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The one-loop six-dimensional hexagon integral with three massive corners

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Abstract

We compute the six-dimensional hexagon integral with three non-adjacent external masses analytically. After a simple rescaling, it is given by a function of six dual conformally invariant cross-ratios. The result can be expressed as a sum of 24 terms involving only one basic function, which is a simple linear combination of logarithms, dilogarithms, and trilogarithms of uniform degree three transcendentality. Our method uses differential equations to determine the symbol of the function, and an algorithm to reconstruct the latter from its symbol. It is known that six-dimensional hexagon integrals are closely related to scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory, and we therefore expect our result to be helpful for understanding the structure of scattering amplitudes in this theory, in particular at two loops.

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1 Introduction

Scalar $n-$point integrals in dimensions $D > 4$ are interesting objects for a number of reasons. They appear in the $\mathcal{O}(\epsilon)$ part of $(D = 4 - 2\epsilon)$-dimensional one-loop amplitudes [1], which are required for computations at higher loop orders.

Quite generally, higher-dimensional scalar integrals are related to tensor integrals in $D = 4$ dimensions [2]. In particular, the $D = 6$ dimensional hexagons are related to finite tensor integrals [3] that appear in $\mathcal{N} = 4$ super Yang-Mills (SYM). More precisely, they appear as derivatives of four-dimensional two-loop tensor integrals. Moreover, applying a further differential operator, the integrals reduce to four-dimensional one-loop tensor integrals [4]. See Ref. [5] for related work on differential equations relevant for integrals in $\mathcal{N} = 4$ SYM.

Finite dual conformal invariant functions [6, 7] are also prototypes of functions that can appear in the remainder function of MHV amplitudes and the ratio function of non-MHV amplitudes in $\mathcal{N} = 4$ SYM [8, 9, 10]. Recently, the massless and one-mass hexagon integrals in $D = 6$ dimensions were computed in Refs. [4, 11, 12]. It was noted that the massless hexagon integral in $D = 6$ resembles very closely the analytical result of the two-loop remainder function for $n = 6$ points [13, 14, 15]. In this note, we extend the computations of hexagon integrals in $D = 6$ dimensions to the case of three non-adjacent external masses.

Our strategy is the following. We derive simple differential equations that relate the three-mass hexagon to known pentagon integrals. These differential equations, together with a boundary condition, completely determine the answer in principle. We find it convenient to first compute the symbol [16] of the answer, and then reconstruct the function from that symbol.

2 Integral representation and differential equations

We consider the hexagon integral with three massive corners,

$$H_9 := \int \frac{d^6 x_i}{i\pi^3} \frac{1}{x_{1i}^2 x_{2i}^2 x_{4i}^2 x_{5i}^2 x_{7i}^2 x_{8i}^2}\,, \tag{1}$$

where we used dual (or region) coordinates $p_j^\mu = x_j^\mu - x_{j+1}^\mu$ (with indices being defined modulo 9), and $x_{ij}^\mu = x_i^\mu - x_j^\mu$. The on-shell conditions read $x_{12}^2 = 0, x_{45}^2 = 0$ and $x_{78}^2 = 0$. As a scalar integral, H_9 is a function of the (non-zero) external Lorentz invariants x_{jk}^2. We work in signature $(-+++)$, so that the Euclidean region has all (non-zero) x_{jk}^2 positive.

Dual conformal covariance [6, 7] of H_9, in particular under the inversion of all dual coordinates, $x^\mu \to x^\mu / x^2$, allows us to write

$$H_9 =: \frac{1}{x_{15}^2 x_{27}^2 x_{48}^2} \Phi_9(u_1, \ldots, u_6)\,, \tag{2}$$

where the cross ratios

$$u_1 := \frac{x_{25}^2 x_{17}^2}{x_{15}^2 x_{27}^2}\,, \quad u_2 := \frac{x_{58}^2 x_{41}^2}{x_{48}^2 x_{15}^2}\,, \quad u_3 := \frac{x_{82}^2 x_{74}^2}{x_{27}^2 x_{48}^2}\,,$$
$$u_4 := \frac{x_{24}^2 x_{15}^2}{x_{14}^2 x_{25}^2}\,, \quad u_5 := \frac{x_{57}^2 x_{48}^2}{x_{47}^2 x_{58}^2}\,, \quad u_6 := \frac{x_{81}^2 x_{72}^2}{x_{82}^2 x_{17}^2}\,, \tag{3}$$

are invariant under dual conformal transformations. Furthermore, the one-loop hexagon integral with three non-adjacent masses is invariant under the action of the dihedral symmetry group $D_3 \simeq S_3$, generated by the cyclic rotation c and the reflection r acting on the dual coordinates via

$$x_j^\mu \xrightarrow{\;c\;} x_{j+3}^\mu \quad \text{and} \quad x_j^\mu \xrightarrow{\;r\;} x_{9-j}^\mu\,, \tag{4}$$

where as usual all indices are understood modulo 9. It is easy to see that under the symmetry the six conformal cross ratios group into two orbits of three elements,

$$\begin{array}{ll} u_1 \xrightarrow{\;c\;} u_2 \xrightarrow{\;c\;} u_3 \xrightarrow{\;c\;} u_1\,, & u_4 \xrightarrow{\;c\;} u_5 \xrightarrow{\;c\;} u_6 \xrightarrow{\;c\;} u_4\,, \\ u_1 \xleftrightarrow{\;r\;} u_3\,, & u_4 \xleftrightarrow{\;r\;} u_5\,, \\ u_2 \xleftrightarrow{\;r\;} u_2\,, & u_6 \xleftrightarrow{\;r\;} u_6\,. \end{array} \tag{5}$$

