SLAC-PUB-14224
HIP-2010-21/TH
CP3-Origins-2010-34

The $\hbar$ Expansion in Quantum Field Theory

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We show how expansions in powers of Planck's constant $\hbar = h/2\pi$ can give new insights into perturbative and nonperturbative properties of quantum field theories. Since $\hbar$ is a fundamental parameter, exact Lorentz invariance and gauge invariance are maintained at each order of the expansion. The physics of the $\hbar$ expansion depends on the scheme; i.e., different expansions are obtained depending on which quantities (momenta, couplings and masses) are assumed to be independent of $\hbar$. We show that if the coupling and mass parameters appearing in the Lagrangian density are taken to be independent of $\hbar$, then each loop in perturbation theory brings a factor of $\hbar$. In the case of quantum electrodynamics, this scheme implies that the classical charge e, as well as the fine structure constant are linear in $\hbar$. The connection between the number of loops and factors of $\hbar$ is more subtle for bound states since the binding energies and bound-state momenta themselves scale with $\hbar$. The $\hbar$ expansion allows one to identify equal-time relativistic bound states in QED and QCD which are of lowest order in $\hbar$ and transform dynamically under Lorentz boosts. The possibility to use retarded propagators at the Born level gives valence-like wave-functions which implicitly describe the sea constituents of the bound states normally present in its Fock state representation.

PACS numbers: 11.15.Tk

I. INTRODUCTION

Planck's constant $\hbar = h/2\pi$ is the fundamental constant of nature related to quantum effects [1]. The most familiar application of the Planck constant is the commutation relation $[x_i, p_j] = i\hbar\delta_{ij}$ which limits simultaneous measurements of position and momentum. More generally, $\hbar$ enters explicitly in the commutation relations of conjugate operators and fields, thus providing the fundamental basis of quantum field theory. Each order of an expansion in $\hbar$ must obey all the underlying symmetries of the theory.

The Planck constant has units of action; *i.e.*, the product of energy and length in units where the velocity of light $c = 1$. One commonly assumes units such that $\hbar = 1$. However, it is illuminating to keep the occurrence of powers of $\hbar$ explicit since this allows one to distinguish quantum versus classical physics. For example, the AdS/CFT prediction for the ratio of shear viscosity to the entropy density of a multi-particle system has the lower limit [2] $\eta/s \geq \hbar/4\pi$. The origin of $\hbar$ in this relation can be traced to the assumed quantum form of the entropy of a black hole in a higher dimensional theory. The empirical observation that η/s in heavy ion collisions at RHIC [3] is not far from the AdS/CFT prediction thus suggests that the dynamics of high energy central heavy-ion collisions is in the quantum domain.

Physical phenomena are usually expected to follow the laws of classical theory in the (hypothetical) limit $\hbar \to 0$. Surprisingly, this is true only for a careful choice of $\hbar$-independent quantities (momenta, couplings and masses). A simple illustration is provided by the standard harmonic oscillator in nonrelativistic quantum mechanics where the potential is $V(x) = \frac{1}{2}m\omega^2 x^2$. The propagation of a particle from (t_i, x_i) to (t_f, x_f) is given by the path integral

$$\mathcal{A}(x_i, x_f; t_f - t_i) = \int [\mathcal{D}x(t)] \exp\left[\frac{im}{2\hbar}\int_{t_i}^{t_f} dt(\dot{x}^2 - \omega^2 x^2)\right] = \int [\mathcal{D}\xi(t)] \exp\left[\frac{im}{2}\int_{t_i}^{t_f} dt(\dot{\xi}^2 - \omega^2\xi^2)\right] \tag{1}$$

