

Large- N volume independence in conformal and confining gauge theories

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ABSTRACT: Consequences of large N volume independence are examined in conformal and confining gauge theories. In the large N limit, gauge theories compactified on $\mathbb{R}^{d-k} \times (S^1)^k$ are independent of the S^1 radii, provided the theory has unbroken center symmetry. In particular, this implies that a large N gauge theory which, on \mathbb{R}^d , flows to an IR fixed point, retains the infinite correlation length and other scale invariant properties of the decompactified theory even when compactified on $\mathbb{R}^{d-k} \times (S^1)^k$. In other words, finite volume effects are $1/N$ suppressed. In lattice formulations of vector-like theories, this implies that numerical studies to determine the boundary between confined and conformal phases may be performed on one-site lattice models. In $\mathcal{N}=4$ supersymmetric Yang-Mills theory, the center symmetry realization is a matter of choice: the theory on $\mathbb{R}^{4-k} \times (S^1)^k$ has a moduli space which contains points with all possible realizations of center symmetry. Large N QCD with massive adjoint fermions and one or two compactified dimensions has a rich phase structure with an infinite number of phase transitions coalescing in the zero radius limit.

KEYWORDS: $1/N$ expansion, lattice QCD, nonperturbative effects.

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1. Introduction

Wide classes of large N gauge theories, when studied on toroidal compactifications of \mathbb{R}^d , have properties which are independent of the compactification radii. For brevity, we will refer to independence on compactification radii as *volume independence*.¹ Examples (and counterexamples) of large N volume independence have been discussed since the 1980's [1, 2, 3, 4, 5], but there has been a recent resurgence of interest in the subject.

In $SU(N)$ gauge theories on $\mathbb{R}^{d-k} \times (S^1)^k$, with only adjoint representation matter fields, large N volume independence holds provided two symmetry realization conditions are satisfied as $N \rightarrow \infty$ [6]:

- Translation symmetry is not spontaneously broken. (1a)

- The $(\mathbb{Z}_N)^k$ center symmetry is not spontaneously broken.² (1b)

¹Our discussion applies to compactifications on $\mathbb{R}^{d-k} \times (S^1)^k$ where k , the number of compactified dimensions, may range from 1 to d . If $k < d$, so that some dimensions remain uncompactified, then it is an abuse of language to refer to independence on the compactification radii as volume independence. We hope that our use of this name, which originated in discussions of compactifications of \mathbb{R}^d to $(S^1)^d$ with cubic symmetry, does not cause confusion. In lattice gauge theories, large N equivalence between the decompactified theory and a single site model is sometimes referred to as “large N reduction” or “Eguchi-Kawai (EK) reduction”.

²Center symmetry transformations are gauge transformations which are periodic only up to an element of the center of the gauge group. For $SU(N)$ theories on $\mathbb{R}^{d-k} \times (S^1)^k$, the group of such transformations, modulo gauge transformations continuously connected to the identity, is $(\mathbb{Z}_N)^k$. Center transformations associated with a particular toroidal cycle multiply Wilson loops by a phase factor z^n where $z \in \mathbb{Z}_N$ and n is the winding number of the loop around the cycle. Hence, topologically non-trivial Wilson loops serve as order parameters for center symmetry. \mathbb{Z}_N center transformations are symmetries of $SU(N)$ gauge theories provided all matter fields are in representations (such as the adjoint) with vanishing N -ality. To simplify the presentation, we limit our discussion to this class of theories. However, it should be noted that there are additional large N equivalences which relate $SO(N)$, $Sp(N)$ and $SU(N)$ gauge theories, and which relate theories with matter fields in rank-2 symmetric, antisymmetric and bifundamental representations to theories with adjoint representation matter [7, 8, 9, 10].

Volume independence applies to the leading large N behavior of expectation values and connected correlators of topologically trivial Wilson loops and similar single-trace observables. See Ref. [6] for more detail.