One can easily derive a differential equation for H_9 by noting that

$$(x_{21}\cdot\partial_{x_2} + 1)\frac{1}{x_{1i}^2 x_{2i}^2} = \frac{1}{(x_{2i}^2)^2}\,. \tag{6}$$

Applying this differential operator to Eq. (1), we find

$$(x_{21}\cdot\partial_{x_2} + 1)H_9 = \int \frac{d^6 x_i}{i\pi^3}\frac{1}{(x_{2i}^2)^2 x_{4i}^2 x_{5i}^2 x_{7i}^2 x_{8i}^2} =: P_8\,. \tag{7}$$

The one-loop pentagon integral P_8 appearing as an inhomogeneous term in this equation is equivalent to a known four-dimensional pentagon integral [4][1],

$$P_8 =: \frac{1}{x_{25}^2 x_{27}^2 x_{48}^2}\Psi_8(u_3, u_4 u_2, u_5)\,. \tag{8}$$

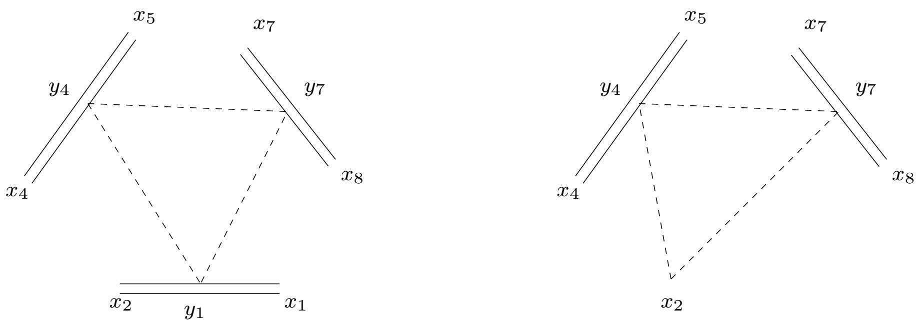
The latter is given by

$$\begin{aligned} \Psi_8(u,v,w) \;=\; \frac{1}{1-u-v+uvw}\Big[& \log u \log v + \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w) \\ & - \mathrm{Li}_2(1-uw) - \mathrm{Li}_2(1-vw)\Big]\,. \end{aligned} \tag{9}$$

We can rewrite Eq. (7) as a differential equation for the rescaled hexagon integral $\Phi_9(u_1,\ldots,u_6)$ that depends on cross-ratios only,

$$D_1\Phi_9(u_1,\ldots,u_6) = \Psi_8(u_3, u_4 u_2, u_5)\,, \tag{10}$$

[1] In Refs. [4, 5], the notation $\tilde{\Psi}$ was used for Ψ_8.



(a) (b)

Figure 1: (a) depicts the representation of H_9 as a line integral, see Eqs. (12) and (13). The differential operator in Eq. (7) localizes the y_1 integration to x_2, yielding $P_8(x_2, x_4, x_5, x_7, x_8)$, see (b).

where

$$D_1 := u_1 + u_1 u_6(u_6 - 1)\partial_6 + (u_4 - 1)\partial_4 + u_1(u_1 - 1)\partial_1 + u_1(1 - u_6)u_3\partial_3 \,, \tag{11}$$

with $\partial_i := \partial/\partial u_i$. By cyclic and reflection symmetry, we have a total of six differential equations. It turns out that only five of them are independent. The remaining freedom can be fixed, *e.g.*, by the boundary condition $H_9(u_1, u_2, u_3, 0, 0, 0) = H_6(u_1, u_2, u_3)$, with H_6 given explicitly in Refs. [4, 11]. (Alternatively, one could derive further differential equations, as in Ref. [4]). Therefore, the set of equations and the boundary condition completely determine H_9.

In the next section, we will use this set of differential equations to determine the symbol $\mathcal{S}(\tilde{\Phi}_9)$, where $\tilde{\Phi}_9$ is obtained from Φ_9 by a simple rescaling, see Eq. (16). Then, we will reconstruct the function $\tilde{\Phi}_9$ (and equivalently H_9) from its symbol.

We note that there is a simple line integral representation of H_9 [4], see Fig. 1(a),

$$H_9 = \int_0^1 d\xi_1 d\xi_4 d\xi_7 \frac{1}{(y_1 - y_4)^2 (y_4 - y_7)^2 (y_7 - y_1)^2} \,, \tag{12}$$

where $y_1^\mu = x_1^\mu + \xi_1 x_{21}^\mu$, $y_4^\mu = x_4^\mu + \xi_4 x_{54}^\mu$ and $y_7^\mu = x_7^\mu + \xi_7 x_{87}^\mu$. The pentagon integral P_8 can be expressed in a similar way, which allows us to write

$$H_9 = \int_0^1 d\xi_1 P_8(y_1(\xi_1), x_4, x_5, x_7, x_8) \,. \tag{13}$$

In this form, the differential equation (7) has the interpretation of localizing one of the line integrals, in this case $y_1(\xi_1) \to x_2$, see Fig. 1(b). It is interesting that similar integrals where certain propagators are localized at cusp points have also appeared in computations of two-loop Wilson loops [17].