In the second equality we have removed the explicit dependence on $\hbar$ by scaling the coordinates as $\xi \equiv x/\sqrt{\hbar}$ (the scaling of the Jacobian is irrelevant for this discussion.). Remarkably, the full quantum mechanical structure of the harmonic oscillator model persists as $\hbar \to 0$ when one uses the scaled variables ξ, since the propagation $\xi_i \to \xi_f$ is independent of $\hbar$, as are the scaled bound-state energies $\epsilon_n \equiv E_n/\hbar = \omega(n + \frac{1}{2})$. Thus there is a domain of positions $x \propto \sqrt{\hbar}$ and momenta $m\dot{x} \propto \sqrt{\hbar}$ where the action S is proportional to $\hbar$ and the system stays quantum mechanical

Work supported in part by US Department of Energy contract DE-AC02-76SF00515.

even in the $\hbar \to 0$ limit. On the other hand, the propagation between fixed ($\hbar$-independent) positions x_i, x_f involves large values of $\xi_i, \xi_f \propto 1/\sqrt{\hbar}$, and thus the transitions between highly excited levels (with n of order $1/\hbar$) correspond to classical dynamics in the $\hbar \to 0$ limit.

In the case of general relativity, $\hbar$ can be eliminated [4] from the equations of motion, and physical phenomena only depend on dimensionless quantities such as α_{QED}. Although mass cancels out of the equations of motion in classical gravity (due to the equivalence principle), it appears in the Schrödinger equation [5], but always in the form $\tilde{m} = m/\hbar$. As we shall show, $\tilde{m}$ is the fundamental mass parameter which appears in the equations of motion for fields when one formulates quantum field theory through the action principle and functional integrals.

In quantum field theory the T-matrix elements of lowest order in $\hbar$ are usually considered to be tree diagrams (Born approximation), with each loop correction introducing one additional power of $\hbar$ [6]. However, Donoghue and Holstein and their collaborators [7] have demonstrated that classical physics (of lowest order in $\hbar$) can also emerge from loop diagrams where zero mass quanta appear. As we shall see, the difference arises from the definition of the $\hbar \to 0$ limit, *i.e.*, the limit depends on which Lagrangian parameters are taken to be independent $\hbar$.

Born diagrams usually provide a good first approximation to scattering amplitudes in quantum field theory. However, the bound-state poles of a scattering amplitude are not present in tree (or any finite number of loop) diagrams, but instead are generated by the divergent perturbative expansion of the covariant Green's function. For example, the Schrödinger and Dirac bound states, which arise from tree-level interactions of an electron in an external Coulomb potential, emerge in field theory from the infinite sum of ladder and crossed-ladder electron-muon Feynman diagram contributions to the electron-muon Green's function in the limit where the muon mass is taken to infinity [8]. The binding is caused by loop momenta which are $\propto \hbar$, thus changing the relation between the number of loops and the power of $\hbar$.

We shall argue that the $\hbar$ expansion can provide a systematic approximation scheme for bound states formed by interactions between particles. This expansion is equally valid for relativistic and non-relativistic dynamics. We find that the lowest order interaction kernel in $\hbar$ indeed defines a viable "Born term for bound states". This Born approximation is insensitive to the $i\varepsilon$ prescription of propagators, which allows a simple Hamiltonian equal-time development in cases where the Coulomb interaction dominates. A hidden Lorentz boost covariance provides a non-trivial test that the approximation correctly includes all contributions of lowest order in $\hbar$.

II. THE $\hbar$ EXPANSIONS

We shall take units where the speed of light $c = \epsilon_0 = 1$. The dimension of Planck's constant $\hbar$ then has dimensions of action: energy E times length L; i.e, $[\hbar] = E \cdot L$. The dimension of the fermion and boson fields are fixed by their commutation and anti-commutation relations. For example, [1]

$$\left\{\psi^\dagger(t,\boldsymbol{x}), \psi(t,\boldsymbol{y})\right\} = \hbar\, \delta^3(\boldsymbol{x}-\boldsymbol{y}) \tag{2}$$

fixes the dimensions $[\psi] = E^{1/2} \cdot L^{-1}$.

