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Noise Amplification in HGHG Seeding*

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Abstract

An essential element of seeded FEL based on high-gain harmonic generation (HGHG) or echo-enabled harmonic generation (EEHG) is an undulator-modulator, in which interaction with a laser beam modulates the beam energy. We study how the interaction of electrons in this undulator-modulator changes the noise properties of the beam.

INTRODUCTION

As was pointed out in Ref. [1], the noise effects in seeded FELs can be amplified in the process of seeding. Moreover, the authors of that paper proposed the scaling in which the noise amplification factor is proportional to the square of the harmonic factor.

A specific mechanism of noise propagation and amplification in HGHG was studied in Ref. [2]. This mechanism takes into account interaction of electrons in the undulator-modulator and the resulting evolution of the bunching factor of the beam in the vicinity of the HGHG harmonics.

In this paper, we extend the analysis of [2] by explicitly considering the energy exchange of the electrons in the undulator-modulator caused by the electron interaction via undulator radiation. We find that this interaction, combined with the passage through a chicane, introduces *correlations* in the position of electrons. The correlations would then lead to increased radiation in the final undulator of the system, which might be interpreted as an increased noise in the beam. Our analysis in this paper is limited to the HGHG seeding only. In a companion paper [3] we expand our analysis for the case of the echo-enabled harmonic generation [4, 5].

NOISE AND SEEDING

The quantity that characterizes the beam as a radiating medium, in a one dimensional model, is $\sum_{j,l=1}^{N} e^{ik(z_j-z_l)}$, where $k = \omega/c$, and z_j is the longitudinal coordinate of j-th particle. This quantity appears in calculation of the intensity of the radiation of the beam at frequency ω, in the limit when the transverse size of the beam can be neglected. As always, we separate the terms with $j = l$ in the sum to obtain $N + \sum_{j\neq l} e^{ik(z_j-z_l)}$. The first term here is the so called shot noise—it results in an incoherent radiation of the beam equal to the sum of intensities of all particles. The second term is responsible for the coherent radiation due to nonuniform distribution of particles in the beam as well as radiation due to correlation of positions of different particles. The relative strength of the second term in comparison with the first one (the shot noise) is denoted here by the factor F

$$F(k) = \frac{1}{N}\sum_{j\neq l} e^{ik(z_j-z_l)}. \tag{1}$$

As written, this factor depends on the exact positions of all particles in the beam. Such information is never available in a macroscopic system, and we will introduce an ensemble-averaged value of F

$$\langle F(k)\rangle = \frac{1}{N}\langle\sum_{j\neq l} e^{ik(z_j-z_l)}\rangle \approx N\langle e^{ik(z_1-z_2)}\rangle. \tag{2}$$

The averaging here should be performed with the help of a two-particle distribution f_2, which depends on coordinates and momenta of both particles. In our case, we assume that the particles are characterized by the longitudinal coordinate z and the relative energy deviation η, so that the distribution function is $f_2(z_1,\eta_1,z_2,\eta_2)$ normalized so that $\int dz_1 dz_2 d\eta_1 d\eta_2 f_2(z_1,\eta_1,z_2,\eta_2) = 1$.

We now consider a seeding system which generates a density modulation in the beam. Our goal will be to compute the factor $\langle F\rangle$ at the exit from the seeding system, before the beam enters the undulator-radiator. The coordinates, momenta and the distribution function at the exit from the seeding system are marked below with a hat, and we have

$$\langle F(k)\rangle = N\int d\hat z_1 d\hat z_2 d\hat\eta_1 d\hat\eta_2 e^{ik(\hat z_1-\hat z_2)}\hat f_2(\hat z_1,\hat\eta_1,\hat z_2,\hat\eta_2).$$

To find the two-particle distribution function $\hat f_2$, one has to know its initial value before the entrance to the system, and to solve the kinetic equation which describes evolution of f_2 through the undulator and the chicane. Note that interaction of particles in the undulator will lead to establishing correlations in the system, which will be reflected in the structure of $\hat f_2$. The corresponding technique was developed in Ref. [6], however it leads to a rather complicated analysis. We will use a different approach, which involves working with the variables and the distribution function at the entrance to the seeding system.