From this discussion it is also clear that the integral reduces further in degree under the action of other differential operators, until one eventually obtains a rational function. More explicitly, the operator $(x_{54} \cdot \partial_{x_5} + 1)$ acting on P_8 similarly gives a first-order differential equation relating

Ψ_8 to a single-log function, namely a 3-mass box integral with two doubled propagators,

$$X_7 := \int \frac{d^6 x_i}{i\pi^3} \frac{1}{(x_{2i}^2)^2 (x_{4i}^2)^2 x_{7i}^2 x_{8i}^2} =: \frac{1}{x_{25}^2 x_{27}^2 x_{58}^2} \chi_7(u_3 u_5) \tag{14}$$

where $\chi_7(y) = \log(y)/(y-1)$. Acting further on X_7 with $(x_{87} \cdot \partial_{x_8} + 1)$ gives the 3-mass triangle with three doubled propagators, which is a constant up to the usual prefactors, $1/(x_{25}^2 x_{58}^2 x_{82}^2)$.

The representation (12) may also be useful for numerical checks. For future reference, it can be rewritten as

$$\begin{aligned} &\Phi_9(u_1, \ldots, u_6) \\ &\quad = \int_0^1 \frac{d\xi_1 d\xi_4 d\xi_7}{(u_2 \bar{\xi}_1 \bar{\xi}_4 + u_4 u_2 \xi_1 \bar{\xi}_4 + \xi_4)(u_3 \bar{\xi}_4 \bar{\xi}_7 + u_5 u_3 \xi_4 \bar{\xi}_7 + \xi_7)(u_1 \bar{\xi}_7 \bar{\xi}_1 + u_6 u_1 \xi_7 \bar{\xi}_1 + \xi_1)}\,, \end{aligned} \tag{15}$$

where $\bar{\xi}_i := 1 - \xi_i$.

3 Symbols from differential equations

We find that the following definition

$$\Phi_9(u_1, \ldots, u_6) =: \frac{1}{\sqrt{\Delta_9}} \tilde{\Phi}_9(u_1, \ldots, u_6)\,. \tag{16}$$

leads to a pure function $\tilde{\Phi}_9(u_i)$, *i.e.*, a function that can be written as a linear combination of transcendental functions, with numerical coefficients only. Here

$$\begin{aligned} \Delta_9 := &(1 - u_1 - u_2 - u_3 + u_4 u_1 u_2 + u_5 u_2 u_3 + u_6 u_3 u_1 - u_1 u_2 u_3 u_4 u_5 u_6)^2 \\ &- 4 u_1 u_2 u_3 (1 - u_4)(1 - u_5)(1 - u_6)\,. \end{aligned} \tag{17}$$

Using this definition, and $D_1(1/\sqrt{\Delta_9}) = 0$, we can rewrite Eq. (10) as

$$\tilde{D}_1 \tilde{\Phi}_9(u_1, \ldots, u_6) = \tilde{\Psi}_8(u_3, u_4 u_2, u_5)\,, \tag{18}$$

where

$$\begin{aligned} \tilde{D}_1 := &\frac{1}{\sqrt{\Delta_9}} (1 - u_3 - u_2 u_4 + u_2 u_3 u_4 u_5) \times \\ &\times [u_1 u_6 (u_6 - 1) \partial_6 + (u_4 - 1) \partial_4 + u_1 (u_1 - 1) \partial_1 + u_1 (1 - u_6) u_3 \partial_3] \\ = &\frac{1}{\sqrt{\Delta_9}} (1 - u_3 - u_2 u_4 + u_2 u_3 u_4 u_5)(D_1 - u_1)\,, \end{aligned} \tag{19}$$

and

$$\tilde{\Psi}_8(u, v, w) := (1 - u - v + uvw)\, \Psi_8(u, v, w)\,. \tag{20}$$

We find it convenient to convert (18) into a differential equation for the symbol of $\tilde{\Phi}_9$, which reads

$$\tilde{D}_1 \mathcal{S}(\tilde{\Phi}_9)(u_1,\ldots,u_6) = \mathcal{S}(\tilde{\Psi}_8)(u_3, u_4u_2, u_5)\,. \tag{21}$$

Here the differentiation of a symbol is defined by

$$\partial_x\,\left(a_1\otimes\ldots\otimes a_{n-1}\otimes a_n\right) = \partial_x \log(a_n)\;\times\; a_1\otimes\ldots\otimes a_{n-1}\,. \tag{22}$$

The following set of variables is useful to describe the solution,

$$W_i := \frac{g_i-\sqrt{\Delta_9}}{g_i+\sqrt{\Delta_9}}\,,\qquad i=1\ldots 6\,, \tag{23}$$

where

$$\begin{aligned} g_1 &:= 1-u_1-u_2+u_3+u_1u_2u_4-u_2u_3u_5-2u_3u_6+u_1u_3u_6+2u_2u_3u_5u_6-u_1u_2u_3u_4u_5u_6\,,\\ g_4 &:= 1-u_1-u_2-u_3+2u_1u_2-u_1u_2u_4+u_2u_3u_5+u_1u_3u_6-2u_1u_2u_3u_5u_6+u_1u_2u_3u_4u_5u_6\,, \end{aligned}$$

and where g_2, g_3 (g_5, g_6) are obtained from g_1 (g_4) by cyclic mappings $1\to 2\to 3\to 1; 4\to 5\to 6\to 4$. These variables have a nice behavior under the differential operators, *e.g.*,