One obtains the same dimensions of the fields directly from the action. In the functional integral formulation of quantum field theory the action is normalized by the Planck constant. In QED,

$$\frac{1}{\hbar}\mathcal{S}_{QED} = \frac{1}{\hbar}\int d^4x \mathcal{L}(x) = \frac{1}{\hbar}\int d^4x \left[\bar{\psi}(i\partial\!\!\!/ - \tilde{e}A\!\!\!/ - \tilde{m})\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \tag{3}$$

is dimensionless given (2). Furthermore, $[A] = E^{1/2} \cdot L^{-1/2}$, $[\tilde{m}] = 1/L$ and $[\tilde{e}] = E^{-1/2} \cdot L^{-1/2}$. Similarly in QCD,

$$-\frac{1}{4\hbar}\int d^4x \,(G^{\mu\nu})^2 = -\frac{1}{4\hbar}\int d^4x \left(\partial^\mu A^\nu - \partial^\nu A^\mu + i\tilde{g}\,[A^\mu, A^\nu]\right)^2 \tag{4}$$

is dimensionless with $[\tilde{g}] = E^{-1/2} \cdot L^{-1/2}$.

Remarkably, the coupling constants $\tilde{e}$, $\tilde{g}$ appearing in the Lagrangian density $\mathcal{L}$ and the action $\mathcal{S}$ in Eqs. (3) and (4) have different dimensions than their classical counterparts $e = \tilde{e}\hbar$ and $g = \tilde{g}\hbar$. The dimensionless fine structure

[1] Since ψ is an anti-commuting field there is no limit in which it would approach a classical field. Thus it is also possible to define its anti-commutation relation without the factor of $\hbar$ [9]. For scalar and gluon fields (see below) the corresponding convention for the commutators of fields would be unconventional. In view of gauge invariance and supersymmetry, we will also retain the factor of $\hbar$ for fermions.

constant is thus

$$\alpha \equiv \frac{e^2}{4\pi\hbar} = \frac{\tilde{e}^2\hbar}{4\pi} \tag{5}$$

Similarly the mass $\tilde{m}$ appearing in the Lagrangian density has the dimension of wave number, whereas the usual classical mass expressed in units of energy is $m = \tilde{m}\hbar$. The distinction between $\tilde{e},\ \tilde{g},\ \tilde{m}$ and their classical counterparts $e,\ g,\ m$ is irrelevant in units where $\hbar = 1$, but it is important if one expands theoretical results in powers of $\hbar$. Since classical physics such as Maxwell's equations is expressed in terms of e and m it is natural to regard those quantities as independent of $\hbar$ in studies of how classical physics emerges from quantum field theory in the $\hbar \to 0$ limit [7]. In this case the QED action S_{QED} in (3) depends on $\hbar$ through $\tilde{e} = e/\hbar$ and $\tilde{m} = m/\hbar$ and loops are found to contribute even in the classical limit.

In standard derivations [6] of the relation between the power of $\hbar$ and the number of loops, one associates a factor of $\hbar$ with each propagator and a factor $1/\hbar$ with each vertex. This counting is consistent with the actions in (3) and (4) *provided* $\tilde{e},\ \tilde{g}$ *and* $\tilde{m}$ *are independent of* $\hbar$. Then each loop is found to add a power of $\hbar$. It is illuminating to re-derive this result by rescaling the fields in analogy with the harmonic oscillator (1). Thus we define

$$\tilde{\psi} \equiv \psi/\sqrt{\hbar}, \quad \tilde{A} \equiv A/\sqrt{\hbar}. \tag{6}$$

