HU-EP-11/17 CERN–PH–TH/2011-075 SLAC–PUB–14434 LAPTH-013/11

The one-loop six-dimensional hexagon integral and its relation to MHV amplitudes in $\mathcal{N} = 4$ SYM

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Abstract

We provide an analytic formula for the (rescaled) one-loop scalar hexagon integral $\tilde{\Phi}_6$ with all external legs massless, in terms of classical polylogarithms. We show that this integral is closely connected to two integrals appearing in one- and two-loop amplitudes in planar $\mathcal{N} = 4$ super-Yang-Mills theory, $\Omega^{(1)}$ and $\Omega^{(2)}$. The derivative of $\Omega^{(2)}$ with respect to one of the conformal invariants yields $\tilde{\Phi}_6$, while another first-order differential operator applied to $\tilde{\Phi}_6$ yields $\Omega^{(1)}$. We also introduce some kinematic variables that rationalize the arguments of the polylogarithms, making it easy to verify the latter differential equation. We also give a further example of a six-dimensional integral relevant for amplitudes in $\mathcal{N} = 4$ super-Yang-Mills.

Work supported in part by US Department of Energy contract DE-AC02-76SF00515.

1 Introduction and outline

Recent years have seen dramatic progress in the understanding of multi-loop and multi-leg scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills theory (SYM), especially in the planar limit. The planar amplitudes have a hidden dual conformal symmetry [1, 2, 3] that leads to powerful constraints. There is also a surprising correspondence between scattering amplitudes and Wilson loops [4, 5, 6]; see refs. [7, 8, 9, 10] for recent developments. A dual conformal Ward identity [11], derived for Wilson loops, can be used to fix the functional form of multi-loop scattering amplitudes, up to *a priori* undetermined functions of dual conformal cross-ratios. For example, the functional form of the four- and five-point amplitudes is uniquely fixed to all orders in the coupling constant, in agreement with explicit computations in field theory [2, 12, 13, 14, 15, 16, 17, 18, 19] and string theory [4]. For maximally-helicity-violating (MHV) amplitudes, the difference between the (logarithms of the) particular solution to the Ward identity (the BDS ansatz [13]) and the amplitude is called the remainder function [16, 20]. For six external particles, this remainder function can depend only on three dual conformal cross ratios u_1, u_2 and u_3.

Another important consequence of dual conformal symmetry is a powerful restriction on the planar loop integrand, which had been observed in dimensional regularization [1, 2, 21], and can be made rigorous on the Coulomb branch of $\mathcal{N}=4$ SYM [22, 23, 24, 25].

The six-point remainder function at two loops is known analytically [26, 27, 28], thanks to the correspondence between scattering amplitudes and Wilson loops. On the amplitude side, so far results are available numerically [16] and analytically in certain kinematical limits [29, 30, 31]. Recently, iterative differential equations were used to directly evaluate integrals that contribute to the scattering amplitudes [32].

The motivation of the present paper is to show how to derive analytical results for loop integrals relevant for multi-leg scattering amplitudes, using differential equations. We concentrate on the six-point case, but our method is also applicable to more external legs.

The "even" part of the planar six-particle MHV scattering amplitude at two loops was first given in ref. [16] in terms of fifteen separate integrals with simple dual conformal properties. It can be represented alternatively [31, 33] in terms of six dual conformal two-loop integrals, five of which are infrared divergent and one of which is finite. The finite integral, denoted by $\Omega^{(2)}$, depends on the three dual conformal cross-ratios u_1, u_2, u_3. It is reasonable to believe that it contains an essential part of the two-loop six-point remainder function. In ref. [32] it was found that $\Omega^{(2)}$ satisfies several simple second-order differential equations, one of which relates it to an analogous one-loop integral, called $\Omega^{(1)}$.

In this paper we observe that the one-loop scalar hexagon integral in six space-time dimensions is related to the aforementioned four-dimensional integrals via first-order differential equations. The relations that we find are (schematically)

$$\Omega^{(2)}(u_1, u_2, u_3) \longrightarrow \tilde{\Phi}_6(u_1, u_2, u_3) \longrightarrow \Omega^{(1)}(u_1, u_2, u_3)\,, \tag{1}$$

where the arrows denote certain first-order differential operators in the u_i. (See fig. 1.) Here $\tilde{\Phi}_6$ stands for the six-dimensional scalar hexagon integral, after two simple rescalings. The first (to Φ_6) makes it invariant under dual conformal transformations. The second removes an algebraic

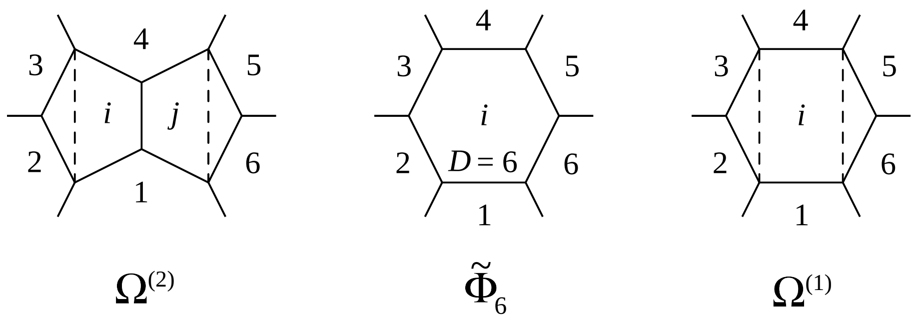

Figure 1: Three dual conformal integrals which are related to each other by the action of first-order differential operators, as discussed in the text. The labels $i, j, 1, 2, \ldots, 6$ are indices k for dual (or region) coordinates x_k. Solid lines indicate propagators; dashed lines indicate numerator factors of x_{ai}^2 or x_{bi}^2, as explained in the text. The central integral $\tilde{\Phi}_6$ has no such numerator factors, but is evaluated in dimension $D = 6$ instead of $D = 4$. The standard hexagon integral H is rescaled to obtain a dual conformal invariant integral Φ_6, which is rescaled once again to obtain the pure degree 3 function $\tilde{\Phi}_6$.

prefactor. It is natural to consider $\tilde{\Phi}_6$ as an intermediate step between $\Omega^{(1)}$ and $\Omega^{(2)}$. Thanks to the high degree of symmetry of the hexagon integral, the first-order differential equation relating $\tilde{\Phi}_6$ (or Φ_6) and $\Omega^{(1)}$ in fact leads to a system of three inequivalent equations. Together with a simple boundary condition, the latter completely determines the three-variable function $\Phi_6(u_1, u_2, u_3)$.