$$\tilde{D}_1\log(W_i) = \begin{cases} -1, & \text{if } i=6\\ 0, & \text{otherwise}\end{cases} \quad\text{and}\quad \tilde{D}_4\log(W_i) = \begin{cases} 1, & \text{if } i=1\\ -1, & \text{if } i=2 \text{ or } 4\\ 0, & \text{otherwise}\end{cases}\,, \tag{24}$$

where $\tilde{D}_4$ is defined as the image of $\tilde{D}_1$ under the reflection $u_4\leftrightarrow u_6$ and $u_2\leftrightarrow u_3$. Given these variables, we can write the solution to Eq. (21) as

$$\mathcal{S}(\tilde{\Phi}_9)(u_1,\ldots,u_6) = \mathcal{S}(\tilde{\Psi}_8)(u_3,u_4u_2,u_5)\otimes W_6 + T\,, \tag{25}$$

where T satisfies $\tilde{D}_1 T = 0$. Taking into account the differential equations related to (21) by symmetry further restricts the form of T. The particular solution we obtained is in general not an integrable symbol. We therefore proceed and add a particular T_h satisfying $\tilde{D}_i T_h = 0$ (for $i=1\ldots 5$) to obtain an integrable symbol. Finally, additional terms satisfying the homogeneous equations $\tilde{D}_i T = 0$ are fixed by demanding that the symbol for $\tilde{\Phi}_6$ for the massless hexagon [4, 11] is reproduced when $u_4=u_5=u_6=0$.

Following this procedure, we find that the symbol $\mathcal{S}(\tilde{\Phi}_9)$ can then be written as

$$\mathcal{S}(\tilde{\Phi}_9) = \sum_{i=1}^{6}\mathcal{S}(f_i)\otimes W_i\,, \tag{26}$$

where f_i are the following degree three functions,

$$\begin{aligned} f_1 &:= \tilde{\Psi}_8(u_2,u_1u_6,u_4)+\tilde{\Psi}_8(u_1,u_2u_5,u_4)+\tilde{\Psi}_8(u_2,u_3u_6,u_5)-F(u_1,u_2,u_3,u_4,u_5,u_6)\,,\\ f_4 &:= -\tilde{\Psi}_8(u_1,u_3u_5,u_6)\,. \end{aligned} \tag{27}$$

Here the quantities f_2, f_3 (f_5, f_6) are obtained from f_1 (f_4) by cyclic mappings $1 \to 2 \to 3 \to 1; 4 \to 5 \to 6 \to 4$. Moreover,

$$F := 2\tilde{\Psi}_8(u_1, u_2, u_4) + \log u_1 \log u_5 + \log u_2 \log u_6 - \log u_3 \log u_4 \,. \tag{28}$$

Note that one can rearrange terms in Eq. (27) because of the identity,

$$\begin{aligned} 0 = &\tilde{\Psi}_8(u_3, u_2u_4, u_5) + \tilde{\Psi}_8(u_1, u_3u_5, u_6) + \tilde{\Psi}_8(u_2, u_1u_6, u_4) \\ &- \tilde{\Psi}_8(u_3, u_1u_4, u_6) - \tilde{\Psi}_8(u_1, u_2u_5, u_4) - \tilde{\Psi}_8(u_2, u_3u_6, u_5) \,. \end{aligned} \tag{29}$$

4 Twistor geometry associated to a three-mass hexagon

The differential equation technique allowed us to obtain the symbol of the one-loop three-mass hexagon integral. If we want to find the analytic expression for the integral, we need to integrate the symbol to a function. We follow here the approach of Ref. [18], which, after making a suitable choice for the functions that should appear in the answer, allows us to reduce the problem of integrating the symbol to a problem of linear algebra. The algorithm of Ref. [18], however, requires the arguments of the symbol to be rational functions (of some parameters). From Eq. (26) it is clear that in our case this requirement is not immediately fulfilled, because the variables W_i are algebraic functions of the cross ratios u_i. In order to bypass this problem, we have to parametrize the six cross ratios such that Δ_9 becomes a perfect square.

A convenient way to find a parametrization that turns Δ_9 into a perfect square is to write the six cross ratios as ratios of twistor brackets. Indeed, even though we work in $D = 6$ dimensions where the link to twistor space is not immediately obvious, we can nevertheless consider the cross ratios as being parametrized by cross ratios in twistor space $\mathbb{CP}^3$, because the functional dependence of Φ_9 is only through the six conformally invariant quantities u_i, which do not make reference to the six-dimensional space. In other words, we can consider the external momenta to lie in a four-dimensional subspace, even as we integrate over six components of loop momentum. Furthermore, in Ref. [15] it was noted that in terms of momentum twistor variables, the equivalent of Δ_9 in the massless case becomes a perfect square. Hence, momentum twistors seem to provide a natural framework to search for a suitable parametrization. We therefore briefly review the geometry of a three-mass hexagon configuration in momentum twistor space.