Assuming that one knows the transformation from the initial coordinates z_i, η_i at the entrance to the seeding device (which are denoted here without the hats), to the final ones $\hat z_i$, $\hat\eta_i$, instead of averaging over the final coordinates, one can use averaging in Eq. (1) over the initial

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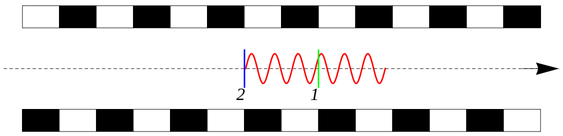

Figure 1: Interaction of two particles in the undulator.

coordinates, assuming that the initial distribution function $f_N(z_1, \eta_1, z_2, \eta_2, \dots, z_N, \eta_N)$ is known,

$$\langle F(k)\rangle = N \int e^{ik[\hat{z}_1(z_1,\eta_1,\dots,z_N,\eta_N)-\hat{z}_2(z_1,\eta_1,\dots,z_N,\eta_N)]}$$
$$f_N(z_1,\eta_1,\dots,z_N,\eta_N)\, dz_1 \dots dz_N\, d\eta_1 \dots d\eta_N. \quad (3)$$

It is reasonable to assume that initially there are no correlations in the beam. This means that the N-particle distribution function is a product of one particle distributions $f_1(z_i, \eta_i)$,

$$f_N(z_1,\eta_1,\dots,z_N,\eta_N) = f_1(z_1,\eta_1)\dots f_1(z_N,\eta_N). \quad (4)$$

In our calculations we will assume a uniform distribution of particles in the bunch with a Gaussian energy distribution:

$$f_1(z,\eta) = \frac{1}{\sqrt{2\pi}\sigma_\eta L} e^{-\eta^2/2\sigma_\eta^2} \text{ for } -\frac{1}{2L} < z < \frac{1}{2L}, \quad (5)$$

and $f_1 = 0$ otherwise, where L is the length of the bunch.

INTERACTION IN UNDULATOR

The key element of the proposed approach is taking into account the interaction of particles in the undulator-modulator. Every electron of the beam (call it the first particle) emits an electromagnetic wave, that propagates in front of it, see Fig. 1). Whenever this wave reaches another electron (the second particle), it starts to change its energy, and depending on the relative position of the two electrons, either increases of decreases the energy of the second electron. This energy change can be easily computed in a 1D model [7]. We will use here the result of Ref. [8] for a helical undulator. According to [8], the accumulated relative energy change $h = \Delta E/\gamma mc^2$ of an electron located at coordinate z_1 in the bunch traveling through an undulator, due to the presence of another electron at $z_2 = z_1 - \zeta$, is given by

$$h(\zeta) = -A\left(1 - \frac{\zeta}{N_u\lambda_0}\right)\cos k_0\zeta, \quad N_u\lambda_0 > \zeta > 0 \quad (6)$$

and $h(\zeta) = 0$ otherwise, with the parameter A

$$A = 2\pi \frac{e^2 K^2 N_u \lambda_u^2}{S\gamma^3 mc^2 \lambda_0}, \quad (7)$$

where N_u is the number of undulator periods, λ_0 the undulator radiation wavelength, K the undulator parameter, S the transverse area of the beam, and λ_u the undulator period. The plot of the function h is shown in Fig. 2.

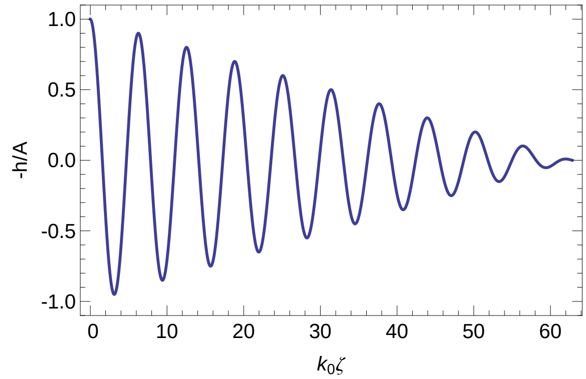

Figure 2: The interaction function h as a function of the argument $k_0\zeta$ for $N_u = 10$.

We will consider now an HGHG seeding in which the beam is first modulated in energy due to interaction with a laser at wavelength λ_0 in an undulator-modulator, and then sent through a chicane with the strength R_{56}. The resulting functions $\hat{\eta}_i$ and $\hat{z}_i$ are

$$\hat{\eta}_i = \eta_i - \Delta\eta \sin(k_0 z_i) + \sum_{j\neq i} h(z_i - z_j),$$
$$\hat{z}_i = z_i + R_{56}\hat{\eta}_i, \quad (8)$$

where $\Delta\eta$ is the amplitude of the energy modulation.

