}\Big(\frac{\kappa}{2}\Big)^4 \frac{\langle 1\,2\rangle^2\,[3\,4]^2}{[1\,2]^2\,\langle 3\,4\rangle^2}\Big[ic_\Gamma s^2 + s(u-t)\Big(I_2(t)-I_2(u)\Big) - 2I_4^{D=6-2\epsilon}(t,u)stu\Big] + \mathcal{O}(\epsilon)\,, \qquad (3.20)$$

matching the result of ref. [19].

C. Five-point one-loop $\mathcal{N}\geq 4$ supergravity amplitudes

Our construction at five points is again directly based on eq. (3.6). We only need to construct $\mathcal{M}^{\text{1-loop}}_{\mathcal{N}\text{susy}}(1^-,2^-,3^+,4^+,5^+)$; the other nonvanishing amplitudes are related by parity and relabeling. Our starting point is the known one-loop five-point amplitudes of $\mathcal{N}=4$ super-Yang-Mills theory. The original construction of the amplitude [11, 18] uses a basis of scalar box integrals. Rearranging these results into the adjoint-representation color basis gives

$$\mathcal{A}^{\text{1-loop}}(1,2,3,4,5) = g^5 \sum_{S_5/(Z_5\times Z_2)} c_{12345}\, A^{\text{1-loop}}(1,2,3,4,5)\,. \qquad (3.21)$$

The sum runs over the distinct permutations of the external legs of the amplitude. This is the set of all 5! permutations, S_5, but with cyclic, Z_5, and reflection symmetries, Z_2, removed, leaving 12 distinct permutations. The color factor c_{12345} is the one of the pentagon diagram shown in fig. 3, with legs following the cyclic ordering as in eq. (3.1). The color-ordered one-loop amplitudes of $\mathcal{N}=4$ super-Yang-Mills theory are

$$A^{\text{1-loop}}_{\mathcal{N}=4}(1,2,3,4,5) \;=\; \frac{i}{2}A^{\text{tree}}(1,2,3,4,5)\Big(s_{34}s_{45}I_4^{(12)345} + s_{45}s_{15}I_4^{1(23)45} + s_{12}s_{15}I_4^{12(34)5} + s_{12}s_{23}I_4^{123(45)} + s_{23}s_{34}I_4^{234(51)}\Big) + \mathcal{O}(\epsilon)\,, \quad (3.22)$$

where $s_{ij}=(k_i+k_j)^2$ and the $I_4^{abc(de)}$ are box integrals where the legs in parenthesis connects to the same vertex, *e.g.* $I_4^{(12)345}$ is the box diagram in fig. 3. The explicit value of $I_4^{(12)345}$ is given in eq. (B8), and the values of the remaining box integrals are obtained by relabeling.

If we insert these explicit expressions in eq. (3.22) then the polylogarithms cancel after using identities (see refs. [11, 18]) leaving the expression for $A^{\text{1-loop}}_{\mathcal{N}=4}$ given in eq. (A1) of appendix A. The representation (3.22) of the amplitude does not manifestly satisfy the duality.

A duality satisfying representation of the amplitude was found in ref. [7]:

$$\mathcal{A}^{\text{1-loop}}_{\mathcal{N}=4}(1^-,2^-,3^+,4^+,5^+) \;=\; g^5 \langle 1\,2\rangle^4 \Bigg(\sum_{S_5/(Z_5\times Z_2)} c_{12345} n_{12345} I_5^{12345} + \sum_{S_5/Z_2^2} c_{[12]345} n_{[12]345} \frac{1}{s_{12}} I_4^{(12)345} \Bigg)\,, \tag{3.23}$$

where I_5^{12345} is the scalar pentagon, and $I_4^{(12)345}$ is the one-mass scalar box integral, as shown in fig. 3. The explicit values of these integrals through $\mathcal{O}(\epsilon^0)$ are collected in appendix B. Each of the two sums runs over the distinct permutations of the external legs of the integrals. For I_5^{12345}, the set $S_5/(Z_5 \times Z_2)$ denotes all permutations but with cyclic and reflection symmetries removed, leaving 12 distinct permutations. For $I_4^{(12)345}$ the set S_5/Z_2^2 denotes all permutations but with the two symmetries of the one-mass box removed, leaving 30 distinct permutations. Note that we pulled out an overall factor $\langle 1\,2\rangle^4$, which we do not include in the numerators. (If promoted to its supersymmetric form it should then be included [7].) The numerators defined in this way are then [7]

$$n_{12345} = -\frac{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]}{4i\epsilon(1,2,3,4)}\,, \tag{3.24}$$

and

$$n_{[12]345} = \frac{[1\,2]^2\,[3\,4]\,[4\,5]\,[5\,3]}{4i\epsilon(1,2,3,4)}\,, \tag{3.25}$$

where $4i\epsilon(1,2,3,4) = 4i\epsilon_{\mu\nu\rho\sigma}k_1^\mu k_2^\nu k_3^\rho k_4^\sigma = [1\,2]\,\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle - \langle 1\,2\rangle\,[2\,3]\,\langle 3\,4\rangle\,[4\,1]$. It is not difficult to confirm that the duality holds for this representation, for example,

$$n_{12345} - n_{21345} = n_{[12]345}\,. \tag{3.26}$$

A nice feature of this representation is that the numerator factors of both the pentagon and box integrals do not depend on loop momentum, allowing us to use eq. (3.7). This will greatly simplify the construction of the corresponding supergravity amplitudes.