In $SU(N)$ Yang-Mills theory, large N volume independence holds as long as all compactification radii are larger than a critical radius $L_c \sim \Lambda^{-1}$ [11]. Volume independence fails below this critical radius due to a center-symmetry breaking phase transition. In the case of a single compactified dimension, this is the usual confinement/deconfinement thermal phase transition. Two approaches for modifying a lattice gauge theory in order to suppress this center-symmetry breaking phase transition, and restore the validity of volume independence down to arbitrarily small radius (or down to a single site in a lattice regulated theory), are known.³ One may add double trace deformations to the action involving absolute squares of Wilson loops wrapping the compactified directions [17]. Such deformations can suppress spontaneous breaking of center symmetry while leaving the large N dynamics in the center-symmetric sector of the theory completely unaffected. Alternatively, one can add one or more massless adjoint representation fermions, with periodic (not antiperiodic) boundary conditions [6]. The fermions modify the effective potential for topologically non-trivial Wilson loops in a manner which prevents the breaking of center symmetry even for arbitrarily small compactification radii.

In this work we discuss the physical basis for large N volume independence and examine its implications in both confining and conformal gauge theories. We highlight the presence of nonuniformities when the gauge group rank $N \rightarrow \infty$ and compactification size $L \rightarrow 0$. For simplicity of presentation, much of our discussion will focus on the case of a single compactified dimension. If center symmetry is unbroken on $\mathbb{R}^{d-1} \times S^1$, we show that the long distance physics for finite values of N is sensitive to L only via the combination NL . Specific consequences we discuss include the following:

1. For asymptotically free theories with a strong scale Λ , the compactified theory differs negligibly from the decompactified theory provided $NLA \gg 1$. Conversely, a dimensionally reduced long distance effective theory characterizes the long distance dynamics only when $NLA \ll 1$.
2. For theories which, on \mathbb{R}^d , have scale-invariant long distance dynamics, compactification on $\mathbb{R}^{d-1} \times S^1$ leads to a correlation length of order NL ; a factor of N times longer than would be naively expected. Maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ SYM) is a special case where the correlation length of the compactified theory remains infinite and all realizations of center symmetry coexist.
3. For QCD with n_f massless adjoint fermions, the lower boundary of the “conformal window” (*i.e.*, the minimal number of flavors n_f^* for which the theory flows to a non-trivial IR fixed point) may, in the large N limit, be determined by studying the compactified theory with arbitrarily small radius.
4. For QCD with massive adjoint fermions, the symmetry realization and phase structure of the theory, compactified on $\mathbb{R}^3 \times S^1$, sensitively depends on the value of NLm .

In a final section, we discuss the situation with multiple compactified dimensions. There are several new issues in this case, but most of the $\mathbb{R}^{d-1} \times S^1$ analysis generalizes in a straightforward fashion.

³Earlier schemes known as quenched Eguchi-Kawai reduction [3] and twisted Eguchi-Kawai [4, 5] have recently been shown to fail due to nonperturbative effects [12, 13, 14, 15]. However, see Ref. [16] for a recent proposed fix for twisted reduction.

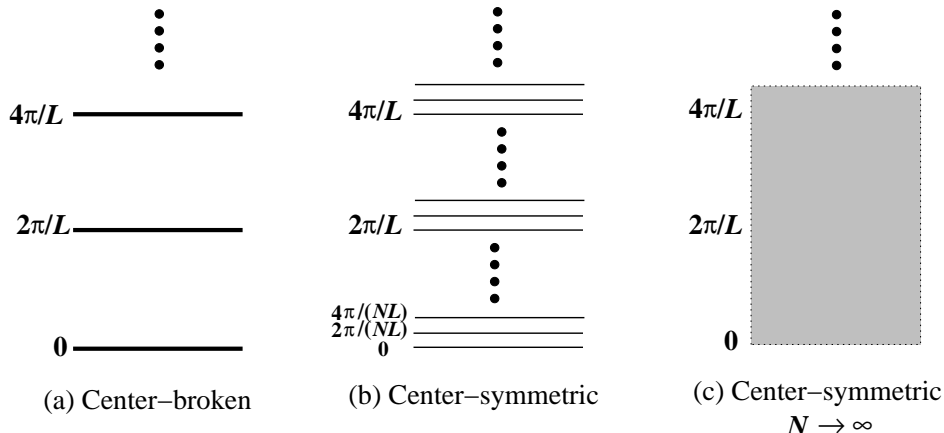


Figure 1: Dependence of the Kaluza-Klein spectrum on the realization of center symmetry. (a) Maximally broken center symmetry with holonomy $\Omega = 1$ gives the usual $2\pi/L$ level spacing. (b) Unbroken center symmetry produces a finer $2\pi/(NL)$ level spacing. (c) Sending $N \rightarrow \infty$ with unbroken center symmetry leads to continuous spectra.