From a practical viewpoint, the intermediate step between $\Omega^{(2)}$ and $\Omega^{(1)}$ in eq. (1) is very useful. It is also very natural, since the $\Omega^{(i)}$ functions are expected to be given by linear combinations of functions defined through iterated (poly)logarithmic integrals, such as $\log^n$, Li_n, and generalizations thereof. If we associate a "degree of transcendentality" with the number of iterated integrals, then $\Omega^{(1)}$, $\tilde{\Phi}_6$ and $\Omega^{(2)}$ are pure functions of degree $2, 3$ and 4, respectively. In some sense, $\tilde{\Phi}_6$ represents a "one-and-a-half" loop function.

We find that the solution for Φ_6 is given by a simple formula in terms of degree three functions, eq. (26) below. It is remarkably similar in structure to the two-loop remainder function.

The six-dimensional hexagon integral Φ_6 also is of inherent interest for a number of reasons. In dimensional regularization with $4 - 2\epsilon$ dimensions, it appears in the $\mathcal{O}(\epsilon)$ part of the one-loop six-particle MHV amplitude [34]. It is generated because a term in the numerator of the one-loop integrand contains a factor of $\ell_{[-2\epsilon]}^2 \equiv \mu^2$, where $\ell_{[-2\epsilon]}$ denotes the components of the loop momenta that lie outside of four dimensions. The integral of such a term yields $-\epsilon$ times the scalar integral in six dimensions. Moreover, in order to determine the remainder function at higher loops, one has to take the logarithm of the amplitude, in which case $\mathcal{O}(\epsilon^i)$ terms at lower loops get multiplied by pole terms ϵ^{-j} (with $j \leq 2L$, where L is the loop order.) The $\mathcal{O}(\epsilon)$ terms must be kept in order to obtain a consistent result at $\mathcal{O}(1)$. As an example, when computing the two-loop remainder function in this way, Φ_6 participates in a cancellation involving certain two-loop "hexabox" integrals [16], where again there is a factor of μ^2 in the numerator for the hexagon loop. This link between higher-order terms in the ϵ expansion and higher-loop integrals also motivates the idea that Φ_6 should already know about some of the structure of the two-loop answer, and our result supports this expectation.

Another motivation for considering six-dimensional integrals in general is the known connection between scalar integrals in $(D+2)$ dimensions and tensor integrals in D dimensions (see e.g. ref. [35].) In particular, many of the finite tensor integrals introduced in ref. [33] can be viewed as higher-dimensional scalar integrals, or are related to them via differential equations. This relation does not depend on dual conformal symmetry. As an example, we will show a six-dimensional integral, and equivalently, a four-dimensional tensor integral, that computes the finite part of the two-mass-easy box integral.

The hexagon integral Φ_6 is a function of three dual conformally invariant cross-ratios u_1, u_2, u_3. Like the two-loop remainder function, it is conveniently expressed in terms of a set of redundant variables $x_{i\pm} = u_i x_\pm$, where

$$x_\pm = \frac{-1+u_1+u_2+u_3 \pm \sqrt{\Delta}}{2u_1u_2u_3}\,, \qquad \Delta = (-1+u_1+u_2+u_3)^2 - 4u_1u_2u_3\,. \tag{2}$$

Later we will give a change of variables from u_i to a set of variables $v_0, v_\pm$. Although these variables do not manifest the cyclic symmetry, they have the feature that the arguments of the polylogarithms in the result for the hexagon integral, and also all terms in the differential equations, are rational functions of $v_0, v_\pm$, with no square roots. This is very convenient for verifying the differential equations. Analogous transformations may be also useful when considering other six-point integrals.

Recently, the notion of symbols was advocated as a tool to think about iterated integrals appearing in $\mathcal{N}=4$ SYM [28]. We compute the symbol of the hexagon integral and find that it is given by a very simple expression. Its simplicity follows from the differential equations that Φ_6 satisfies.

This paper is organized as follows. We begin by defining the hexagon integral Φ_6 and discussing its symmetry properties in section 2.1. We also explain how dual conformal symmetry helps in obtaining a simple Feynman parametrization, which is a general feature. Another example is given in Appendix C. We then point out the relation between Φ_6 and integrals appearing in the two-loop six-point MHV amplitude in $\mathcal{N}=4$ SYM, in the representation of ref. [33]. This relation takes the form of first-order differential equations. We present the analytic solution to these equations in section 2.3. In section 2.4 we introduce a convenient set of variables that renders the arguments of the functions appearing in Φ_6 rational, and directly verify the differential equations. In section 2.5 we discuss the symbol of Φ_6. We conclude and give an outlook in section 3.

2 Six-dimensional hexagon integral

2.1 Preliminaries

We consider the on-shell six-dimensional scalar hexagon integral H in $D=6$ dimensions, with external momenta p_j^μ satisfying momentum conservation, $\sum_{j=1}^6 p_j^\mu = 0$, and masslessness, $p_j^2 = 0$

for $j = 1, 2, \ldots, 6$. In terms of dual (or region) coordinates $p_j^\mu = x_j^\mu - x_{j+1}^\mu$, it is defined by

$$H = \int \frac{d^6 x_i}{i\pi^3} \frac{1}{\prod_{j=1}^6 x_{ij}^2}\,, \tag{3}$$

where $x_{ij}^\mu = x_i^\mu - x_j^\mu$, and x_i^μ is the dual coordinate corresponding to the loop momentum (see fig. 1 for the labeling). The integral is both ultraviolet (UV) and infrared (IR) finite. As a scalar integral, H is a function of the external Lorentz invariants $x_{j,j+2}^2 = s_{j,j+1}$ and $x_{j,j+3}^2 = s_{j,j+1,j+2}$. Here $s_{j,j+1} = (p_j + p_{j+1})^2$ and $s_{j,j+1,j+2} = (p_j + p_{j+1} + p_{j+2})^2$, and external indices are defined modulo 6. We work in signature $(-{+}{+}{+})$, so that the Euclidean region has all $s_{j,j+1}$ and $s_{j,j+1,j+2}$ positive. The on-shell conditions, $p_j^2 = 0$, are expressed in dual coordinates as $x_{j,j+1}^2 = 0$. Momentum conservation translates to $x_{j+6}^\mu \equiv x_j^\mu$ in the dual space.