We first consider the one-loop five-point $\mathcal{N}=8$ amplitude. In this case we have several useful representations. Proceeding as in section III B, using eq. (3.7), we can obtain the five-point amplitude for $\mathcal{N}=8$ by replacing the color factors in eq. (3.21) with the numerator factors of eq. (3.24), multiplying by the overall factor $\langle 1\,2\rangle^4$, and putting in the gravitational couplings. This yields

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8}(1^-,2^-,3^+,4^+,5^+) \;=\; \frac{i}{2}\Big(\frac{\kappa}{2}\Big)^5 \langle 1\,2\rangle^4 \sum_{S_5/Z_2} n_{12345} A^{\text{tree}}(1^-,2^-,3^+,4^+,5^+) s_{12}s_{23} I_4^{123(45)} + \mathcal{O}(\epsilon)\,, \tag{3.27}$$

where the sum runs over all permutations of external legs, denoted by S_5, but with reflections Z_2 removed. To obtain a second representation, we can instead replace the color factors in eq. (3.23) with their corresponding numerator factors, yielding an alternative expression for the amplitude,

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8}(1^-,2^-,3^+,4^+,5^+) = \left(\frac{\kappa}{2}\right)^5 \langle 1\,2\rangle^8 \Big(\sum_{S_5/(Z_5\times Z_2)} (n_{12345})^2 I_5^{12345} + \sum_{S_5/Z_2^2} (n_{[12]345})^2 \frac{1}{s_{12}} I_4^{(12)345} \Big)\,, \tag{3.28}$$

where the sums run over the same permutations as in eq. (3.23). We have checked that in $D=4$ both formulas (3.27) and (3.28) are equivalent to the known five-point amplitude from ref. [45] (after reducing the scalar pentagon integrals to one-mass box integrals),

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=8}(1^-,2^-,3^+,4^+,5^+) = \left(\frac{\kappa}{2}\right)^5 \langle 1\,2\rangle^8 \sum_{S_5/Z_2^2} d^{123(45)}_{\mathcal{N}=8}\, I_4^{123(45)} + \mathcal{O}(\epsilon)\,, \tag{3.29}$$

where the box coefficient is given by

$$d^{123(45)}_{\mathcal{N}=8} \equiv -\frac{1}{8} h(1,\{2\},3) h(3,\{4,5\},1)\, \text{tr}^2[\not k_1 \not k_2 \not k_3 (\not k_4 + \not k_5)]\,, \tag{3.30}$$

and the "half-soft" functions are

$$h(a,\{2\},b) \equiv \frac{1}{\langle a\,2\rangle^2 \langle 2\,b\rangle^2}\,, \qquad h(a,\{4,5\},b) \equiv \frac{[4\,5]}{\langle 4\,5\rangle \langle a\,4\rangle \langle 4\,b\rangle \langle a\,5\rangle \langle 5\,b\rangle}\,. \tag{3.31}$$

Indeed it is straightforward to check that

$$\langle 1\,2\rangle^4 d^{123(45)}_{\mathcal{N}=8} = \frac{i}{2} s_{12} s_{23} \Big(n_{12345} A^{\text{tree}}(1^-,2^-,3^+,4^+,5^+) + n_{12354} A^{\text{tree}}(1^-,2^-,3^+,5^+,4^+) \Big)\,, \tag{3.32}$$

where the pentagon numerator is given in eq. (3.24).

Let us now study amplitudes with fewer supersymmetries starting with the five-graviton amplitude with the $\mathcal{N}=6$ matter multiplet running around the loop. We pick the helicities $(1^-,2^-,3^+,4^+,5^+)$ for the gravitons; as noted above all other helicity or particle configurations can be obtained from this. For the $\mathcal{N}=6$ and $\mathcal{N}=4$ matter multiplets from eq. (3.7) we have

$$\begin{aligned} M_{\mathcal{N}=6,\text{mat.}}(1^-,2^-,3^+,4^+,5^+) &= \left(\frac{\kappa}{2}\right)^5 \langle 1\,2\rangle^4 \sum_{S_5/(Z_5\times Z_2)} n_{12345}\, A^{\text{1-loop}}_{\mathcal{N}=1,\text{mat.}}(1^-,2^-,3^+,4^+,5^+)\,, \\ M_{\mathcal{N}=4,\text{mat.}}(1^-,2^-,3^+,4^+,5^+) &= \left(\frac{\kappa}{2}\right)^5 \langle 1\,2\rangle^4 \sum_{S_5/(Z_5\times Z_2)} n_{12345}\, A^{\text{1-loop}}_{\text{scalar}}(1^-,2^-,3^+,4^+,5^+)\,, \end{aligned} \tag{3.33}$$

where n_{12345} is given in eq. (3.24) and the sums run over all permutations, but with cyclic ones and the reflection removed.