2. Compactification and center symmetry

Consider an asymptotically free $SU(N)$ gauge theory on $\mathbb{R}^3 \times S^1$. Let Λ denote the strong scale of the theory, and let Ω denote the holonomy of the gauge field around the compactified direction. [In other words, $\Omega(\mathbf{x})$ is the path-ordered exponential of the line integral of the gauge field around the S^1 at spatial position \mathbf{x} .] We assume that the theory has a \mathbb{Z}_N center symmetry, so our discussion applies to QCD with adjoint representation fermions [denoted QCD(adj)],⁴ provided the number of flavors n_f is below the asymptotic freedom limit, $n_f < n_f^{\text{AF}} \equiv 5.5$. A center symmetry transformation multiplies the holonomy by a phase factor, $\Omega \rightarrow z\Omega$ with $z \in \mathbb{Z}_N$. Hence, Wilson line expectations $\langle \text{tr } \Omega^k \rangle$, for any non-zero $k \bmod N$, are order parameters for the realization of center symmetry.

Compactification produces Kaluza-Klein (KK) towers — discrete frequency spectra of field modes. The form of the KK spectra is critically dependent on the realization of center symmetry, as illustrated in Figure 1. Fig. 1a shows the “standard” form of Kaluza-Klein towers, in which allowed momenta are spaced at integer (or half-integer) multiples of $2\pi/L$. For adjoint representation fields, each level has an $O(N^2)$ degeneracy. This is the situation when the holonomy $\Omega = 1$ and the center symmetry is completely broken. For physics sensitive to energies small compared to the inverse compactification radius,

$$E \ll \frac{1}{L}, \quad (2.1)$$

⁴The results of this discussion also apply to QCD-like theories with fermions in symmetric or antisymmetric rank-two tensor representations, or ordinary QCD with n_f fundamental representation fermions provided n_f is held fixed as $N \rightarrow \infty$. As noted in Ref. [6] and discussed more fully in Ref. [18], an “emergent” \mathbb{Z}_N center symmetry appears in the large N limit of theories with symmetric or antisymmetric tensor representation matter fields. For these theories, the addition of explicit double-trace center-symmetry stabilizing terms is needed to prevent center symmetry breaking at small L [17].

or length scales large compared to L , the non-zero frequency components of all fields may be integrated out, leading to a dimensionally reduced $3d$ effective theory describing long distance dynamics [19, 20, 21].

In contrast, when the center symmetry is unbroken the eigenvalues of Ω are (on average) evenly distributed around the unit circle. This follows from the vanishing of traces of Ω^k for all non-zero $k \bmod N$. Such a non-degenerate eigenvalue distribution of Ω produces a finer KK spectrum with spacing of $2\pi/(NL)$ and $O(N)$ degeneracies, as illustrated in Fig. 1b. To see this, recall that a non-trivial holonomy shifts the phase acquired by an excitation propagating around the S^1 , or equivalently shifts the frequency moding of field components. When the eigenvalues of Ω are non-degenerate and uniformly distributed, every field breaks up into N pieces with different offsets in the frequency quantization. The range of energies for which a dimensionally reduced effective description is valid is now

$$E \ll \frac{1}{NL}. \quad (2.2)$$

The resulting long distance effective theory, relevant only for distances large compared to NL , corresponds perturbatively to a $SU(N) \rightarrow U(1)^N$ Higgsing of the theory. The holonomy Ω acts as an effective $3d$ adjoint representation Higgs field which gives masses in the $[2\pi/NL, 2\pi/L)$ range to all off-diagonal field components. In addition to perturbative fluctuations, there are also non-perturbative topological defects, both self-dual monopoles and non-self-dual magnetic bions (or monopole-anti-monopole bound states) of various charges [22]. A semiclassical analysis (generalizing Polyakov's classic treatment [23] to gauge theory on $\mathbb{R}^3 \times S^1$) shows that these topological defects, even when arbitrarily dilute, can generate a mass gap and area law behavior for spatial Wilson loops, as discussed in detail for QCD(adj) in Ref. [22].