Covariance of H under dual conformal symmetry [36, 1], in particular under the inversion of all dual coordinates, $x^\mu \to x^\mu / x^2$, allows us to write

$$s_{123} s_{234} s_{345} H(s_{i,i+1}, s_{i,i+1,i+2}) \;\equiv\; \Phi_6(u_1, u_2, u_3)\,, \tag{4}$$

where the cross-ratios

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}\,, \qquad u_2 = \frac{x_{24}^2 x_{51}^2}{x_{25}^2 x_{41}^2} = \frac{s_{23} s_{56}}{s_{234} s_{123}}\,, \qquad u_3 = \frac{x_{35}^2 x_{62}^2}{x_{36}^2 x_{52}^2} = \frac{s_{34} s_{61}}{s_{345} s_{234}}\,, \tag{5}$$

are invariant under dual conformal transformations.

We observe that Φ_6 has both cyclic and reflection symmetries. This leads to a full permutation symmetry in the $\{u_1, u_2, u_3\}$, i.e.

$$\Phi_6(u_1, u_2, u_3) = \Phi_6(u_3, u_1, u_2) = \Phi_6(u_2, u_3, u_1)\,, \qquad \Phi_6(u_1, u_2, u_3) = \Phi_6(u_2, u_1, u_3)\,. \tag{6}$$

We will compute Φ_6 in the Euclidean region, i.e. where $s_{j,j+1} > 0,\;\; s_{j,j+1,j+2} > 0$. Although we will eventually compute Φ_6 from differential equations, it is useful to have a simple parametric representation for Φ_6, for example for numerical checks. Here we give an instructive example that highlights technical simplifications brought about by dual conformal symmetry that may be of more general interest.

Introducing Feynman parameters in the standard way [37], we have

$$\Phi_6(u_1, u_2, u_3) = 2\, x_{14}^2 x_{25}^2 x_{36}^2 \int_0^\infty \prod_{i=1}^6 d\alpha_i \, \frac{\delta(\sum_{i=1}^6 c_i \alpha_i - 1)}{\left[\sum_{i<j} x_{ij}^2 \alpha_i \alpha_j\right]^3}\,. \tag{7}$$

Note that we can choose the c_i arbitrarily, as long as at least one of them is different from zero [37].

We have already seen that dual conformal symmetry leads to the simplified variable dependence (4). Moreover, dual conformal symmetry often leads to further simplifications in the evaluation of loop integrals. For example, it is well known [36] that in the off-shell case, a combination of a translation and an inversion in the dual space of the x_i can be used to send one of the dual points to infinity, thereby reducing the number of propagators by one. In this way, Broadhurst demonstrated the equivalence of an infinite class of off-shell three- and four-point ladder integrals.

In the present case, we cannot immediately use the same idea, due to the light-like constraints $p_j^2 = x_{j,j+1}^2 = 0$, which would make the above-mentioned inversion singular. However, we can nevertheless exploit technical simplifications that dual conformal symmetry entails.

For a generic one-loop integral, a factor of $(\sum_i \alpha_i)^{a-D}$, where a is the number of propagators, would be present under the integral sign on the right-hand side of eq. (7). Here, this factor is absent since $a = D = 6$, which is precisely the condition for dual conformal symmetry. In this case it is often convenient to choose one or more $c_i = 0$, because the resulting integrals from 0 to ∞ in eq. (7) are easy to carry out. We will set $c_6 = c_1 = c_2 = 0$, $c_3 = c_4 = c_5 = 1$. We will also use the redundancy in eq. (5) to set $x_{46}^2 = u_1, x_{24}^2 = u_2, x_{26}^2 = u_3$. (All other x_{jk}^2 appearing in eq. (5) are set to 1.)

Performing the $\alpha_6, \alpha_1, \alpha_2$ integrations, we readily obtain

$$\Phi_6(u_1, u_2, u_3) = \int_0^1 d\alpha_{3,4,5} \log\left(\frac{ad}{bc}\right) \frac{\delta(\sum_{i=3}^5 \alpha_i - 1)}{ad - bc}\,, \tag{8}$$

where $ad = \alpha_3\alpha_5 u_3$, $b = \alpha_4 u_2 + \alpha_5$ and $c = \alpha_4 u_1 + \alpha_3$. In this form, it is easy to see that the answer will be built from degree three functions.

2.2 Relation to integrals appearing in the six-point MHV amplitude

As was mentioned in the introduction, Φ_6 appears in the $O(\epsilon)$ part of the one-loop six-particle MHV amplitude in dimensional regularization [34]. Moreover, when computing the logarithm of that amplitude to two loops, Φ_6 participates in a cancellation involving certain two-loop hexabox integrals [16]. It is therefore not unreasonable to think that Φ_6 already contains some of the structure of the two-loop result.