There are a number of simplifications that occur because of the permutation sum in eq. (3.33) and because of the algebraic properties of the $\mathcal{N}=4$ sYM numerators (n_{12345} and permutations). Because the matter multiplet contributions have neither infrared nor ultraviolet divergences [46], all $1/\epsilon^2$ and $1/\epsilon$ divergences cancel. In $\mathcal{N}=6$ supergravity, this manifests itself by the cancellation of all bubble and triangle integral contributions, as noted in ref. [20]. In the case of $\mathcal{N}=4$ supergravity, the cancellation is not complete but the sum over bubble-integral coefficients vanishes to prevent the appearance of a $1/\epsilon$ singularity. A rational function remains which can be written in a relatively simple form once the terms are combined and simplified. Our results match those obtained in ref. [20].

The final form of the $\mathcal{N}=6$ results after simplifications are then [20]

$$
\begin{aligned}
&\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{mat.}}(1^-,2^-,3^+,4^+,5^+) \\
&\quad = -\left(\frac{\kappa}{2}\right)^5 \langle 1\,2\rangle^8 \sum_{Z_3(345)} \left(\frac{\langle 1\,3\rangle\,\langle 2\,3\rangle\,\langle 1\,4\rangle\,\langle 2\,4\rangle}{\langle 3\,4\rangle^2\,\langle 1\,2\rangle^2}\right)\left(d^{324(51)}_{\mathcal{N}=8}\, I^{324(51)}_{4,\text{trunc}} + d^{314(52)}_{\mathcal{N}=8}\, I^{314(52)}_{4,\text{trunc}}\right) \\
&\qquad + \mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{3.34}
$$

where the summation runs over the three cyclic permutations of legs 3, 4, 5 in the box integrals and coefficients. The factor $d^{123(45)}_{\mathcal{N}=8}$ is exactly the coefficient (3.30) of the $\mathcal{N}=8$ theory and the integral $I^{123(45)}_{4,\text{trunc}}$ given in eq. (B9) of appendix B is the one-mass box integral but with its infrared divergent terms subtracted out. Similarly, the simplified $\mathcal{N}=4$ supergravity results are

$$
\begin{aligned}
&\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{mat.}}(1^-,2^-,3^+,4^+,5^+) \\
&\quad = \left(\frac{\kappa}{2}\right)^5 \Bigg[\langle 1\,2\rangle^8 \sum_{Z_3(345)} \left(\frac{\langle 1\,3\rangle\,\langle 2\,3\rangle\,\langle 1\,4\rangle\,\langle 2\,4\rangle}{\langle 3\,4\rangle^2\,\langle 1\,2\rangle^2}\right)^2\left(d^{324(51)}_{\mathcal{N}=8}\, I^{324(51)}_{4,\text{trunc}} + d^{314(52)}_{\mathcal{N}=8}\, I^{314(52)}_{4,\text{trunc}}\right) \\
&\qquad + ic_\Gamma \sum_{i=3}^{5} (c_{1i}\ln(-s_{1i}) + c_{2i}\ln(-s_{2i})) + ic_\Gamma R_5\Bigg] + \mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{3.35}
$$

where the coefficient of $\log(-s_{13})$ coming from the bubble integrals is

$$
\begin{aligned}
c_{13} \;=\; & \frac{1}{2}\,\frac{\langle 1\,2\rangle^4\,[3\,1]\,[5\,2]}{\langle 1\,3\rangle\,\langle 2\,5\rangle\,\langle 4\,5\rangle}\Bigg[-\frac{\langle 2\,4\rangle^2\,\langle 4|2+5|4]\ \langle 1|3|4]^2}{\langle 3\,4\rangle^2\,\langle 4\,5\rangle\,\langle 4|1+3|4]^2} - \frac{\langle 2\,3\rangle}{\langle 3\,4\rangle}\left(\frac{\langle 1\,5\rangle\,\langle 2\,5\rangle\,\langle 1|3|5]\ \langle 5|2|4]}{\langle 3\,5\rangle^2\,\langle 4\,5\rangle\,\langle 5|1+3|5]}\right. \\
& \left. - \frac{\langle 1\,4\rangle\,\langle 2\,4\rangle\,\langle 1|3|4]\ \langle 4|2+5|4]}{\langle 3\,4\rangle^2\,\langle 4\,5\rangle\,\langle 4|1+3|4]}\right) + \frac{\langle 2\,4\rangle}{\langle 3\,4\rangle}\left(\frac{\langle 1\,4\rangle\,\langle 2\,3\rangle\,\langle 1|3|4]\ \langle 3|2+5|4]}{\langle 3\,4\rangle^2\,\langle 3\,5\rangle\,\langle 4|1+3|4]}\right. \\
& \left. + \frac{\langle 2\,5\rangle\,\langle 5|2|4]}{\langle 3\,5\rangle\,\langle 4\,5\rangle}\left(\frac{\langle 1\,5\rangle\,\langle 1|3|5]}{\langle 4\,5\rangle\,\langle 5|1+3|5]} - \frac{\langle 1\,4\rangle\,\langle 1|3|4]}{\langle 4\,5\rangle\,\langle 4|1+3|4]}\right)\right)\Bigg] + (4\leftrightarrow 5)\,,
\end{aligned}
\tag{3.36}
$$