We then have

$$\mathcal{S}_{QED}/\hbar = \int d^4x \Big[\bar{\tilde{\psi}}(i\partial\!\!\!/ - \tilde{e}\sqrt{\hbar}\tilde{A\!\!\!/} - \tilde{m})\tilde{\psi} - \tfrac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\Big] \tag{7}$$

where $\tilde{F}^{\mu\nu} = \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu$. Now all of the dependence on $\hbar$ in the QED generating functional

$$Z = \int [\mathcal{D}\bar{\tilde{\psi}}][\mathcal{D}\tilde{\psi}][\mathcal{D}\tilde{A}] \exp\Big\{i\int d^4x \Big[\bar{\tilde{\psi}}(i\partial\!\!\!/ - \tilde{e}\sqrt{\hbar}\tilde{A\!\!\!/} - \tilde{m})\tilde{\psi} - \tfrac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\Big]\Big\} \tag{8}$$

appears only through the coupling $\tilde{e}\sqrt{\hbar}$ of the rescaled Lagrangian density:

$$\tilde{\mathcal{L}} = \bar{\tilde{\psi}}(i\partial\!\!\!/ - \tilde{e}\sqrt{\hbar}\tilde{A\!\!\!/} - \tilde{m})\psi - \tfrac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \tag{9}$$

Each increasing order in $\tilde{e}^2$ of the perturbative expansion brings one power of $\hbar$ through $\alpha = \tilde{e}^2\hbar/4\pi$. Feynman diagrams with n loops are then of order $\alpha^n \propto \hbar^n$ (compared to the Born diagrams). Thus the perturbative (loop) expansion is equivalent to an expansion in $\hbar$. In dimensionless ratios of momenta and masses one may use units of either wave number or energy.

We can perform an analogous rescaling in QCD. For instance, for the gluon field strength we have

$$\tilde{G}^{\mu\nu} = G^{\mu\nu}/\sqrt{\hbar} = \partial^\mu \tilde{A}^\nu - \partial^\nu \tilde{A}^\mu - \tilde{g}\sqrt{\hbar}\,[\tilde{A}^\mu, \tilde{A}^\nu] \tag{10}$$

where $\tilde{g}$ is independent of $\hbar$. Again the $\hbar$ dependence of the Dirac, 3-gluon and 4-gluon couplings causes the perturbative (loop) expansion in $\alpha_s = \tilde{g}^2\hbar/4\pi$ to be equivalent to an expansion in $\hbar$.

III. APPLICATION OF THE $\hbar$ EXPANSION TO BOUND STATES

Bound states in quantum field theory can be identified from the divergence of the perturbative series of the T-matrix at each bound-state energy. In the familiar case of non-relativistic QED atoms, the divergence arises from ladder diagrams with loop momenta which scale with α. The binding energy of Hydrogen thus scales as $E_n/m_e = -\frac{1}{2}\alpha^2/n^2 \propto \hbar^2$. Hence an $\hbar$ expansion at bound-state poles requires that the external momenta in the T-matrix scale with $\hbar$. In such a limit the usual relation between the number of loops and power of $\hbar$ need not hold. The ladder diagrams are in fact sensitive to infrared loop momenta that contribute inverse powers of $\hbar$. The lowest-order contribution in α to the Hydrogen atom is also of lowest order in $\hbar$, *i.e.*, it is a "Born term" for the bound state. The loop momenta do not scale with $\hbar$ in the higher-order corrections to propagators and vertices, implying the usual relation between the number of such loops and the power of $\hbar$.

The bound-state poles of non-relativistic QED atoms are obtained by summing ladder diagrams, the first two of which are shown in Fig. 1. The tree diagram in (a) generally dominates the loop diagram (b) at small coupling strength α. However, for external momenta within the range of the atomic wave function the loop diagram is unsuppressed even in the $\alpha \to 0$ limit. In the CM system this requires $|\boldsymbol{p}_1| = |\boldsymbol{p}_2|$ to be of $\mathcal{O}(\alpha m)$ (m being the mass of the

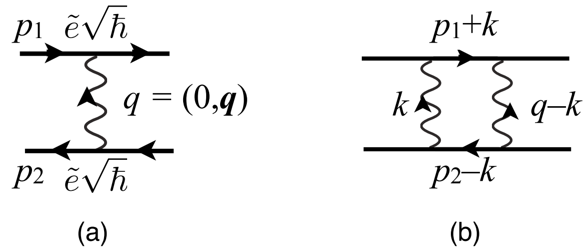

FIG. 1: The first two ladder diagrams contributing to non-relativistic atoms in the limit of small coupling α.