In fact, one can find a very direct relation between integrals relevant for MHV scattering amplitudes and Φ_6. In refs. [33, 38], dual conformal integrals with a tensor structure in the numerator were introduced for the description of scattering amplitudes in $\mathcal{N} = 4$ SYM. One of them is given by

$$\Omega^{(1)}(u_1, u_2, u_3) = -\frac{x_{35}^2 x_{26}^2 x_{14}^2}{x_{ab}^2} \int \frac{d^4 x_i}{i\pi^2} \frac{x_{ai}^2 x_{bi}^2}{\prod_{j=1}^6 x_{ji}^2}\,, \tag{9}$$

where x_a^μ is a solution to the four-cut condition $x_{1a}^2 = x_{2a}^2 = x_{3a}^2 = x_{4a}^2 = 0$, and x_b^μ is obtained from x_a^μ by a rotation by 3 units (see fig. 1). The two choices for x_a are related by parity. For the finite integrals we consider here the result is independent of the choice. The numerator factor $x_{ai}^2 x_{bi}^2$ is crucial in order to make the integral IR finite [33, 38]. The definition of the numerator and the normalization in eq. (9) are easy to write out explicitly in twistor-space notation. We refer the interested reader to refs. [33, 32] for further details.

One might think that $\Omega^{(1)}$ would be a rather complicated hexagon integral. However, dual conformal symmetry and the specific choice of the numerator in eq. (9) allow it to be given by a remarkably simple formula,

$$\Omega^{(1)}(u_1, u_2, u_3) = \log u_1 \log u_2 + \mathrm{Li}_2(1-u_1) + \mathrm{Li}_2(1-u_2) + \mathrm{Li}_2(1-u_3) - 2\zeta_2 \,. \tag{10}$$

The integral $\Omega^{(1)}$ also plays an important role as the source term for a second-order differential equation for $\Omega^{(2)}$, an integral appearing in the two-loop six-particle MHV amplitude [32]. The latter integral is defined by

$$\Omega^{(2)}(u_1, u_2, u_3) = -\frac{x_{35}^2 x_{26}^2 (x_{14}^2)^2}{x_{ab}^2} \int \frac{d^4 x_i}{i\pi^2} \int \frac{d^4 x_j}{i\pi^2} \frac{x_{ai}^2 x_{bj}^2}{x_{1i}^2 x_{2i}^2 x_{3i}^2 x_{4i}^2 x_{ij}^2 x_{4j}^2 x_{5j}^2 x_{6j}^2 x_{1j}^2} \,, \tag{11}$$

where the definition of x_a^μ and x_b^μ is the same as for $\Omega^{(1)}$ in eq. (9). This integrals is also depicted in fig. 1.

The differential equation obeyed by $\Omega^{(2)}$ is [32]

$$u_3 \partial_{u_3} \tilde{D}^{(1)} \Omega^{(2)} = \Omega^{(1)} \,, \tag{12}$$

where $\tilde{D}^{(1)}$ is the first-order differential operator

$$\tilde{D}^{(1)} = -u_1(1-u_1)\partial_{u_1} - u_2(1-u_2)\partial_{u_2} + (1-u_1-u_2)(1-u_3)\partial_{u_3} \,. \tag{13}$$

Given the factorized structure of the second-order differential operator in eq. (12), it is natural to search for an object which sits “between” $\Omega^{(2)}$ and $\Omega^{(1)}$. The $D=6$ scalar hexagon integral, with transcendentality degree 3, is a particularly good candidate for such an object.

Inspecting the Feynman parametrization[1] of $\Omega^{(1)}$, it is easy to see that it is related to Φ_6 in the following way,

$$D^{(1)} \Phi_6 = \Omega^{(1)} \,, \tag{14}$$

where $D^{(1)}$ is the first-order differential operator

$$D^{(1)} = \frac{u_3}{u_1 u_2} \left[u_1(1-u_1)\partial_{u_1} + u_2(1-u_2)\partial_{u_2} - (1-u_1-u_2)(1-u_3)\partial_{u_3} - 1 \right] u_1 u_2 \,. \tag{15}$$

This relation is not particularly surprising, since it is well known that tensor integrals in D dimensions are often related to scalar integrals in $(D+2)$ dimensions [35]. We give a further example in Appendix B. Relation (14) is easy to understand: when acting on the scalar integrand of Φ_6 in Feynman parameter form, see eq. (3), the differential operator (15) creates terms that are equivalent to those coming from the numerator of $\Omega^{(1)}$. Further, the increase in the power of the denominator due to the differentiation can be absorbed by a shift in the dimension from 6 to 4.

[1] J. M. Henn thanks N. Arkani-Hamed and J. Bourjaily for collaboration on Feynman parametrizations of twistor integrals.

Let us comment further on the remarkable link between Φ_6 and $\Omega^{(2)}$. We can commute the two first-order operators in eq. (12). Using

$$\left[u_3\partial_{u_3}, \tilde{D}^{(1)}\right] \;=\; -(1-u_1-u_2)\partial_{u_3}\,, \tag{16}$$

$$D^{(1)} \;=\; -\tilde{D}^{(1)}u_3 + (1-u_1-u_2)\,, \tag{17}$$

we have

$$D^{(1)}\partial_{u_3}\Omega^{(2)} \;=\; -\Omega^{(1)}\,. \tag{18}$$

Comparing eq. (18) with eq. (14), we find that

$$\partial_{u_3}\Omega^{(2)} \;=\; -\Phi_6 + K\,, \tag{19}$$

where K satisfies $D^{(1)}K = 0$. In fact we find numerically that $K = 0$. Thus Φ_6 can be considered as an intermediate step between $\Omega^{(1)}$ and $\Omega^{(2)}$. Only one more integration of Φ_6 is required to obtain $\Omega^{(2)}$. Consistent with these differential equations, the degree of transcendentality increases from $\Omega^{(1)}$ to Φ_6 to $\Omega^{(2)}$ in steps of one. Considering its links to the six-particle MHV amplitudes in $\mathcal{N}=4$ super Yang-Mills, it is of interest to understand better the function Φ_6.