and the others are given by the natural label swaps, $c_{1i} = c_{13}|_{3\leftrightarrow i}$ and $c_{2i} = c_{1i}|_{1\leftrightarrow 2}$. The rational terms follow the notation of ref. [20],

$$
R_5 = R_5^b + \sum_{Z_2(12)\times Z_3(345)} R_5^a\,,
\tag{3.37}
$$

where

$$
R_5^a \;=\; -\frac{1}{2}\,\langle 1\,2\rangle^4\,\frac{[3\,4]^2\,[2\,5]\,\langle 2\,3\rangle\,\langle 2\,4\rangle}{\langle 3\,4\rangle^2\,\langle 2\,5\rangle\,\langle 3\,5\rangle\,\langle 4\,5\rangle}\,, \qquad R_5^b = -\,\langle 1\,2\rangle^4\,\frac{[3\,4]\,[3\,5]\,[4\,5]}{\langle 3\,4\rangle\,\langle 3\,5\rangle\,\langle 4\,5\rangle}\,.
\tag{3.38}
$$

The sum in eq. (3.37) corresponds to the composition of the two permutations of negative-helicity legs 1 and 2 and the three cyclic permutations over the positive-helicity legs 3, 4 and 5, giving six terms in total. (Results for general MHV amplitudes may be found in ref. [20].)

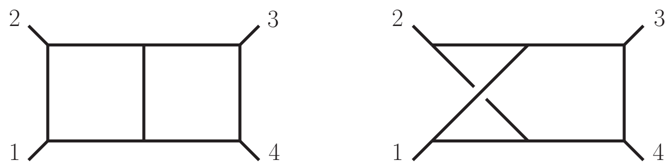

FIG. 4: The two-loop cubic diagrams appearing in the two-loop four-point $\mathcal{N}=4$ and $\mathcal{N}=8$ supergravity amplitudes.

Inserting the results from eq. (3.33) into eq. (2.8) immediately converts the results we obtained for the matter multiplets into those for the $\mathcal{N}=4,5,6$ gravity multiplets (the pure supergravities). For the $\mathcal{N}=4$ and $\mathcal{N}=6$ gravity multiplets these match the results of ref. [20].

Thus we have succeeded in expressing the four- and five-point integrated amplitudes of $\mathcal{N}\geq 4$ supergravity amplitudes as simple linear combination of corresponding gauge-theory ones. To generalize this construction to higher points, one would need to find duality satisfying representations of m-point one-loop $\mathcal{N}=4$ super-Yang-Mills amplitudes.

D. Comments on two loops

An interesting question is whether the same considerations hold at higher loops. Consider the two-loop four-point amplitude of $\mathcal{N}=4$ super-Yang-Mills theory [41, 43]:

$$\begin{aligned}\mathcal{A}_4^{\text{2-loop}}(1,2,3,4) \;=\; -g^6 st\, A_4^{\text{tree}}(1,2,3,4)\Big(&c^{\text{P}}_{1234}\, s\, I_4^{\text{2-loop,P}}(s,t)+c^{\text{P}}_{3421}\, s\, I_4^{\text{2-loop,P}}(s,u) \\ &+c^{\text{NP}}_{1234}\, s\, I_4^{\text{2-loop,NP}}(s,t)+c^{\text{NP}}_{3421}\, s\, I_4^{\text{2-loop,NP}}(s,u)+\text{cyclic}\Big)\,,\end{aligned} \tag{3.39}$$

where '+ cyclic' instructs one to add the two cyclic permutations of (2,3,4) and the integrals correspond to the scalar planar and nonplanar double-box diagrams displayed in fig. 4. As at one loop, the color factor for each diagram is obtained by dressing each cubic vertex with an $\tilde{f}^{abc}$. It is then simple to check that all duality relations (2.3) hold.

According to the double-copy prescription (2.5), we obtain the corresponding $\mathcal{N}=8$ supergravity amplitude by replacing the color factor with a numerator factor,

$$c^{\text{P}}_{1234}\to is^2tA^{\text{tree}}(1,2,3,4)\,,\qquad\qquad c^{\text{NP}}_{1234}\to is^2tA^{\text{tree}}(1,2,3,4)\,, \tag{3.40}$$

including relabelings and then swapping the gauge coupling for the gravitational one. Indeed, this gives the correct $\mathcal{N}=8$ supergravity amplitude, as already noted in ref. [43].

As explained in section II, generalized gauge invariance implies that we need have only one of the two copies in a form manifestly satisfying the duality (2.3). The color Jacobi identity allows us to express any four-point color factor of an adjoint representation in terms of the ones in fig. 4 [37]. If the duality and double-copy properties hold we should then be able to obtain integrated $\mathcal{N}\geq 4$ supergravity amplitudes starting from $\mathcal{N}\leq 4$ super-Yang-Mills theory and applying the replacement rule (3.40). Indeed, in ref. [47], explicit expressions for the four-point two-loop $\mathcal{N}\geq 4$ supergravity amplitudes, including the finite terms, are obtained in this manner.