Substituting (8) into (3) and using (5) we obtain

$$\langle F(k)\rangle = \frac{N}{L^N} e^{-k^2 R_{56}^2 \sigma_\eta^2} \int_{-L/2}^{L/2} dz_1 \dots \int_{-L/2}^{L/2} dz_N$$
$$\exp\Big[ik\{\zeta - R_{56}\Delta\eta\,[\sin(k_0 z_1) - \sin(k_0 z_2)] \quad (9)$$
$$+ R_{56}\left(\tilde{h}(\zeta) + {\sum}'\,[h(z_1 - z_n) - h(z_2 - z_n)]\right)\}\Big],$$

where $\zeta = z_1 - z_2$, $\tilde{h}(\zeta) = h(\zeta) - h(-\zeta)$, $\sum' = \sum_{n\neq 1,2}$. This equation establishes a mathematical framework for calculation of the noise properties in the system.

NOISE AMPLIFICATION

In this paper we use the approximation

$$kR_{56}h \sim AkR_{56} \ll 1, \quad (10)$$

and Taylor expand the exponentials in (9). For typical undulators used in practice, Eq. (10) is well satisfied. Note that integrals over variables $z_3, \dots z_N$ in (9) are separable and identical. The product of $N-2$ integrals can be written as

$$\left(\frac{1}{L}\int_{-L/2}^{L/2} dz_i e^{ikR_{56}[h(z_1-z_i)-h(z_2-z_i)]}\right)^{N-2}. \quad (11)$$

Using (10) and expanding the exponential in the Taylor series up to the quadratic terms in h, we notice that the linear in h term vanishes after the integration, and we are left with

$$\left(1 - \frac{k^2 R_{56}^2}{2L} \int dz_i \left[h(z_1 - z_i) - h(z_2 - z_i)\right]^2\right)^{N-2}$$
$$\approx \left(1 - n_0 \frac{k^2 R_{56}^2}{2N} \int dz_i \left[h(z_1 - z_i) - h(z_2 - z_i)\right]^2\right)^{N}, \quad (12)$$

and we have defined the particle density $n_0 = N/L$. Taking the limit $N \to \infty$ and using $\lim_{N\to\infty}(1 + x/N)^N = e^x$, one obtains

$$\langle F(k)\rangle = \frac{n_0}{L} e^{-k^2 R_{56}^2 \sigma_\eta^2} \int_{-L/2}^{L/2} dz_1 dz_2 e^{ik\zeta}(1 + \Gamma_1)e^{\Gamma_2}$$
$$\exp\left[ikR_{56}\{\tilde{h}(\zeta) - \Delta\eta\left[\sin(k_0 z_1) - \sin(k_0 z_2)\right]\}\right] \quad (13)$$

where $\Gamma_1 = ikR_{56}\tilde{h}(\zeta)$ and

$$\Gamma_2 = -\frac{n_0}{2} k^2 R_{56}^2 \int dz \left[h(z_1 - z) - h(z_2 - z)\right]^2. \quad (14)$$

The integral in (14) can be easily computed using (6). In the limit $N_u \gg 1$ the calculation gives

$$\Gamma_2 = \frac{n_0}{2} k^2 R_{56}^2 A^2 N_u \lambda_0 \left[R\left(\frac{\zeta}{N_u \lambda_0}\right) \cos k_0 \zeta - \frac{1}{3}\right], \quad (15)$$

with $R(x) = \left(\frac{1}{3} - \frac{1}{2}|x| + \frac{1}{6}|x|^3\right)$ for $x \le 1$ and $R(x) = 0$ otherwise.

We will further limit our analysis by assumption $\Gamma_2 \ll 1$ or $n_0 k^2 R_{56}^2 A^2 N_u \lambda_0 \ll 1$. We then have

$$(1 + \Gamma_1)e^{\Gamma_2} \approx 1 + \Gamma_1 + \Gamma_2. \quad (16)$$

The first term in this equation (unity) corresponds to the limit of no interaction in the undulator, and gives $\langle F(k)\rangle$ corresponding to the standard HGHG density modulation of the beam. The second and the third terms are responsible for the noise amplification. Using (16), due to the assumed smallness of Γ_2, we can neglect the addendum $-1/3$ in (15) in comparison with 1 in (16). In what follows, we redefine Γ_2 dropping $-1/3$ in (15). That makes the function Γ_2 equal to zero outside of the interval $|\zeta| < N_u \lambda_0$.