In order to describe this geometry, we assume that the dual coordinates x_i are elements of four-dimensional Minkowski space $\mathbb{M}^4$. As the dependence of Φ_9 is solely through cross ratios, we can assume that this condition is satisfied, as long as the 'projection' to the four-dimensional space leaves the cross ratios invariant. The twistor correspondence then associates to each point x_i in $\mathbb{M}^4$ a projective line X_i in momentum twistor space, and two points x_i and x_j in $\mathbb{M}^4$ are lightlike separated if and only if the corresponding lines X_i and X_j intersect. In our case this implies that the six lines must intersect pairwise (See Fig. 2). Denoting the intersection points by Z_1, Z_4 and Z_7, we can define six more twistors by

$$X_i = Z_i \wedge Z_{i-1}, \quad i \in \{1, 2, 4, 5, 7, 8\} \,. \tag{30}$$

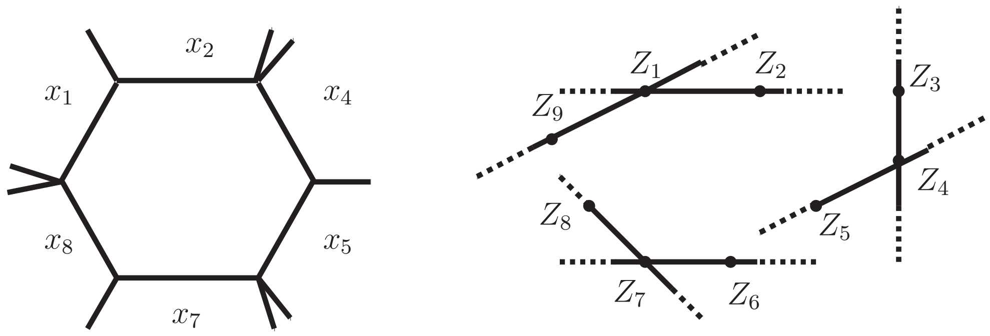


Figure 2: The one-loop three-mass hexagon integral (left), and its geometric configuration in momentum twistor space $\mathbb{CP}^3$ (right). Only the intersection points Z_1, Z_4 and Z_7 have an intrinsic geometrical meaning, whereas all other twistors can be moved freely along the lines.

Note that the only points in twistor space that have an intrinsic geometric meaning are Z_1, Z_4 and Z_7, whereas the other six points are defined through Eq. (30), which is left invariant by the redefinitions

$$\begin{aligned} &Z_2 \to Z_2 + \alpha_2 Z_1\,, \quad &&Z_5 \to Z_5 + \alpha_5 Z_4\,, \quad &&Z_8 \to Z_8 + \alpha_8 Z_7\,, \\ &Z_9 \to Z_9 + \alpha_9 Z_1\,, \quad &&Z_3 \to Z_3 + \alpha_3 Z_4\,, \quad &&Z_6 \to Z_6 + \alpha_6 Z_7\,, \end{aligned} \tag{31}$$

where α_i are non-zero complex numbers. These shifts simply express the fact that we can move the points along the line without altering the geometric configuration. Furthermore, the intersection of two lines X_i and X_j can be expressed through the condition,

$$\langle X_i X_j \rangle := \langle (i-1)\, i\, (j-1)\, j \rangle = \langle Z_{i-1}\, Z_i\, Z_{j-1}\, Z_j \rangle = \epsilon_{IJKL}\, Z_{i-1}^I\, Z_i^J\, Z_{j-1}^K\, Z_j^L = 0\,. \tag{32}$$

Using the twistor brackets, the cross ratios u_i can be parametrized as

$$\begin{aligned} u_1 &= \frac{\langle X_2 X_5 \rangle \langle X_1 X_7 \rangle}{\langle X_1 X_5 \rangle \langle X_2 X_7 \rangle}\,, \qquad & u_2 &= \frac{\langle X_5 X_8 \rangle \langle X_4 X_1 \rangle}{\langle X_4 X_8 \rangle \langle X_1 X_5 \rangle}\,, \qquad & u_3 &= \frac{\langle X_8 X_2 \rangle \langle X_7 X_4 \rangle}{\langle X_2 X_7 \rangle \langle X_4 X_8 \rangle}\,, \\ u_4 &= \frac{\langle X_2 X_4 \rangle \langle X_1 X_5 \rangle}{\langle X_1 X_4 \rangle \langle X_2 X_5 \rangle}\,, \qquad & u_5 &= \frac{\langle X_5 X_7 \rangle \langle X_4 X_8 \rangle}{\langle X_4 X_7 \rangle \langle X_5 X_8 \rangle}\,, \qquad & u_6 &= \frac{\langle X_8 X_1 \rangle \langle X_7 X_2 \rangle}{\langle X_8 X_2 \rangle \langle X_1 X_7 \rangle}\,. \end{aligned} \tag{33}$$

It is clear that the dihedral symmetry of the integral is reflected at the level of the twistors by

$$Z_i \xrightarrow{c} Z_{i+3} \quad \text{and} \quad Z_i \xrightarrow{r} Z_{8-i}\,, \tag{34}$$

where again all indices are understood modulo 9. This action on the twistors induces an action on the lines X_i and the planes $\overline{Z}_i = Z_{i-1} \wedge Z_i \wedge Z_{i+1}$ via

$$\begin{aligned} X_i &\xrightarrow{c} X_{i+3} \quad \text{and} \quad X_i \xrightarrow{r} -X_{9-i}\,, \\ \overline{Z}_i &\xrightarrow{c} \overline{Z}_{i+3} \quad \text{and} \quad \overline{Z}_i \xrightarrow{r} -\overline{Z}_{8-i}\,. \end{aligned} \tag{35}$$

We now choose a particular representation for the twistors. Since the points Z_1, Z_4 and Z_7 play a special role, we choose their homogeneous coordinates as

$$Z_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad Z_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad Z_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{36}$$

As the other six points do not carry any intrinsic geometric meaning, we prefer not to fix them, but choose their homogeneous coordinates to be

$$Z_i = \begin{bmatrix} 1 \\ x_i \\ y_i \\ z_i \end{bmatrix}, \qquad \text{for } i \in \{2,3,5,6,8,9\}. \tag{37}$$