Two-loop supergravity amplitudes are UV finite and their IR behavior is given in terms of the square of the one-loop amplitude [21]:

$$\mathcal{M}_4^{(2\text{-loop})}(\epsilon) = \frac{1}{2}\Big[\mathcal{M}_4^{1\text{-loop}}(\epsilon)\Big]^2 + \text{finite}\,. \tag{3.41}$$

The amplitudes of ref. [47] satisfy this relation and the finite remainders are given in a relatively simple form. These two-loop results then provide a rather nontrivial confirmation of the duality and double-copy properties for cases with less than maximal supersymmetry.

IV. CONCLUSIONS

The duality between color and kinematic numerators offers a powerful means for obtaining loop-level gauge and gravity amplitudes and for understanding their structure. A consequence of the duality conjecture is that complete amplitudes are controlled by a set of master diagrams; once the numerators are known in a form that makes the duality between color and kinematics manifest, all others are determined from Jacobi-like relations. In this form we immediately obtain gravity integrands via the double-copy relation.

In the present paper, we used the duality to find examples where *integrated* supergravity amplitudes are expressed directly as linear combinations of gauge-theory amplitudes. In particular, we constructed the integrated four- and five-point one-loop amplitudes of $\mathcal{N} \geq 4$ supergravity directly from known gauge-theory amplitudes. This construction was based on identifying representations of $\mathcal{N} = 4$ super-Yang-Mills four- and five-point amplitudes that satisfy the duality. Because the relations are valid in D dimensions, by using known D-dimensional forms of gauge-theory four-point amplitudes we obtain corresponding ones for supergravity. The agreement of our four- and five-point $\mathcal{N} \geq 4$ supergravity results with independent evaluations [19, 20] in $D = 4$ provides evidence in favor of these conjectures holding for less than maximal supersymmetry. The two-loop results in ref. [47] provide further nontrivial evidence.

The examples we presented here are particularly simple because the numerator factors of one copy of the gauge-theory amplitudes were independent of loop momenta. In more general cases, we expect useful constraints to arise at the integrated level. These constraints, for example, lead to KLT-like relations visible in box-integral coefficients, such as those found in refs. [45, 48]. It would be very interesting to further explore relations between gravity and gauge theory after having carried out the loop integration.

There are a number of other interesting related problems. It would of course be important to unravel the underlying group-theoretic structure responsible for the duality between color and kinematics. Some interesting progress has recently made for self-dual field configurations and for MHV tree amplitudes, identifying an underlying diffeomorphism Lie algebra [31]. Another key problem is to find better means for finding representations that automatically satisfy the duality and double-copy properties. Such general representations are known at tree level for any choice of helicities [26]. We would like to have similar constructions at loop level, instead of having to find duality satisfying forms case by case. In particular, no examples have as yet been constructed at loop level at six and higher points.

In summary, using the duality between color and kinematics we exposed a surprising relation between integrated four- and five-point one-loop amplitudes of $\mathcal{N} \geq 4$ supergravity and those of gauge theory. We look forward to applying these ideas to further unravel the structure of gauge and gravity loop amplitudes.

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Appendix A: The one-loop five-point Yang-Mills amplitudes

This appendix collects the five-point one-loop Yang-Mills amplitudes used to construct the five-point supergravity amplitudes. The external states are gluons and all amplitudes can be obtained from two configurations, $(1^-, 2^-, 3^+, 4^+, 5^+)$ and $(1^-, 2^+, 3^-, 4^+, 5^+)$, using relabeling and parity. These results are from ref. [18] which the reader is invited to consult for further details. The results are presented in the four-dimension helicity (FDH) regularization scheme [16], which is known to preserve supersymmetry at one loop.

The five-gluon color-ordered and coupling-stripped amplitudes with the $\mathcal{N} = 4$, $\mathcal{N} = 1$ matter multiplet and a *real* scalar going around the loop can be expressed as:

$$\begin{aligned} A^{\text{1-loop}}_{\mathcal{N}=4}(1,2,3,4,5) &= c_\Gamma V^g A_5^{\text{tree}}\,, \\ A^{\text{1-loop}}_{\mathcal{N}=1,\text{mat.}}(1,2,3,4,5) &= -c_\Gamma (V^f A_5^{\text{tree}} + iF^f)\,, \\ A^{\text{1-loop}}_{\text{scalar}}(1,2,3,4,5) &= \frac{1}{2} c_\Gamma (V^s A_5^{\text{tree}} + iF^s)\,, \end{aligned} \tag{A1}$$

where the tree amplitudes are

$$\begin{aligned} A_5^{\text{tree}}(1^-, 2^-, 3^+, 4^+, 5^+) &= \frac{i\langle 1\,2\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\,, \\ A_5^{\text{tree}}(1^-, 2^+, 3^-, 4^+, 5^+) &= \frac{i\langle 1\,3\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\,. \end{aligned} \tag{A2}$$