A key point is that integrating out the non-zero frequency modes perturbatively, and analyzing the monopole/bion dynamics using semiclassical methods, is only valid when the theory is weakly coupled on the scale of NL . In other words, the dimensionally reduced description of the long distance dynamics, with semi-classical Abelian confinement, is only valid when $NL\Lambda \ll 1$.

As $N \rightarrow \infty$, with unbroken center symmetry, the frequency spacing in Kaluza-Klein towers approaches zero and the spectrum approaches the continuous frequency spectrum of the decompactified theory, as illustrated in Fig. 1c. The domain of validity (2.2) of the $3d$ long distance description shrinks to zero energy (and diverging distances). Observables probing any fixed energy scale become unable to resolve the vanishing discreteness in the frequency spectrum. Compactification effects are $1/N$ suppressed and the leading large N behavior of expectations, or connected correlators, of single trace, topologically trivial observables is volume independent. For dynamics on the scale Λ , the effect of compactification is negligible when $NL\Lambda \gg 1$. In this regime, there is no weakly coupled description of the long distance physics.

Recognition of the connection between unbroken center symmetry and a $1/N$ suppressed spacing in the KK spectrum does not constitute a proof of large N volume independence. For that, one must compare the large N loop equations in lattice regularized theories [8], or the $N = \infty$ classical dynamics generated by appropriate coherent states [24]. But consideration of the KK spectrum does provide a simple physical understanding of the origin of large N volume independence, and clarifies the relevant scales which control the approach to the $N = \infty$ limit.

To reiterate, for theories on $\mathbb{R}^3 \times S^1$, when the center symmetry is not spontaneously broken the physically relevant length scale appearing in finite volume effects is not L , but rather NL . For confining theories with a strong scale Λ , there are two distinct characteristic regimes:

$$NL\Lambda \ll 1, \quad \text{semi-classical, Abelian confinement} \implies \text{volume dependence}; \quad (2.3)$$

$$NL\Lambda \gg 1, \quad \text{non-Abelian confinement} \implies \text{volume independence}. \quad (2.4)$$

3. Conformal theories and conformal windows

Interest in possible extensions to the standard model involving strongly coupled, nearly conformal sectors [25, 26, 27, 28, 29, 30] has stimulated substantial efforts to determine “conformal windows” in QCD-like gauge theories with varying fermion content [31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. Numerical simulations must, of course, work with UV and IR regulated theories, and finite volume toroidal compactifications are virtually always used as the IR regular of choice. As with all numerical simulations of lattice gauge theories, efforts to find the boundary between confining and conformal behavior must adequately control multiple sources of systematic error: finite volume effects, non-zero lattice spacing artifacts, and extrapolations to the chiral limit.

For theories with infinite correlation length (when defined on \mathbb{R}^d), compactification of a spatial dimension introduces a new length scale, the compactification size L . This modifies the behavior of fluctuations with wavelengths comparable or larger than L and typically produces a finite correlation length of order L . But for theories satisfying large N volume independence, for the reasons discussed in the previous section, the characteristic size of a compactification-induced correlation length is not the physical size L , but instead equals L times a positive power of N . This means that finite volume effects are $1/N$ suppressed as long as the center symmetry is unbroken. Even for simulations at modest values of N , this additional suppression of finite volume effects decreases the lattice size required to achieve acceptably small systematic errors. For larger values of N , it implies that very small lattices will be sufficient to extract infinite volume observables. The basic point we wish to stress is that large N volume independence is not restricted to confining phases; it is equally applicable to conformal phases provided the symmetry realization conditions (1) are satisfied.