(The x_i and y_i defined here should not be confused with the previous definitions, where they were dual coordinates.) In this parametrization the cross ratios then take the form

$$\begin{aligned} u_1 &= \frac{(y_9-y_6)\,(z_2-z_5)}{(y_2-y_6)\,(z_9-z_5)}, & u_2 &= \frac{(x_5-x_8)\,(z_3-z_9)}{(x_3-x_8)\,(z_5-z_9)}, & u_3 &= \frac{(x_6-x_3)\,(y_8-y_2)}{(x_8-x_3)\,(y_6-y_2)}, \\ u_4 &= \frac{(z_2-z_3)\,(z_9-z_5)}{(z_2-z_5)\,(z_9-z_3)}, & u_5 &= \frac{(x_5-x_6)\,(x_3-x_8)}{(x_3-x_6)\,(x_5-x_8)}, & u_6 &= \frac{(y_6-y_2)\,(y_8-y_9)}{(y_8-y_2)\,(y_6-y_9)}. \end{aligned} \tag{38}$$

Note that the cross ratios only depend on 12 out of the 18 homogeneous coordinates defined in Eq. (37), which is a consequence of the shift invariance (31). The action of the dihedral symmetry that permutes the cross ratios is implemented in this parametrization via

$$x_i \xrightarrow{c} y_{i+3} \xrightarrow{c} z_{i+6} \xrightarrow{c} x_i \qquad \text{and} \qquad x_i \xleftrightarrow{r} z_{8-i} \;\; \text{and} \;\; y_i \xleftrightarrow{r} y_{8-i}\,. \tag{39}$$

This action seems to be inconsistent with Eq. (34). However, we have broken the symmetry by freezing Z_1, Z_4 and Z_7 to constant values, and the symmetry is now reflected at the level of the cross ratios via Eq. (39). Finally, we note that Δ_9 becomes a perfect square in these variables,

$$\Delta_9 = \frac{\left((x_6-x_8)\,(y_9-y_2)\,(z_3-z_5)+(x_5-x_3)\,(y_8-y_6)\,(z_2-z_9)\right)^2}{(x_3-x_8)^2\,(y_6-y_2)^2\,(z_9-z_5)^2}, \tag{40}$$

and Eq. (40) is manifestly invariant under the transformations (39). Having obtained a parametrization that makes Δ_9 into a perfect square, we can write the symbol in a form in which all the entries are rational functions of the variables we just defined, and hence the symbol now takes a form which allows it to be integrated using the algorithm of Ref. [18]. Furthermore, using this parametrization it is trivial to check that the symbol of $\tilde{\Phi}_9$ obtained in the previous section has the correct dihedral symmetry. In particular, we find that

$$c[\mathcal{S}(\tilde{\Phi}_9)] = \mathcal{S}(\tilde{\Phi}_9) \;\; \text{and} \;\; r[\mathcal{S}(\tilde{\Phi}_9)] = -\mathcal{S}(\tilde{\Phi}_9)\,. \tag{41}$$

The parametrization (38) also makes it very easy to check the various soft limits of H_9. Indeed, we have

$$u_4 \to 0 \Leftrightarrow z_3 \to z_2\,, \qquad u_5 \to 0 \Leftrightarrow x_6 \to x_5\,, \qquad u_6 \to 0 \Leftrightarrow y_9 \to y_8\,. \tag{42}$$

We checked that in taking these limits $\mathcal{S}(\tilde{\Phi}_9)$ reduces to the symbols for the massless and one-mass hexagon integrals [4, 11, 12].

5 Integrating the symbol: the one-loop three-mass hexagon integral

As the parametrization of the cross ratios in terms of momentum twistors introduced in the previous section turns Δ_9 into a perfect square, we can now integrate the symbol using the algorithm of Ref. [18]. However, even though the parametrization (38) makes all the symmetries manifest, it uses a redundant set of parameters. We therefore choose a minimal set of parameters by breaking the S_3 symmetry down to its alternating subgroup $A_3 \simeq \mathbb{Z}_3$ by fixing six of the twelve parameters,

$$x_6 = y_9 = z_3 = 0 \quad\text{and}\quad x_3 = y_6 = z_9 = 1\,. \tag{43}$$

The cross ratios then take the form

$$\begin{aligned}
u_1 &= \frac{z_2 - z_5}{(1-y_2)\,(1-z_5)}\,, & u_2 &= \frac{x_5 - x_8}{(1-x_8)\,(1-z_5)}\,, & u_3 &= \frac{y_8 - y_2}{(1-x_8)\,(1-y_2)}\,,\\
u_4 &= \frac{z_2\,(1-z_5)}{z_2 - z_5}\,, & u_5 &= \frac{x_5\,(1-x_8)}{x_5 - x_8}\,, & u_6 &= \frac{y_8\,(1-y_2)}{y_8 - y_2}\,,
\end{aligned} \tag{44}$$

and Δ_9 can now be written as

$$\Delta_9 = \frac{\left(x_8 y_2 z_5 + (1-x_5)\,(1-y_8)\,(1-z_2)\right)^2}{(1-x_8)^2\,(1-y_2)^2\,(1-z_5)^2}\,. \tag{45}$$

We note in passing that the Jacobian of the parametrization (44) is non-zero for generic values of the parameters.