The function,

$$\text{V}^g = -\frac{1}{\epsilon^2}\sum_{j=1}^{5}(-s_{j,j+1})^{-\epsilon} + \sum_{j=1}^{5} \ln\left(\frac{-s_{j,j+1}}{-s_{j+1,j+2}}\right) \ln\left(\frac{-s_{j+2,j-2}}{-s_{j-2,j-1}}\right) + \frac{5}{6}\pi^2\,. \tag{A3}$$

is independent of the helicity configuration. In contrast to ref. [18], we have set the dimensional-regularization scale parameter, μ, to unity. For the $(1^-,2^-,3^+,4^+,5^+)$ helicity configuration we have,

$$\begin{aligned}
V^f &= -\frac{1}{\epsilon}+\frac{1}{2}\left[\ln(-s_{23})+\ln(-s_{51})\right]-2\,, \qquad V^s=-\frac{1}{3}V^f+\frac{2}{9}\,,\\
F^f &= -\frac{1}{2}\frac{\langle 1\,2\rangle^2\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle+\langle 2\,4\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\frac{\mathrm{L}_0\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}}\,,\\
F^s &= -\frac{1}{3}\frac{[3\,4]\,\langle 4\,1\rangle\,\langle 2\,4\rangle\,[4\,5]\left(\langle 2\,3\rangle\,[3\,4]\,\langle 4\,1\rangle+\langle 2\,4\rangle\,[4\,5]\,\langle 5\,1\rangle\right)}{\langle 3\,4\rangle\,\langle 4\,5\rangle}\frac{\mathrm{L}_2\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}^3}-\frac{1}{3}F^f\\
&\quad -\frac{1}{3}\frac{\langle 3\,5\rangle\,[3\,5]^3}{[1\,2]\,[2\,3]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[5\,1]}+\frac{1}{3}\frac{\langle 1\,2\rangle\,[3\,5]^2}{[2\,3]\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,[5\,1]}+\frac{1}{6}\frac{\langle 1\,2\rangle\,[3\,4]\,\langle 4\,1\rangle\,\langle 2\,4\rangle\,[4\,5]}{s_{23}\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,s_{51}}\,,
\end{aligned} \tag{A4}$$

and the corresponding functions for the $(1^-,2^+,3^-,4^+,5^+)$ helicity configuration,

$$\begin{aligned}
V^f &= -\frac{1}{\epsilon}+\frac{1}{2}\left[\ln(-s_{34})+\ln(-s_{51})\right]-2\,, \qquad V^s=-\frac{1}{3}V^f+\frac{2}{9}\,,\\
F^f &= -\frac{\langle 1\,3\rangle^2\langle 4\,1\rangle[2\,4]^2}{\langle 4\,5\rangle\langle 5\,1\rangle}\frac{\mathrm{Ls}_1\left(\frac{-s_{23}}{-s_{51}},\frac{-s_{34}}{-s_{51}}\right)}{s_{51}^2}+\frac{\langle 1\,3\rangle^2\langle 5\,3\rangle[2\,5]^2}{\langle 3\,4\rangle\langle 4\,5\rangle}\frac{\mathrm{Ls}_1\left(\frac{-s_{12}}{-s_{34}},\frac{-s_{51}}{-s_{34}}\right)}{s_{34}^2}\\
&\quad -\frac{1}{2}\frac{\langle 1\,3\rangle^3\left(\langle 1\,5\rangle\,[5\,2]\,\langle 2\,3\rangle-\langle 3\,4\rangle\,[4\,2]\,\langle 2\,1\rangle\right)}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\frac{\mathrm{L}_0\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}}\,,\\
F^s &= -\frac{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle^2[2\,4]^2}{\langle 4\,5\rangle\langle 5\,1\rangle\langle 2\,4\rangle^2}\frac{2\,\mathrm{Ls}_1\left(\frac{-s_{23}}{-s_{51}},\frac{-s_{34}}{-s_{51}}\right)+\mathrm{L}_1\left(\frac{-s_{23}}{-s_{51}}\right)+\mathrm{L}_1\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}^2}\\
&\quad +\frac{\langle 3\,2\rangle\langle 2\,1\rangle\langle 1\,5\rangle\langle 5\,3\rangle^2[2\,5]^2}{\langle 5\,4\rangle\langle 4\,3\rangle\langle 2\,5\rangle^2}\frac{2\,\mathrm{Ls}_1\left(\frac{-s_{12}}{-s_{34}},\frac{-s_{51}}{-s_{34}}\right)+\mathrm{L}_1\left(\frac{-s_{12}}{-s_{34}}\right)+\mathrm{L}_1\left(\frac{-s_{51}}{-s_{34}}\right)}{s_{34}^2}\\
&\quad +\frac{2}{3}\frac{\langle 2\,3\rangle^2\langle 4\,1\rangle^3[2\,4]^3}{\langle 4\,5\rangle\langle 5\,1\rangle\langle 2\,4\rangle}\frac{\mathrm{L}_2\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}^3}-\frac{2}{3}\frac{\langle 2\,1\rangle^2\langle 5\,3\rangle^3[2\,5]^3}{\langle 5\,4\rangle\langle 4\,3\rangle\langle 2\,5\rangle}\frac{\mathrm{L}_2\left(\frac{-s_{12}}{-s_{34}}\right)}{s_{34}^3}\\
&\quad +\frac{\mathrm{L}_2\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}^3}\Bigg(\frac{1}{3}\frac{\langle 1\,3\rangle\,[2\,4]\,[2\,5]\left(\langle 1\,5\rangle\,[5\,2]\,\langle 2\,3\rangle-\langle 3\,4\rangle\,[4\,2]\,\langle 2\,1\rangle\right)}{\langle 4\,5\rangle}\\
&\quad +\frac{2}{3}\frac{\langle 1\,2\rangle^2\langle 3\,4\rangle^2\,\langle 4\,1\rangle\,[2\,4]^3}{\langle 4\,5\rangle\,\langle 5\,1\rangle\,\langle 2\,4\rangle}-\frac{2}{3}\frac{\langle 3\,2\rangle^2\langle 1\,5\rangle^2\,\langle 5\,3\rangle\,[2\,5]^3}{\langle 5\,4\rangle\,\langle 4\,3\rangle\,\langle 2\,5\rangle}\Bigg)\\
&\quad +\frac{1}{6}\frac{\langle 1\,3\rangle^3\left(\langle 1\,5\rangle\,[5\,2]\,\langle 2\,3\rangle-\langle 3\,4\rangle\,[4\,2]\,\langle 2\,1\rangle\right)}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle}\frac{\mathrm{L}_0\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}}+\frac{1}{3}\frac{[2\,4]^2[2\,5]^2}{[1\,2][2\,3][3\,4]\langle 4\,5\rangle[5\,1]}\\
&\quad -\frac{1}{3}\frac{\langle 1\,2\rangle\langle 4\,1\rangle^2[2\,4]^3}{\langle 4\,5\rangle\langle 5\,1\rangle\langle 2\,4\rangle[2\,3][3\,4]s_{51}}+\frac{1}{3}\frac{\langle 3\,2\rangle\langle 5\,3\rangle^2[2\,5]^3}{\langle 5\,4\rangle\langle 4\,3\rangle\langle 2\,5\rangle[2\,1][1\,5]s_{34}}+\frac{1}{6}\frac{\langle 1\,3\rangle^2\,[2\,4]\,[2\,5]}{s_{34}\,\langle 4\,5\rangle\,s_{51}}\,.
\end{aligned} \tag{A5}$$