CONTRIBUTION OF Γ_1 AND Γ_2

Let us first focus on the contribution of Γ_1 in Eq. (16) to $\langle F\rangle$, which we denote by $\langle F_1\rangle$

$$\langle F_1(k)\rangle = \frac{1}{L} ikn_0 R_{56} e^{-k^2 R_{56}^2 \sigma_\eta^2} \int_{-L/2}^{L/2} dz_1 \int_{-L/2}^{L/2} dz_2$$
$$e^{ik\{\zeta - R_{56}\Delta\eta[\sin(k_0 z_1) - \sin(k_0 z_2)]\}} \tilde{h}(\zeta). \quad (17)$$

Using the expansion $e^{-ia\sin(x)} = \sum_{p=-\infty}^{\infty} J_p(a) e^{-ipx}$, where J_p is the Bessel function of order p, one finds

$$\langle F_1(k)\rangle = \frac{1}{L} ikn_0 R_{56} e^{-k^2 R_{56}^2 \sigma_\eta^2}$$
$$\times \sum_{p,r} J_p(kR_{56}\Delta\eta) J_r(kR_{56}\Delta\eta) \int_{-L/2}^{L/2} dz_1 \int_{-L/2}^{L/2} dz_2$$
$$\times e^{ik\zeta - ipk_0 z_1 + irk_0 z_2} \tilde{h}(\zeta). \quad (18)$$

In the limit when L is large, $L \gg 1/k_0$, the dominant contribution to the integral comes from the terms with $p = r$, in which case the whole integrand depends on the difference $z_1 - z_2$ only. Using this simplification we arrive at

$$\langle F_1(k)\rangle = \frac{1}{L} n_0 ikR_{56} e^{-k^2 R_{56}^2 \sigma_\eta^2} \sum_p J_p(kR_{56}\Delta\eta)^2$$
$$\times \int_{-L/2}^{L/2} dz_1 \int_{-L/2}^{L/2} dz_2 e^{i(k - pk_0)\zeta} \tilde{h}(\zeta). \quad (19)$$

We can now transform the integration to the variables $\zeta = z_1 - z_2$ and $Z = (z_1 + z_2)/2$. The integration over ζ is limited by a small range of $0 < \zeta < N_u \lambda_0$, much less than L, hence approximately $\int dZ \to L$, which gives

$$\langle F_1(k)\rangle = ikn_0 R_{56} e^{-k^2 R_{56}^2 \sigma_\eta^2}$$
$$\times \sum_p J_p(kR_{56}\Delta\eta)^2 \int d\zeta e^{i(k - pk_0)\zeta} \tilde{h}(\zeta). \quad (20)$$

The function $\tilde{h}$ (as well as h) is an oscillating function with the period λ_0 and slowly changing amplitude, hence its spectrum is localized near the wavenumbers $\pm k_0$. This means that, for a given value of p, the integral in Eq. (20) is localized in the two narrow spectral regions around the frequencies $\omega_0(p \pm 1)$. Let us assume that the radiator-undulator is tuned to the frequency $m\omega_0$ and we are interested in the noise in the frequency interval around this value, $k = mk_0 + \Delta k$, with $\Delta k \ll k$. The main contribution to this interval comes from two terms in (20) with $p = m \pm 1$. Leaving only these terms, we obtain

$$\langle F_1(k)\rangle = in_0 mk_0 R_{56} e^{-m^2 k_0^2 R_{56}^2 \sigma_\eta^2}$$
$$\times \left(J_{m-1}(mk_0 R_{56}\Delta\eta)^2 \int d\zeta e^{i(\Delta k + k_0)\zeta} \tilde{h}(\zeta)\right.$$
$$\left. + J_{m+1}(mk_0 R_{56}\Delta\eta)^2 \int d\zeta e^{i(\Delta k - k_0)\zeta} \tilde{h}(\zeta)\right). \quad (21)$$

Using the notation

$$b_{q,p} = e^{-q^2 k_0^2 R_{56}^2 \sigma_\eta^2/2} |J_{q-p}(qk_0 R_{56}\Delta\eta)| \quad (22)$$

we can also write

$$\langle F_1(k)\rangle = n_0 mk_0 R_{56}$$
$$\times \left(b_{m,1}^2 H_1(\Delta k) - b_{m,-1}^2 H_1(-\Delta k)\right), \quad (23)$$

where

$$H_1(\Delta k) = i \int_{-\infty}^{\infty} d\zeta e^{i(\Delta k + k_0)\zeta} \tilde{h}(\zeta)$$
$$= 2AN_u\lambda_0 g_1\left(2\pi N_u \frac{\Delta k}{k_0}\right), \qquad (24)$$

with $g_1(x) = (x - \sin x)/2x^2$. In derivation of (24) we assumed $N_u \gg 1$ and $\Delta k \ll k_0$. Eq. (23) can also be written as

$$\langle F_1(k)\rangle = 2An_0 mk_0 R_{56} N_u \lambda_0 g_1\left(2\pi N_u \frac{\Delta k}{k_0}\right)$$
$$\times (b_{m,1}^2 + b_{m,-1}^2). \qquad (25)$$