constituents) and $|p_i^0 - m|$ to be of $\mathcal{O}\left(\alpha^2 m\right)$. The restricted range of the loop momentum, $d^4k \propto \alpha^5$ in the bound-state domain, as well as the additional factor of α in the loop diagram, is balanced (relative to the tree diagram) by its three additional propagators, each of which is of $\mathcal{O}\left(1/\alpha^2\right)$. The situation repeats for all of the higher-order, multi-loop ladder diagrams, allowing the ladder series to diverge and bound-state poles to occur at any value of α.

Since $\alpha = \tilde{e}^2\hbar/4\pi \propto \hbar$ the previous argument demonstrates that the loop diagram in Fig. 1(b) is, in the bound-state domain, of the same order also in $\hbar$ as the tree diagram of Fig. 1(a). The loop does not bring an extra factor of $\hbar$ since we consider a limit where the (external and internal) momenta depend on $\hbar$, similarly to the case of the harmonic oscillator (1). Repeating the argument for ladder diagrams with more rungs, we may conclude that the Schrödinger equation defines the Born term for non-relativistic atoms. The standard higher order corrections in α to the binding energies and wave functions are also of higher order in $\hbar$.

The energy $k^0 \propto \alpha^2 m$ exchanged in the ladder loops is of higher order in $\hbar$ than the 3-momentum $|\boldsymbol{k}| \propto \alpha m$. Thus in the Born approximation we may set $k^0 = 0$ in the photon propagators. This makes the non-relativistic bound-state dynamics equivalent to tree-level scattering from an external, instantaneous potential.

The relativistic Dirac-Coulomb electron bound-states are obtained by summing $e\mu$ scattering diagrams which include all ladder and crossed-ladder photon exchanges [8]. In the limit where the muon mass is large, the sum of crossed and uncrossed photon exchanges gives a $\delta(k^0)$ which suppresses energy exchange and reduces the dynamics to tree-level scattering. In this sense the relativistic Dirac-Coulomb bound states are also of lowest order in $\hbar$.

IV. TIME EVOLUTION AT BORN LEVEL

The $i\varepsilon$ prescription in propagators is related to the boundary condition in time and is irrelevant at lowest order in $\hbar$. Heuristically, this is seen from the damping factors $\exp(\pm t\delta)$ which give energy denominators $E_{initial} - E_{intermediate} \pm i\epsilon$ where $\epsilon = \delta\hbar$. In the perturbative expansion tree diagrams are in fact independent of $i\varepsilon$ (all internal lines being off-shell by definition). Conversely, the $i\varepsilon$ prescription determines the discontinuities of diagrams which are of higher order in $\hbar$ through the pinching of loop integrals. For Born level bound states this prescription independence allows alternative but equivalent wave functions as we shall next discuss.

Consider first the Dirac-Coulomb bound states in a static external potential. In four-momentum space the interactions with the Coulomb potential $A^0(\boldsymbol{k})$ do not change the energy component p^0 of the particle's momentum. Denoting a single Coulomb photon interaction by K the Green's function G of the particle can be expanded as

$$G(p^0,\boldsymbol{p}) = S + SKS + SKSKS + \ldots = S + SKG = \frac{R(E,\boldsymbol{p})}{p^0 - E} + \ldots \tag{11}$$