In contrast to ref. [18], in eqs. (A4) and (A5) we use unrenormalized amplitudes; this distinction actually has no effect on the corresponding gravity amplitudes since the difference drops out in eq. (3.33). The functions appearing in the above expressions are

$$\mathrm{L}_0(r)=\frac{\ln(r)}{1-r}\,,\qquad \mathrm{L}_1(r)=\frac{\ln(r)+1-r}{(1-r)^2}\,,\qquad \mathrm{L}_2(r)=\frac{\ln(r)-(r-1/r)/2}{(1-r)^3}\,,$$

$$\mathrm{Ls}_1(r_1,r_2) \;=\; \frac{1}{(1-r_1-r_2)^2}\Big[\,\mathrm{Li}_2(1-r_1)+\mathrm{Li}_2(1-r_2)+\ln r_1\,\ln r_2-\frac{\pi^2}{6} + (1-r_1-r_2)(\mathrm{L}_0(r_1)+\mathrm{L}_0(r_2))\Big]\,. \tag{A6}$$

As discussed in section III C, these gauge-theory amplitudes serve as building blocks for the corresponding $\mathcal{N}\geq 4$ supergravity amplitudes.

Appendix B: Integrals

In this appendix we collect the integrals used in our expressions from various sources and adjust normalization to match our conventions. The m-point scalar integrals in D dimensions are defined as:

$$I_m = \int \frac{d^Dp}{(2\pi)^D}\frac{1}{p^2(p-K_1)^2(p-K_1-K_2)^2\ldots(p-K_1-K_2-\ldots-K_{m-1})^2}\,, \tag{B1}$$

where the K_i's are the external momenta which can be on- or off-shell.

The $D=4-2\epsilon$ bubble with momentum K is

$$I_2(K^2) = \frac{ic_\Gamma}{\epsilon(1-2\epsilon)}(-K^2)^{-\epsilon}\,, \tag{B2}$$

where

$$c_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{B3}$$

The $D=4-2\epsilon$ one-mass triangle is

$$I_3(K_1^2) = \frac{-ic_\Gamma}{\epsilon^2}(-K_1^2)^{-1-\epsilon}\,, \tag{B4}$$

where K_1 is the massive leg momentum and the two-mass triangle is

$$I_3(K_1^2,K_2^2) = \frac{-ic_\Gamma}{\epsilon^2}\,\frac{(-K_1^2)^{-\epsilon}-(-K_2^2)^{-\epsilon}}{(-K_1^2)-(-K_2^2)}\,, \tag{B5}$$

where K_1 and K_1 are the two massive leg momenta.