Let us now focus on the contribution of Γ_2 in Eq. (16) to $\langle F\rangle$ which we denote by $\langle F_2\rangle$. Calculations analogous to that of the previous section lead to the following result

$$\langle F_2(k)\rangle = \frac{1}{2} n_0^2 m^2 k_0^2 R_{56}^2 A^2 N_u \lambda_0 H_2(\Delta k)(b_{m,1}^2 + b_{m,-1}^2)$$
$$H_2(\Delta k) = \int_{-\infty}^{\infty} d\zeta \tilde{d}(\zeta) e^{i(\Delta k + k_0)\zeta}, \qquad (26)$$

where $\tilde{d}(\zeta) = R(|\zeta|/N_u\lambda_0)\cos k_0\zeta$.

The function $H_2(\Delta k)$ can be easily computed

$$H_2(\Delta k) = \frac{1}{8} N_u \lambda_0 g_2\left(2\pi N_u \frac{\Delta k}{k_0}\right), \qquad (27)$$

with

$$g_2(x) = \frac{4}{x^4}[x^2 - 2x\sin(x) - 2\cos(x) + 2]. \qquad (28)$$

The function g_2 is always positive and leads to the amplification of the noise. The quadratic term (26) was also derived in [2].

DISCUSSION AND NUMERICAL EXAMPLE

Note that in contrast to the statement of Refs. [1, 2] that the noise scales as m^2 we found in our model two noise terms (25) and (26). The first one is linear in m, and the second one is quadratic. Moreover, the first term being antisymmetric with respect to the central frequency $m\omega_0$ takes both negative and positive values, and as demonstrated in [9] can lead to the noise suppression in some cases.

As a numerical example, let us consider the nominal parameter of the seeded HGHG FEL at the Fermi@Elettra project [10]. The electron beam energy is 1.2 GeV, the slice energy spread is 150 keV and the peak current is 800 A. We assume the wavelength of the seed laser $\lambda_0 = 240$ nm and consider generation of the 6-th harmonic, with the wavelength of 40 nm ($k = 0.16$ nm^{-1}). The modulator undulator parameters are: $N_u = 19$, $\lambda_u = 16$ cm, with the energy modulation amplitude $\Delta E = 2$ MeV. Although FERMI will have a plane undulator-modulator, here

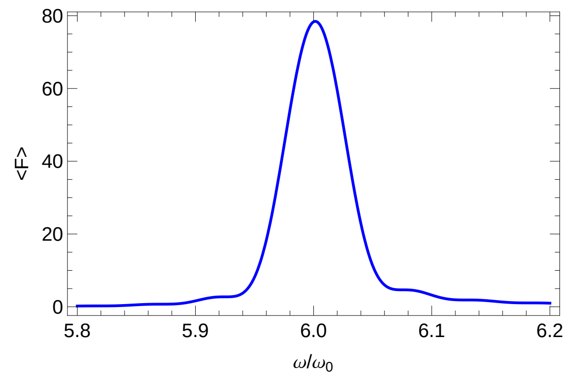

Figure 3: Noise amplification factor in the vicinity of 40 nm (6th harmonic) for FERMI FEL.

we use for estimates our model that assumes a helical undulator with the value of K, which can be inferred from the above parameters, $K = 2.7$. The chicane strength is $R_{56} = 36$ μm. We assume that the transverse size of the beam in the modulator-undulator is $\sigma_x = \sigma_y = 100$ μm and use for the parameter S in (7) $S = 2\pi\sigma_z\sigma_y$.

First we estimate the parameter A in (7) to obtain $A = 3.2 \times 10^{-10}$. We then find $\Gamma_1 \sim kR_{56}A \approx 1.8 \times 10^{-6}$, $\Gamma_2 \sim \frac{n_0}{2} k^2 R_{56}^2 A^2 N_u \lambda_0 \approx 1.2 \times 10^{-4}$, which shows that both Γ_1 and Γ_2 are much smaller than one, and the Taylor expansion used above is valid. The plot of the noise amplification factor calculated using (25) and (26) is shown in Fig. 3. We see that at the center of the line, the noise is amplified by almost two orders of magnitude. The dominant contribution to the noise amplification in this case is due to the quadratic term (26). Note that is scales as N_u^4 with the number of periods in the undulator.

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