In a nutshell, the algorithm of Ref. [18] proceeds in two steps:

1. Given the symbol of $\tilde{\Phi}_9$ computed in Section 3, it constructs a set of rational functions $\{R_i(x_5, x_8, y_2, y_8, z_2, z_5)\}$ such that, *e.g.*, symbols of the form $\mathcal{S}(\mathrm{Li}_n(R_i))$ span the vector space of which $\mathcal{S}(\tilde{\Phi}_9)$ is an element.

2. Once a suitable set of rational functions has been obtained, it makes an ansatz

$$\tilde{\varphi} = \sum_i c_i\,\mathrm{Li}_3(R_i) + \sum_{i,j} c_{ij}\,\mathrm{Li}_2(R_i)\,\log R_j + \sum_{i,j,k} c_{ijk}\,\log R_i\,\log R_j\,\log R_k\,, \tag{46}$$

where the c_i, c_{ij} and c_{ijk} are rational numbers to be determined such that

$$\mathcal{S}(\tilde{\varphi}) = \mathcal{S}(\tilde{\Phi}_9)\,. \tag{47}$$

As the objects appearing in this last equation are tensors (*i.e.*, elements of a vector space), the coefficients c_i, c_{ij} and c_{ijk} can equally well be seen as coordinates in a vector space, and the problem of finding the coefficients reduces to a problem of linear algebra.

We have implemented the algorithm of Ref. [18] into a Mathematica code, which we have applied to the function $\tilde{\Phi}_9(x_5, x_8, y_2, y_8, z_2, z_5)$. The result we obtain takes a strikingly simple form,

$$\Phi_9(u_1, \ldots, u_6) = \frac{1}{\sqrt{\Delta_9}} \sum_{i=1}^{4} \sum_{g\in S_3} \sigma(g)\, \mathcal{L}_3(x^+_{i,g}, x^-_{i,g})\,, \tag{48}$$

where $\sigma(g)$ denotes the signature of the permutation ($+1$ for $\{1, c, c^2\}$, -1 for $\{r, rc, rc^2\}$), and where we defined

$$\mathcal{L}_3(x^+, x^-) := \frac{1}{18}\left(\ell_1(x^+) - \ell_1(x^-)\right)^3 + L_3(x^+, x^-)\,, \tag{49}$$

and

$$L_3(x^+, x^-) := \sum_{k=0}^{2} \frac{(-1)^k}{(2k)!!} \log^k(x^+ x^-)\,\left(\ell_{3-k}(x^+) - \ell_{3-k}(x^-)\right)\,, \tag{50}$$

with

$$\ell_n(x) := \frac{1}{2}\left(\mathrm{Li}_n(x) - \mathrm{Li}_n(1/x)\right)\,. \tag{51}$$

The arguments appearing in the polylogarithms can be written in the form $x^\pm_{i,g} := g(x^\pm_i)$, for $g \in S_3$, with

$$\begin{aligned} &x^+_1 := \chi(1,4,7)\,, \quad x^+_2 := \chi(2,5,7)\,, \quad x^+_3 := \chi(2,4,8)\,, \quad x^+_4 := \chi(1,5,8)\,,\\ &x^-_1 := \overline{\chi}(1,4,7)\,, \quad x^-_2 := \overline{\chi}(2,5,7)\,, \quad x^-_3 := \overline{\chi}(2,4,8)\,, \quad x^-_4 := \overline{\chi}(1,5,8)\,, \end{aligned} \tag{52}$$

where we defined

$$\chi(i,j,k) := -\frac{\langle 4\overline{7}\rangle\langle X_i X_k\rangle\langle X_j 17\rangle}{\langle 1\overline{7}\rangle\langle X_j X_k\rangle\langle X_i 47\rangle}\,, \tag{53}$$

with $\langle i\bar{\jmath}\rangle = \langle i(j-1)j(j+1)\rangle$. The function $\overline{\chi}$ is related to χ by Poincaré duality,

$$\overline{\chi}(i,j,k) := -\frac{\langle \overline{4}7\rangle\langle X_i X_k\rangle\langle X_j \overline{1}\cap\overline{7}\rangle}{\langle \overline{1}7\rangle\langle X_j X_k\rangle\langle X_i \overline{4}\cap\overline{7}\rangle}\,. \tag{54}$$

The function Φ_9 manifestly has the cyclic symmetry. The reflection symmetry however needs some explanation, because $\tilde{\Phi}_9$ is odd under reflection. In twistor variables, Δ_9 becomes a perfect square, and so we can remove the square root and rewrite $\sqrt{\Delta_9}$ as a rational function of twistor brackets. This procedure however introduces an ambiguity for the sign of the square root. In particular, the rational function we obtained is now odd under the reflection (34), so that Φ_9 is again even.

We stress that Eq. (48) is only valid in the region where $\Delta_9 < 0$. In this region, since χ and $\overline{\chi}$ are related by Poincaré duality, the function Eq. (48) is manifestly real, and we checked numerically that Eq. (48) agrees with the parametric integral representation for Φ_9 given in Eq. (15). Note that, as multiple zeta values are in the kernel of the symbol map, we could a priori add to Eq. (48) terms proportional to ζ_2 without altering its symbol[2]. The numerical agreement with the integral representation (15) however shows that such terms are absent in the present case.