For amplitudes with four massless external particles we have the zero-mass box $I_4^{1234}\equiv I_4(s,t)$ where $s=(k_1+k_2)^2$, $t=(k_2+k_3)^2$ and the k_i are massless momenta. An all-order in ϵ expansion in terms of hypergeometric functions is [49]:

$$I_4(s,t) = \frac{2ic_\Gamma}{\epsilon^2 st}\left[t^{-\epsilon}\,{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;1+\frac{t}{s}\right) \;+\; s^{-\epsilon}\,{}_2F_1\left(-\epsilon,-\epsilon;1-\epsilon;1+\frac{s}{t}\right)\right]\,, \tag{B6}$$

which through order ϵ^0 is

$$I_4(s,t) = \frac{ic_\Gamma}{st}\left[\frac{2}{\epsilon^2}\Big((-s)^{-\epsilon}+(-t)^{-\epsilon}\Big)-\ln^2\left(\frac{-s}{-t}\right)-\pi^2\right]+\mathcal{O}(\epsilon)\,. \tag{B7}$$

Similarly, the one-mass box through ϵ^0 is [49],

$$I_4^{(12)345} = -\frac{2ic_\Gamma}{s_{34}s_{45}}\Bigg\{ -\frac{1}{\epsilon^2}\Big[(-s_{34})^{-\epsilon} + (-s_{45})^{-\epsilon} - (-s_{12}^2)^{-\epsilon}\Big] + \mathrm{Li}_2\left(1-\frac{s_{12}}{s_{34}}\right) + \mathrm{Li}_2\left(1-\frac{s_{12}}{s_{45}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{34}}{s_{45}}\right) + \frac{\pi^2}{6}\Bigg\} + \mathcal{O}(\epsilon)\,, \qquad \text{(B8)}$$

where legs 1 and 2 are at the massive corner. An all orders in ϵ form in terms of hypergeometric functions may be found in ref. [49]. The integral $I_{4,\text{trunc}}^{(12)345}$ is given by dropping the term multiplied by $1/\epsilon^2$,

$$I_{4,\text{trunc}}^{(12)345} = -\frac{2ic_\Gamma}{s_{34}s_{45}}\Bigg\{ \mathrm{Li}_2\left(1-\frac{s_{12}}{s_{34}}\right) + \mathrm{Li}_2\left(1-\frac{s_{12}}{s_{45}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{34}}{s_{45}}\right) + \frac{\pi^2}{6}\Bigg\} + \mathcal{O}(\epsilon)\,. \qquad \text{(B9)}$$

Finally, we use the pentagon integral whose expansion to order ϵ^0 is [49]

$$I_5^{12345} = \sum_{Z_5} \frac{-ic_\Gamma(-s_{51})^\epsilon(-s_{12})^\epsilon}{(-s_{23})^{1+\epsilon}(-s_{34})^{1+\epsilon}(-s_{45})^{1+\epsilon}}\left[\frac{1}{\epsilon^2} + 2\,\mathrm{Li}_2\Big(1-\frac{s_{23}}{s_{51}}\Big) + 2\,\mathrm{Li}_2\Big(1-\frac{s_{45}}{s_{12}}\Big) - \frac{\pi^2}{6}\right] + \mathcal{O}(\epsilon)\,, \qquad \text{(B10)}$$

where the sum is over the five cyclic permutations of external legs.

We also need integrals in higher dimensions. The triangle and bubble integrals are obtained by direct integration and the box integrals by dimension-shifting relations [49]. Explicitly, the $D = 6 - 2\epsilon$ bubble is

$$I_2^{D=6-2\epsilon}(K^2) = \frac{-ic_\Gamma}{2\epsilon(1-2\epsilon)(3-2\epsilon)}(-K^2)^{1-\epsilon}\,, \qquad \text{(B11)}$$

whereas the $D = 6 - 2\epsilon$ one-mass triangle is

$$I_3^{D=6-2\epsilon}(K_1^2) = \frac{-ic_\Gamma}{2\epsilon(1-\epsilon)(1-2\epsilon)}(-K_1^2)^{-\epsilon}\,. \qquad \text{(B12)}$$

The zero-mass $D = 6 - 2\epsilon$ box can be expressed as a linear combination of the four-dimensional one-mass boxes and one-mass triangles:

$$I_4^{D=6-2\epsilon}(s,t) = \frac{1}{s+t}\left(\frac{st}{2}I_4(s,t) - i\frac{c_\Gamma}{\epsilon^2}\Big((-s)^{-\epsilon} + (-t)^{-\epsilon}\Big)\right)\,. \qquad \text{(B13)}$$

Note that it is finite and equal to

$$I_4^{D=6-2\epsilon}(s,t) = -i\frac{c_\Gamma}{2(s+t)}\left[\ln^2\left(\frac{-s}{-t}\right) + \pi^2\right] + \mathcal{O}(\epsilon)\,. \qquad \text{(B14)}$$

We also make use of the integral combination from ref. [17],

$$J_m = -\epsilon I_m^{D=6-2\epsilon}\,, \qquad K_m = -\epsilon(1-\epsilon)I_m^{D=8-2\epsilon}\,. \qquad \text{(B15)}$$

Through order ϵ^0, these become

$$J_4 = 0 + \mathcal{O}(\epsilon)\,, \qquad K_4 = -\frac{i}{6(4\pi)^2} + \mathcal{O}(\epsilon)\,, \qquad J_3 = \frac{i}{2(4\pi)^2} + \mathcal{O}(\epsilon)\,. \tag{B16}$$

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