Let us proceed to evaluate the hexagon integral. The idea is to use eq. (14) in order to determine Φ_6. We will first put the equation into a more useful form. The zeroth-order piece in eq. (15) suggests that Φ_6 has some algebraic prefactor. Indeed, let us define

$$\tilde{\Phi}_6 \;:=\; \sqrt{\Delta}\,\Phi_6\,, \tag{20}$$

where $\Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$. Then, thanks to $D^{(1)}(1/\sqrt{\Delta}) = 0$, it is straightforward to commute the first-order part of $D^{(1)}$ around $u_1u_2/\sqrt{\Delta}$, and one obtains

$$-\frac{u_3}{\sqrt{\Delta}}\tilde{D}^{(1)}\tilde{\Phi}_6 = \Omega^{(1)}\,, \tag{21}$$

where the operator $\tilde{D}^{(1)}$ given in eq. (13) no longer contains zeroth-order terms. Due to the permutation symmetry (6) in the arguments of Φ_6, eq. (21) leads to two further non-trivial first-order differential equations. This set of differential equations determines Φ_6 up to one integration constant. The latter can be fixed by the requirement that Φ_6 should be non-singular at $\Delta = 0$, which implies the vanishing of $\tilde{\Phi}_6$ on that locus.

Diagonalizing the set of differential equations generated by eq. (21), we have

$$\begin{aligned}\partial_{u_1}\tilde{\Phi}_6(u_1,u_2,u_3) \;=\;& -\frac{1-u_1+u_2-u_3}{(1-u_1)\sqrt{\Delta}}\Omega^{(1)}(u_1,u_2,u_3)\\ &-\frac{1-u_1-u_2-u_3}{u_1\sqrt{\Delta}}\Omega^{(1)}(u_2,u_3,u_1) - \frac{1-u_1-u_2+u_3}{(1-u_1)\sqrt{\Delta}}\Omega^{(1)}(u_3,u_1,u_2)\,,\end{aligned} \tag{22}$$

plus the two cyclically related equations. In the next subsection, we will present the full solution for $\Phi_6(u_1,u_2,u_3)$.

2.3 Result for $\Phi_6(u_1, u_2, u_3)$

Here we present the solution to the differential equations (21), or equivalently (22). We first define the variables

$$x_{i\pm} = u_i x_\pm \,, \tag{23}$$

where $x_\pm$ and Δ are given in eq. (2). The appearance of the $x_{i\pm}$ should not come as a surprise, since they played a prominent role in the two-loop remainder function [28], and we have already argued that Φ_6 should capture some of its structure.

Further, we define

$$\begin{aligned} L_3(x_+, x_-) &= \sum_{m=0}^{2} \frac{(-1)^m}{(2m)!!} \log^m(x_+ x_-) \left[\ell_{3-m}(x_+) - \ell_{3-m}(x_-)\right] , & (24)\\ \ell_m(x) &= \frac{1}{2}(\mathrm{Li}_m(x) - (-1)^m \mathrm{Li}_m(1/x)) \,, & (25) \end{aligned}$$

which is very similar to the function L_4 defined in ref. [28]. As in ref. [28], the branch cuts of $\mathrm{Li}_n(x_+)$ and $\mathrm{Li}_n(1/x_-)$ are taken to lie below the real axis, i.e. $\mathrm{Li}_n(x_+) := \mathrm{Li}_n(x_+ + i\epsilon)$, etc., and the branch cuts of $\mathrm{Li}_n(x_-)$ and $\mathrm{Li}_n(1/x_+)$ are taken to lie above the real axis.[2]

We found the following formula for Φ_6,

$$\Phi_6(u_1, u_2, u_3) = \frac{\tilde{\Phi}_6(u_1, u_2, u_3)}{\sqrt{\Delta}} = \frac{1}{\sqrt{\Delta}} \left[-2 \sum_{i=1}^{3} L_3(x_{i+}, x_{i-}) + 2\zeta_2 J + \frac{1}{3} J^3 \right] , \tag{26}$$

where

$$J = \sum_{i=1}^{3} \left[\ell_1(x_{i+}) - \ell_1(x_{i-})\right] . \tag{27}$$

Although individual terms in eq. (26) can be complex, their sum is always real in the Euclidean region $u_i > 0$.

In the next section, we prove directly that eq. (26) satisfies the differential equations (21). In section 2.5, we will see another way to justify eq. (26) based on the differential equations for its symbol.

2.4 Direct verification of the differential equations

We found the following change of variables to be convenient,

$$u_1 = \frac{v_0 - v_+ v_-}{1 + v_0 - v_+ - v_-} \,, \quad u_2 = \frac{v_0 - v_+ v_-}{(1 + v_0 - v_+ - v_-) v_0} \,, \quad u_3 = \frac{v_+ v_-}{v_0} \,. \tag{28}$$

[2] We are grateful to M. Spradlin and C. Vergu for discussions and correspondence on the branch cut structure of L_4 in ref. [28].

This definition is symmetric in v_+ and v_-. Choosing $v_+ > v_-$ without loss of generality, the inverse transformation is given by

$$v_+ = u_1 u_3 x_+ \,, \quad v_- = u_1 u_3 x_- \,, \quad v_0 = \frac{u_1}{u_2} \,. \tag{29}$$

We also have the following useful expressions for the $x_{i\pm}$,

$$x_{1\pm} = \frac{v_0}{v_\mp} \,, \quad x_{2\pm} = \frac{1}{v_\mp} \,, \quad x_{3\pm} = \frac{v_\pm(1 + v_0 - v_+ - v_-)}{v_0 - v_+ v_-} \,. \tag{30}$$

In terms of the variables $v_{0,+,-}$, Δ is a perfect square,

$$\Delta = \frac{(v_+ - v_-)^2 (v_0 - v_+ v_-)^2}{(1 + v_0 - v_+ - v_-)^2 v_0^2} \,. \tag{31}$$

In the Euclidean region $u_i > 0$ that we are considering, we can take the square root $\sqrt{\Delta}$ without sign ambiguities, see eq. (28).