6 Conclusion

Using a differential equation method to determine the symbol of a function, and an algorithm to reconstruct the function from its symbol, we have computed analytically the one-loop non-adjacent three-mass hexagon integral in $D = 6$ dimensions. Just as for the massless and one-mass hexagon integrals, the result is given in terms of classical polylogarithms of uniform transcendental weight three, which are functions of six dual conformally invariant cross-ratios. Because of the high degree of symmetry of the integral, the result is extremely compact: it can be expressed as a sum of 24 terms involving only one basic function, which is a simple linear combination of logarithms, dilogarithms, and trilogarithms. Given the relation between one-loop hexagon integrals in $D = 6$ dimensions and higher-loop amplitudes in $D = 4$ dimensions, we expect that our result will help to understand the structure of $\mathcal{N} = 4$ SYM amplitudes and Wilson loops, particularly at two loops.

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Note added: After this calculation was completed, we were informed of an independent computation of the symbols of hexagon integrals, using a different method [19].

[2] Note that a constant term proportional to ζ_3 is excluded because of the reality condition on the function.

A Special cases

For $u_4 = u_5 = u_6 = 1$, the differential equations simplify considerably. We have

$$[u_1 + u_1(u_1-1)\partial_1]\,\Phi_9(u_1,u_2,u_3,1,1,1) = \Psi_8(u_2,u_3,1)\,, \tag{55}$$

where $\Psi_8(u,v,1) = \log u \log v/(u-1)/(v-1)$, and the two cyclically related equations. The solution is simply

$$\Phi_9(u_1,u_2,u_3,1,1,1) = \prod_{i=1}^{3} \frac{\log u_i}{u_i - 1}\,. \tag{56}$$

The case $u_5 = u_6 = 1$ is also very simple,

$$\Phi_9(u_1,u_2,u_3,u_4,1,1) = \frac{\log u_3}{u_3-1}\Psi_8(u_1,u_2,u_4)\,. \tag{57}$$

B Arguments in terms of space-time cross ratios

In this appendix we present the expressions of the functions x_i^+ defined in Eq. (52) in terms of the space-time cross ratios u_i,

$$\begin{aligned}
x_1^+ &= \frac{2u_3\,(1-u_6)\,[1-u_3u_6-u_2\,(1-u_3u_5u_6)]-(1-u_3u_6)\left(g_1-\sqrt{\Delta_9}\right)}{2u_3\,(1-u_6)\,[1-u_2-u_3\,(1-u_2u_5)\,u_6]}\,,\\
x_2^+ &= \frac{2u_1u_3\,(1-u_6)\,[1-u_2u_4-u_3\,(1-u_2u_4u_5)]-(1-u_3)\left(g_6-\sqrt{\Delta_9}\right)}{2u_1\,(1-u_6)\,[1-u_2u_4-u_3\,(1-u_2u_4u_5)]}\,,\\
x_3^+ &= \frac{2u_3\,(1-u_6)\,[(1-u_2u_5)\,(1-u_3u_5)-u_1\,(1-u_5)]-(1-u_3u_5)\left(g_1-\sqrt{\Delta_9}\right)}{2u_1u_3u_5\,(1-u_6)\,[1-u_2u_4-u_3\,(1-u_2u_4u_5)]}\,,\\
x_4^+ &= -u_6\,\frac{2u_3\,(1-u_6)\,[1-u_5-u_1\,(1-u_2u_4u_5)\,(1-u_3u_5u_6)]+(1-u_3u_5u_6)\left(g_6-\sqrt{\Delta_9}\right)}{2\,(1-u_6)\,[1-u_2-u_3\,(1-u_2u_5)\,u_6]}\,.
\end{aligned} \tag{58}$$

The variables x_i^- are obtained from x_i^+ by replacing $\sqrt{\Delta_9}$ by $-\sqrt{\Delta_9}$. Also, in Eq. 48 we define the action of the odd permutations g to include the replacement $\sqrt{\Delta_9} \to -\sqrt{\Delta_9}$.

The twistor variables x_i, y_i and z_i rationalize the $x_i^{\pm}$, so that they take the form,

$$\begin{aligned}
x_1^+ &= \frac{x_8}{1-y_8}\,, \\
x_2^+ &= -\frac{x_8(y_2-y_8)}{(1-x_8)(1-y_8)}\,, \\
x_3^+ &= \frac{x_8(1-y_2)}{x_5(1-y_8)}\,, \\
x_4^+ &= \frac{x_8 y_8}{(1-y_8)(x_5-x_8)}\,, \\
x_1^- &= \frac{(1-x_5)[1-x_8(1-y_2)-y_8-z_2(1-x_8-y_8)]}{y_2[(1-x_5)(1-y_8)-z_5(1-x_8-y_8)]}\,, \\
x_2^- &= -\frac{(1-x_5)(y_2-y_8)[1-x_8(1-y_2)-y_8-z_2(1-x_8-y_8)]}{y_2(1-x_8)[(z_2(1-x_5)-z_5)(1-y_8)+z_5x_8(1-y_2)]}\,, \\
x_3^- &= \frac{(1-y_2)(1-x_5)[(x_5(1-y_8)-x_8)(1-z_2)+x_8y_2(1-z_5)]}{y_2x_5[(z_2(1-x_5)-z_5)(1-y_8)+z_5x_8(1-y_2)]}\,, \\
x_4^- &= \frac{y_8(1-x_5)[(x_5(1-y_8)-x_8)(1-z_2)+x_8y_2(1-z_5)]}{y_2(x_5-x_8)[(1-x_5)(1-y_8)-z_5(1-x_8-y_8)]}\,.
\end{aligned} \tag{59}$$

Note that these expressions correspond to a particular choice for the sign of square root.

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