In the last equality we displayed the pole contribution of a bound state with energy E, whose residue $R(E,\boldsymbol{p})$ is easily seen to satisfy the Dirac equation. Since K preserves p^0 the Green's function G is independent of the $i\varepsilon$ prescription at the negative-energy pole of the Dirac propagator $S(p)$. In other words, for $p^0 > 0$ we have $p^0 + E_p > 0$, where $E_p \equiv \sqrt{\boldsymbol{p}^2 + m^2}$. In particular, the bound-state energy E is the same whether we use a Feynman S_F or retarded S_R propagator,

$$S_{F/R}(p^0,\boldsymbol{p}) \equiv i\frac{\not{p} + m}{(p^0 - E_p + i\varepsilon)(p^0 + E_p \mp i\varepsilon)} \tag{12}$$

If we time-order the interactions through a Fourier transform $p^0 \to t$ the bound state gives a stationary contribution $G(t,\boldsymbol{p}) = \exp(-iEt)R(E,\boldsymbol{p}) + \ldots$ However, the Fock state decomposition of its equal-time wave function depends on the choice of propagator (12). In the Feynman propagator $S_F(t,\boldsymbol{p})$ the negative energy components move backward in time. This gives rise to "Z"-diagrams describing particle-antiparticle pair fluctuations in the time-ordered interactions with the static potential. The equal-time wave function of a Dirac-Coulomb bound state thus has Fock components with any number of pairs, and its explicit expression is, to the best of our knowledge, not known even in simple cases such as a $1/r$ potential.

In the case of the retarded propagator S_R (12), the negative-energy components move forward in time,

$$S_R(t,\boldsymbol{p}) = \frac{\theta(t)}{2E_p}\left[(E_p\gamma^0 - \boldsymbol{p}\cdot\boldsymbol{\gamma} + m)e^{-iE_pt} + (E_p\gamma^0 + \boldsymbol{p}\cdot\boldsymbol{\gamma} - m)e^{iE_pt}\right] \tag{13}$$

The corresponding time-ordered interactions have no Z-contributions, thus only the single bound particle is present at any intermediate time. As a consequence of this, the retarded propagator is local in space at infinitesimal times,

$$\lim_{t\to 0^+} S_R(t,\boldsymbol{x}) = \gamma^0\delta^3(\boldsymbol{x}) \tag{14}$$

By contrast, the right-hand side would be non-local in $\boldsymbol{x}$ for the Feynman propagator S_F. This may be understood by invoking a completeness sum over states at an earlier time: the particle first moves backward in time and then returns to $\boldsymbol{x}' \neq \boldsymbol{x}$ at $t = 0$.

The absence of pair production in retarded propagation as well as the locality property (14) allows a one-particle Hamiltonian description as in relativistic quantum mechanics [10]. The bound-state wave functions are then given by the Dirac equation. These single-particle wave functions describe the same bound states, which using Feynman propagators, contain an indefinite number of particle pairs arising from Z-diagrams. As we have seen from (11), the bound-state energies E of both pictures agree at lowest order in $\hbar$.

V. RELATIVISTIC BOUND STATES IN FIELD THEORY

The Hamiltonian description of relativistic bound states in an external Coulomb potential can in certain cases be extended to field theory [10]. According to the above arguments the binding energies will at Born level be independent of the $i\varepsilon$ prescription used in the bound particle propagators. This allows the use of the retarded propagator S_R (13) which suppresses Z-diagrams and makes the time evolution local as in (14). Coulomb exchange gives an instantaneous potential due to the absence of $\partial_0 A^0$ terms in the Lagrangian density. Transversely polarized photons propagate in time and thus they generate higher Fock states. Here we only consider cases where Coulomb exchange is dominant.

In $D = 1+1$ dimensions only Coulomb exchange contributes in a gauge where $A^1 = 0$. For an $e^-\mu^+$ state where the electron is at x_1 and the muon at x_2

$$|E, t=0\rangle = \int dx_1 dx_2\, \psi_e^\dagger(t=0,x_1)\chi(x_1,x_2)\psi_\mu(t=0,x_2)|0\rangle \tag{15}$$

the equation of motion (Gauss' law),

$$-\partial_x^2 A^0(x;x_1,x_2) = e\left[\delta(x-x_1) - \delta(x-x_2)\right] \tag{16}$$