In the remainder of this section, we will assume $\Delta > 0$ for simplicity, so that the $v_\pm$ are real. Note that the factor J defined in eq. (27) becomes simply

$$J \;=\; -\frac{1}{2} \log \frac{v_+}{v_-} \,. \tag{32}$$

The differential equations (22) are easily expressed in the new variables, using Jacobian factors such as

$$\frac{\partial u_1}{\partial v_+} = \frac{(v_0 - v_-)(1 - v_-)}{(1 + v_0 - v_+ - v_-)^2} \,. \tag{33}$$

The differential equation in v_0 turns out to be the simplest one, namely

$$\partial_{v_0} \tilde{\Phi}_6(v_\pm, v_0) = \frac{v_+ - v_-}{(v_0 - v_-)(v_0 - v_+)} \log \frac{(v_0 - v_+ v_-)}{(1 + v_0 - v_+ - v_-) v_0} \log \frac{(v_0 - v_+ v_-) v_0}{(1 + v_0 - v_+ - v_-) v_+ v_-} \,. \tag{34}$$

Using eqs. (30) and (32), it is easy to show that

$$\partial_{v_0} L_3(x_{1+}, x_{1-}) \;=\; \frac{1}{8} \frac{v_+ - v_-}{(v_0 - v_-)(v_0 - v_+)} \log^2 \left(\frac{v_+ v_-}{v_0^2} \right) \,, \tag{35}$$

$$\partial_{v_0} L_3(x_{3+}, x_{3-}) \;=\; -\frac{1}{8} \frac{v_+ - v_-}{(v_0 - v_-)(v_0 - v_+)} \log^2 \left(\frac{v_+ v_- (1 + v_0 - v_+ - v_-)^2}{(v_0 - v_+ v_-)^2} \right) \,, \tag{36}$$

$$\partial_{v_0} L_3(x_{2+}, x_{2-}) \;=\; 0 \,, \tag{37}$$

$$\partial_{v_0} J \;=\; 0 \,. \tag{38}$$

Hence $\tilde{\Phi}_6$ as defined in eq. (26) satisfies eq. (34). We have checked numerically that the differential equations with respect to v_+ and v_- are satisfied as well.

2.5 Symbol of $\tilde{\Phi}_6(u_1, u_2, u_3)$

The notion of symbols has proven to be a useful tool for thinking about transcendental functions appearing in $\mathcal{N} = 4$ SYM; see ref. [28] and references therein.

The symbol $[\tilde{\Phi}_6]$ of $\tilde{\Phi}_6$ is very simple, namely,

$$[\tilde{\Phi}_6(u_1, u_2, u_3)] = -[\Omega^{(1)}(u_1, u_2, u_3)] \otimes \frac{x_+(1 - x_{3-})}{x_-(1 - x_{3+})} + \text{cyclic}\,, \tag{39}$$

where

$$[\Omega^{(1)}(u_1, u_2, u_3)] = u_1 \otimes u_2 + u_2 \otimes u_1 - \sum_{i=1}^{3} u_i \otimes (1 - u_i)\,. \tag{40}$$

Note that the first of the three entries in $[\tilde{\Phi}_6]$ is always either u_1, u_2 or u_3. Because the u_i are ratios of the distances x_{ij}^2, using standard properties of the symbol the first entry can always be expressed as a distance. This property has been argued to follow from the branch-cut structure of loop integrals [10].

In order to see directly that eq. (39) is the symbol of eq. (26) it is helpful to introduce to some projective variables $w_i \in \mathbb{CP}^1$ for $i = 1, \ldots, 6$. Choosing homogeneous coordinates $w_i = (1, z_i)$, they coincide with the z_i variables of [28]. We can represent the three cross-ratios as follows,

$$u_1 = \frac{(23)(56)}{(25)(36)}\,, \qquad u_2 = \frac{(34)(61)}{(36)(41)}\,, \qquad u_3 = \frac{(45)(12)}{(41)(52)}\,, \tag{41}$$

where $(ij) = -(ji) = \epsilon_{ab} w_i^a w_j^b$. In terms of these variables Δ is a perfect square,

$$\Delta = \left[\frac{(12)(34)(56) + (23)(45)(61)}{(14)(25)(36)}\right]^2 \tag{42}$$

and all entries of the symbol factorize into two-brackets (ij). Thus one can canonically represent the symbol as a sum of terms of the form

$$(ab) \otimes (cd) \otimes (ef)\,. \tag{43}$$

Performing this on the symbol (39) and the symbol of (26) one finds immediately the same expression.

One can easily check that the symbol of $\tilde{\Phi}_6$ is consistent with the differential equation (22) for $\tilde{\Phi}_6$. We simply replace the functions $\tilde{\Phi}_6$ and $\Omega^{(1)}$ in eq. (22) by their symbols, and use the following simple identities,

$$\partial_{u_1} \log \frac{x_+(1 - x_{1-})}{x_-(1 - x_{1+})} = \frac{1 - u_1 - u_2 - u_3}{u_1\sqrt{\Delta}}\,, \tag{44}$$

$$\partial_{u_1} \log \frac{x_+(1 - x_{2-})}{x_-(1 - x_{2+})} = \frac{1 - u_1 - u_2 + u_3}{(1 - u_1)\sqrt{\Delta}}\,, \tag{45}$$

$$\partial_{u_1} \log \frac{x_+(1 - x_{3-})}{x_-(1 - x_{3+})} = \frac{1 - u_1 + u_2 - u_3}{(1 - u_1)\sqrt{\Delta}}\,, \tag{46}$$

and the differentiation rule for symbols,

$$\partial_x \, (a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n) = \partial_x \log(a_n) \, \times \, a_1 \otimes \ldots \otimes a_{n-1} \, . \tag{47}$$

This analysis can be used to justify the solution (26), following ref. [28]: We have already seen that eq. (26) has the correct symbol. This leaves two ambiguities in Φ_6, firstly where to place the branch cuts, and secondly the freedom to add constants multiplied by functions of lower transcendentality than three. The first ambiguity is resolved by requiring that Φ_6 be real-valued and smooth in the entire Euclidean region $u_i > 0$. We have numerical evidence that this is the case for Φ_6 in eq. (26). The second ambiguity has to be fixed by other means. The ζ_2 term in eq. (10) for $\Omega^{(1)}$, which enters the differential equation (14), suggests the corresponding term in eq. (26). We have also checked that the resulting formula is in agreement with the parametric representation (8) for several numerical values, which cover different regions according to the signs of Δ, $u_i - 1$ and $u_1 + u_2 + u_3 - 1$.