determines the instantaneous Coulomb field,

$$-A^0(x;x_1,x_2) = \tfrac{1}{2}e\big[\,|x-x_1| - |x-x_2|\,\big] \tag{17}$$

It is then straightforward [10] to determine the condition on the bound-state wave function $\chi(x_1,x_2)$ which ensures a stationary time development,

$$\gamma^0(-i\overrightarrow{\partial}_{x_1}\gamma^1 + m_e)\chi(x_1,x_2) - \chi(x_1,x_2)\gamma^0(i\overleftarrow{\partial}_{x_2}\gamma^1 + m_\mu) = [E - V(x_1-x_2)]\chi(x_1,x_2) \tag{18}$$

Here the kinetic term of the electron operates on $\chi(x_1,x_2)$ from the left, and that of the muon from the right. The potential following from (17) is

$$V(x) = \tfrac{1}{2}e^2|x| \tag{19}$$

In $D = 1+1$ dimensions the Dirac matrices as well as the wave function $\chi(x_1,x_2)$ may be taken to be 2×2 matrices.

Not surprisingly, the bound-state equation (18) has a "double Dirac" form, and as such was proposed already by Breit [11]. Now this equation is seen to provide an approximation of lowest order in $\hbar$ to relativistic bound states, with the potential (19) in 1+1 dimensions fixed by QED.

A stringent test that (18) actually represents the exact Born level result is that it is consistent with the Poincaré invariance of QED. Translational invariance is explicit, but the boost invariance is hidden (dynamic), since the constituents are at equal time in all frames. It turns out [12] that for the linear potential (19) (and for no other form of the potential) the bound-state energy indeed has the correct dependence on the CM momentum P of the bound state,

$$E = \sqrt{P^2 + M^2} \tag{20}$$

The P-dependence of the wave function $\chi(x_1, x_2)$ is non-trivial. It resembles an ordinary Lorentz transformation, but with $E \to E - V(x)$ (the canonical energy) in the boost parameter. Thus the wave function contracts at a rate which depends on the separation x between the constituents.

The properties of the relativistic bound states defined by (18) merit further study. At large separations between the constituents, where $V(x) \gg E$, the wave function has a constant, non-vanishing density in x. This may reflect the particle pairs which are polarized from the vacuum in a formulation using Feynman propagators.

The two very different but equivalent pictures of bound states that we find here using retarded *vs.* Feynman time development may be related to the well-known puzzle of the quark model *vs.* parton model views of real hadrons: Hadron excitation spectra reflect mainly their valence quark degrees of freedom, even though sea quarks contribute importantly to deep inelastic scattering at low x_{Bj} and low Q^2 [13].

A bound-state equation analogous to (18) may be derived in $D = 3 + 1$ dimensions by assuming a non-trivial boundary condition in Gauss' law (16). The homogenous solution for A^0 then gives rise to a linear potential. Poincaré invariance is respected similarly to the 1+1-dimensional case. It is also possible to derive the analogous Born level meson and baryon bound states in QCD [10]. The qqq potential is gauge covariant and confines the three quarks in a symmetric way. In some respects this resembles the soft-wall AdS/QCD models [14, 15] which utilize a linear potential in an effective Dirac equation in AdS space.

VI. CONCLUSIONS

The Planck constant $\hbar$ is a fundamental constant of nature which gives a measure of quantum effects and appears as a parameter in quantum field theories. It is common to set $\hbar = 1$ since there is a general belief that the power of $\hbar$ is given by the number of loops in the perturbative expansion [6]. In the functional integral $\hbar$ appears only in the integrand $\exp(i\mathcal{S}/\hbar)$, where the action $\mathcal{S}$ is usually assumed to be independent of $\hbar$. It is then argued that classical physics emerges in the $\hbar \to 0$ limit since the rapidly varying phase $\mathcal{S}/\hbar$ selects field configurations for which the action $\mathcal{S}$ is stationary. This argument is, however, somewhat oversimplified since there may be field configurations which make $\mathcal{S} \propto \hbar$. We have illustrated this with the harmonic oscillator, whose full quantum mechanical bound-state spectrum persists in the $\hbar \to 0$ limit.

Conversely, it has been demonstrated through explicit examples that loops containing massless quanta, such as the gravitational form factors of the electron, can contribute to classical physics in the $\hbar \to 0$ limit [7]. The underlying reason is that an $\hbar$ expansion is not uniquely defined without specifying which quantities (momenta, couplings and masses) are to be regarded as independent of $\hbar$. We have noted that the charge $\tilde{e}$ appearing in the QED action (3) has a dimension different from the classical charge, $e = \tilde{e}\hbar$. Similarly, the mass $\tilde{m}$ in the action is related to the mass m with units of energy as $m = \tilde{m}\hbar$.

It is natural to fix the classical quantities e and m in order to obtain the standard classical limit as is done in [7], although this implies that the free and interacting components of the action have a different dependence on $\hbar$. We also note that the fine structure constant $\alpha = e^2/4\pi\hbar c$ diverges in the $\hbar \to 0$ limit if the classical coupling e is held constant. If on the other hand the parameters of the Lagrangian density $\mathcal{L}$, $\tilde{e}$ and $\tilde{m}$ are taken to be independent of $\hbar$, the fine structure constant $\alpha = \tilde{e}^2\hbar/4\pi \propto \hbar$ and the $\hbar$ expansion is equivalent to the perturbative expansion. For example, in the $\hbar \to 0$ limit, the loops which define the β function vanish, so that the running coupling $\tilde{e}(\mu^2)$ is constant as in a conformal theory.

The $\hbar$ expansion is particularly non-trivial and illuminating in the case of bound states. We have considered whether one can define a Born term for bound states, which is equivalent to a tree (no loop) approximation when $\tilde{e}$ and $\tilde{m}$ are fixed. This would allow an unambiguous and physically motivated approximation scheme for relativistic bound states, maintaining all symmetries of the theory at each order in $\hbar$. In the familiar case of non-relativistic QED atoms the binding energy $E_{bind} \propto \alpha^2 \propto \hbar^2$ depends on $\hbar$. In order to stay on the bound state pole of a Green's function in the $\hbar \to 0$ limit we must allow the momenta to scale with $\hbar$, which introduces additional sources of $\hbar$. This is why the sum of multi-loop ladder diagrams in perturbation theory reduces to a Born approximation for atomic states.

Born terms are insensitive to the $i\varepsilon$ prescription of perturbative propagators. This is explicitly seen for tree diagrams in perturbation theory, and we have shown that it holds also for bound states in a Coulomb potential.

Using Feynman or retarded propagators does, however, make a consequential difference for the wave functions of the equal-time bound states. Relativistic bound states given by the Dirac equation have an indefinite number of particle pair constituents in their Fock expansion which arise from Z-diagrams of the Feynman propagators. In contrast, if one chooses retarded propagators, the bound state appears to contain just a single particle (of positive or negative energy) whose distribution is given by the standard Dirac wave function.

Our understanding of bound states formed by the mutual interactions of relativistic particles is still very limited. This may be due to the lack of a physically well motivated and manageable first approximation akin to the tree diagrams of scattering amplitudes. In this paper we have shown that there is a well-defined Born approximation for relativistic bound states. With retarded propagation one obtains simple and specific bound-state equations. The two-body equations in QED and QCD have a hidden boost invariance which strongly suggests that they include all effects of lowest order in $\hbar$. Just as for the Dirac equation, the resulting valence wave functions implicitly describe the multiple pair constituents generated by Feynman propagation.

Acknowledgments

We thank J. F. Donoghue and B. R. Holstein for a critical reading of our manuscript and a helpful discussion concerning the definition of $\hbar$ expansions. We have also benefitted from discussions with C. Carlson and S. Peigné.

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