3 Conclusions and outlook

In this paper, we have computed the six-dimensional one-loop on-shell scalar hexagon integral Φ_6, giving its full kinematical dependence in the Euclidean region. The result is a remarkably simple formula, eq. (26). Interestingly, its structure is almost identical to that of the two-loop remainder function in planar $\mathcal{N} = 4$ SYM [28], although the latter is of transcendentality degree 4, while Φ_6 is of degree 3.

Our calculation was based on the observation that Φ_6 is related to a known four-dimensional one-loop tensor hexagon integral through first-order differential equations. The latter uniquely determine the answer. It is interesting to note that both the two-loop remainder function and Φ_6 are best expressed in terms of a set of (redundant) variables $x_{i\pm}$. For Φ_6, one is led to these variables in a very natural way when solving the aforementioned differential equations. This approach should be very helpful when computing other integrals of this kind. In particular an extension to degree five and six functions should provide valuable insight into the structure of the remainder function at higher loops. Another interesting extension of this work could be to consider the hexagon integral with massive corners, which may give hints about good sets of kinematic variables for amplitudes with $n > 6$ external legs.

The procedure for finding a relation between $\Omega^{(2)}$ and $\Omega^{(1)}$ in ref. [32] was based on a Laplace equation, which is second-order in nature, as are typical field equations for bosonic fields. On the other hand, fermionic field equations are typically first order. One might speculate that the first-order relations (1) between $\Omega^{(1)}, \Phi_6$ and $\Omega^{(2)}$ found in the present paper could have an explanation based on supersymmetry. What is somewhat mysterious from this point of view is why the function Φ_6 which sits between $\Omega^{(1)}$ and $\Omega^{(2)}$ should have a full cyclic symmetry, when neither $\Omega^{(1)}$ nor $\Omega^{(2)}$ do.

Finally, we comment that the fully off-shell version of H has a conventional conformal symmetry in addition to its dual conformal symmetry. This is the case simply because it is built from ϕ^3 vertices, and ϕ^3 theory in $D = 6$ dimensions is classically conformal. By Fourier transforming the coordinate space conformal generators d, k^μ, and accounting for a change in conformal dimension coming from the amputation of external legs, we find their form in momentum space, acting on H,

$$d = \sum_{i=1}^{n} \Big[p_i^\nu \partial_{i\nu} + 2\Big]\,, \quad k^\mu \;=\; \sum_{i=1}^{n} \Big[-\frac{1}{2} p_i^\mu \partial_i^\nu \partial_{i\nu} + 2\partial_i^\mu + p_i^\nu \partial_{i\nu}\partial_i^\mu \Big]\,. \tag{48}$$

Invariance under these operators then implies homogeneous second-order differential equations. If one takes some or all external legs on shell, as in the case of H (or Φ_6), it can happen that the action of the conformal generators becomes anomalous.

4 Acknowledgments

We thank M. Spradlin and C. Vergu for useful discussions. This research was supported by the US Department of Energy under contract DE–AC02–76SF00515.

Note added. After this calculation was completed, we were informed by V. Del Duca, C. Duhr and V. Smirnov of an independent computation of the hexagon integral presented here, using a different method [39].

A A special case of Φ_6

The differential equations simplify considerably in the special case $u_3 = 1$, for which $\sqrt{\Delta} = u_1 - u_2$. (This is true for $u_1 > u_2$, which we can assume without loss of generality since Φ_6 is symmetric in u_1 and u_2.) Starting from eq. (22), and using $\Omega^{(1)}(u_2, 1, u_1) = \Omega^{(1)}(1, u_1, u_2)$, we find

$$\partial_{u_1} \tilde{\Phi}_6(u_1, u_2, 1) = \frac{\Omega^{(1)}(u_1, u_2, 1)}{1 - u_1} - \frac{\Omega^{(1)}(1, u_1, u_2)}{u_1(1 - u_1)}\,. \tag{49}$$

One can easily find the solution

$$\Phi_6(u_1, u_2, 1) = \frac{\tilde{\Phi}_6(u_1, u_2, 1)}{u_1 - u_2} = \frac{h(u_1, u_2) - h(u_2, u_1)}{u_1 - u_2}\,, \tag{50}$$

where

$$h(u_1, u_2) = \log u_1 \left(\zeta_2 - \mathrm{Li}_2(u_1) - \mathrm{Li}_2(1 - u_2)\right) + 2\,\mathrm{Li}_3(u_1)\,. \tag{51}$$

B Relations between $D = 6$ integrals and $D = 4$ tensor integrals

Here we give another example of a relation between a four-dimensional tensor integral and a six-dimensional scalar integral. While the relation in the main text involved a first-order differential operator, the relation we present here is simply an equality of two integrals.

Let us consider the finite, dual conformal pentagon integral $\tilde{\Psi}$ [33, 32] that appears in the representation of [33] of one-loop MHV amplitudes in $\mathcal{N}=4$ SYM. Up to a normalization factor, it is given by

$$\tilde{\Psi} \propto \int \frac{d^4 x_i}{i\pi^2} \frac{x_{ia}^2}{x_{2i}^2 x_{3i}^2 x_{5i}^2 x_{6i}^2 x_{8i}^2} \,, \tag{52}$$

where x_a^μ is defined as one of the two solutions to the four-cut conditions $x_{2a}^2 = x_{3a}^2 = x_{5a}^2 = x_{6a}^2 = 0$. As in the case of $\Omega^{(1)}$, the numerator factor makes the integral IR finite.

We remark that dual conformal transformations can be used to remove the $1/x_{8i}^2$ propagator, by letting $x_8^\mu \to \infty$, as in ref. [36]. This is possible in this case because there are no light-like constraints between x_8^μ and the neighboring x_2^μ and x_6^μ. In this way we obtain the equivalent integral

$$I = \int \frac{d^4 x_i}{i\pi^2} \frac{x_{ia}^2}{x_{2i}^2 x_{3i}^2 x_{5i}^2 x_{6i}^2} \,. \tag{53}$$

This integral is not dual conformally invariant, and is a function of $x_{25}^2, x_{26}^2, x_{35}^2, x_{36}^2$. Up to a normalization factor, it equals the finite part of the two-mass easy box integral [35, 40]

$$I = \frac{-1}{x_{26}^2 + x_{35}^2 - x_{25}^2 - x_{36}^2} \left[\mathrm{Li}_2(1-\xi x_{26}^2) + \mathrm{Li}_2(1-\xi x_{35}^2) - \mathrm{Li}_2(1-\xi x_{25}^2) - \mathrm{Li}_2(1-\xi x_{36}^2) \right] \,, \tag{54}$$

where $\xi = (x_{26}^2 + x_{35}^2 - x_{25}^2 - x_{36}^2)/(x_{26}^2 x_{35}^2 - x_{25}^2 x_{36}^2)$. Since the finite part of the one-loop MHV amplitude in $\mathcal{N}=4$ SYM is governed by this function (the divergent parts correspond to one-mass and two-mass triangle integrals), this gives a very direct relation between six-dimensional integrals and four-dimensional amplitudes.

In order to see the relation of I to a scalar integral in $D=6$ dimensions, one can introduce Feynman parameters, treating the numerator x_{ia}^2 as an inverse propagator $1/(x_{ia}^2)^{-1+\delta}$ with some auxiliary analytic regularization δ. Integrating out the Feynman parameter corresponding to this inverse propagator and letting $\delta \to 0$, one readily obtains

$$I = \int_0^1 d\alpha_{2,3,5,6} \frac{\delta(1-\sum_{i=2,3,5,6}\alpha_i)}{\alpha_2\alpha_5 x_{25}^2 + \alpha_3\alpha_5 x_{35}^2 + \alpha_2\alpha_6 x_{26}^2 + \alpha_3\alpha_6 x_{36}^2} \,, \tag{55}$$

which is nothing else than the Feynman parametrization of the following scalar integral in $D=6$ dimensions,

$$I = \int \frac{d^6 x_i}{i\pi^3} \frac{1}{x_{2i}^2 x_{3i}^2 x_{5i}^2 x_{6i}^2} \,. \tag{56}$$

We remark that by combining propagators pairwise (see appendix C) and integrating out the resulting finite bubble integral, one obtains a Wilson-loop type of representation for this integral [6, 41].

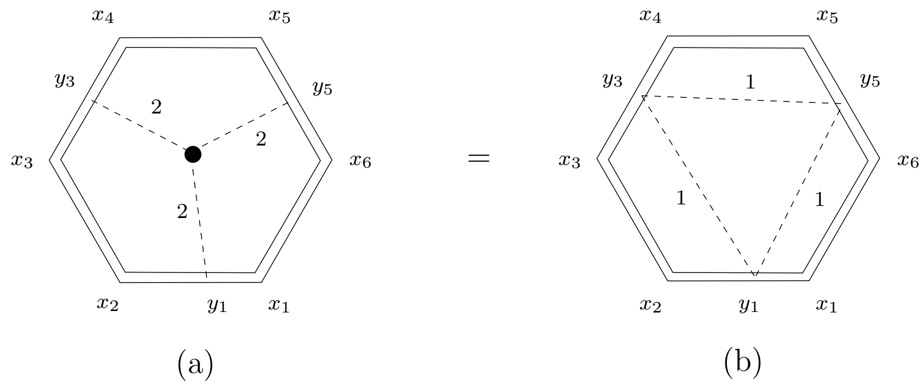

Figure 2: Interpretation of the hexagon integral as a line integral, according to eqs. (59) and (60).

C Wilson-loop representation of Φ_6

In section 2.1, we explained how dual conformal symmetry helps to obtain a convenient Feynman parametrization for H, where in particular the number of parameter integrals is equal to the degree of the function. Here, we present a second way of exploiting dual conformal symmetry, which in addition allows for an interpretation of H as a Wilson-loop integral.

Let us start from the definition of H given in eq. (3). It is well-known that for on-shell integrals it is often desirable to introduce Feynman parameters in steps, i.e. to combine two adjacent propagators at a time, using the formula

$$\frac{1}{x_{1i}^2 x_{2i}^2} = \int_0^1 d\xi_1 \frac{1}{[(y_1 - x_i)^2]^2}\,, \qquad x_{12}^2 = 0\,, \tag{57}$$

where

$$y_1^\mu(\xi_1) = x_1^\mu(1-\xi_1) + x_2^\mu \xi_1\,. \tag{58}$$

For example, the two-mass easy box integral is “easy” precisely because it contains two pairs of propagators separated by a massless leg; eq. (57) can be applied to each pair.

Repeating this procedure for the other two pairs of adjacent propagators leads to

$$H = \int_0^1 d\xi_{1,3,5} \int \frac{d^6 x_i}{i\pi^3} \frac{1}{[(y_1 - x_i)^2]^2[(y_3 - x_i)^2]^2[(y_5 - x_i)^2]^2}\,, \tag{59}$$

where y_3^μ (y_5^μ) is defined like y_1^μ in eq. (58), but with $i \to i+2$ ($i \to i+4$). At the cost of having introduced three parameter integrals, the innermost integral now depends on three “effective propagators” only, see Fig. 2(a). For a triangle integral, however, dual conformal symmetry fixes the answer uniquely to be a constant multiple of $1/[(y_1-y_3)^2(y_1-y_5)^2(y_3-y_5)^2]$. The constant can be determined from a boundary condition, e.g. $y_5 \to \infty$. This is nothing else than the star-triangle (or uniqueness) relation [42], of course. Hence the answer is simply

$$H = \int_0^1 d\xi_{1,3,5} \frac{1}{(y_1-y_3)^2(y_1-y_5)^2(y_3-y_5)^2}\,, \tag{60}$$

which is depicted in see Fig. 2(b). More explicitly, we have $(y_1-y_3)^2 = x_{13}^2\bar{\xi}_1\bar{\xi}_3 + x_{14}^2\xi_3\bar{\xi}_1 + x_{24}^2\xi_1\xi_3$, where $\bar{\xi} := 1-\xi$, etc. In this form, the Feynman loop integral is reminiscent of a Wilson-loop integral in the dual space of the x_i. See ref. [41] for a similar discussion of related integrals.

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