As an example, consider QCD(adj) with massless fermions, and one dimension periodically compactified, as a function of the number of fermions n_f and compactification size L . Figure 2 illustrates the situation.⁵ In the decompactified limit, $L = \infty$, one expects a confining phase with spontaneously broken chiral symmetry for sufficiently small n_f , and a conformal phase (*i.e.*, a phase in which the theory flows to a non-trivial IR fixed point) with unbroken chiral symmetry in some window $n_f^* \leq n_f < n_f^{\text{AF}}$, where n_f^{AF} is the asymptotic freedom limit. Consequently, as L decreases from infinity, there should be both chirally symmetric and chirally asymmetric phases extending into the (L, n_f) -plane phase diagram.⁶

⁵This figure treats the number of fermion flavors n_f as a continuous variable. In the Euclidean theory, one can define non-integer n_f by taking the fermion determinant, which is real and positive in QCD(adj), to a fractional power.

⁶More precisely, there is a continuous non-Abelian chiral symmetry when $n_f > 1$, plus a flavor-independent discrete chiral symmetry. Our discussion focuses on the continuous chiral symmetry. Unlike the non-Abelian chiral symmetry, in the compactified theory the discrete chiral symmetry can be spontaneously broken even at weak coupling [41]. An unconventional order parameter for the discrete chiral symmetry is a topological disorder (or monopole) operator.

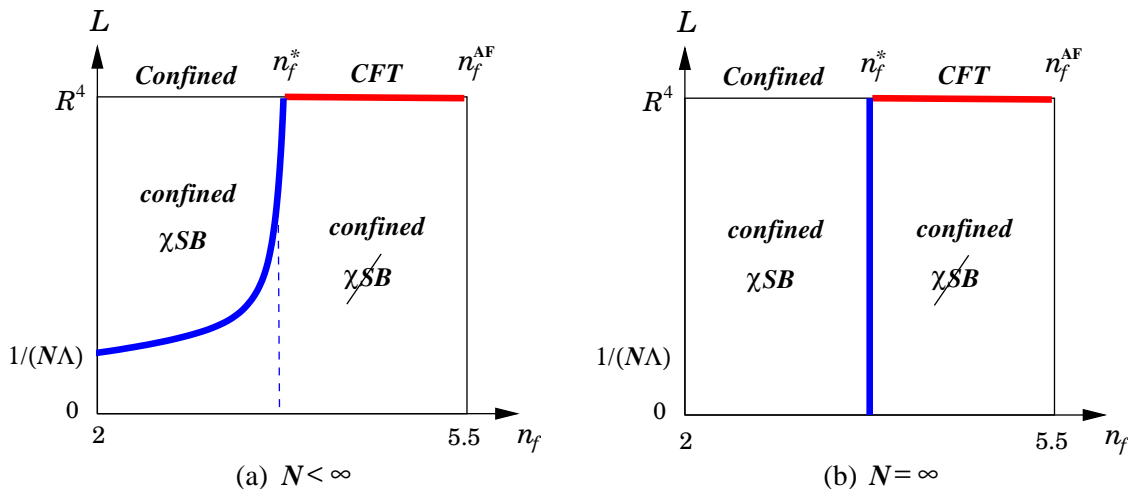


Figure 2: Contrasting finite N and infinite N phase diagrams of massless QCD(adj) on $\mathbb{R}^3 \times S^1$, as a function of the compactification size L and the number of fermion flavors n_f . In the decompactified limit, one expects a confining phase with spontaneously broken chiral symmetry for sufficiently small n_f , and a conformal phase with unbroken chiral symmetry in the window $n_f^* \leq n_f < n_f^{\text{AF}}$ (with n_f^{AF} the asymptotic freedom limit). For finite N (left), as one decreases L the chiral transition line must bend and approach an intercept at an “unconventional” scale of $1/(N\Lambda)$. To the right of this line is a phase with unbroken continuous chiral symmetry and finite, compactification-induced correlation length, smoothly connecting the analytically tractable $N\Lambda \ll 1$ region to the conformal portion of the $L = \infty$ boundary. At $N = \infty$ (right), the theory exhibits volume independence in both the chirally broken and chirally symmetric phases. The phase transition line extends straight down from $L = \infty$ and $n_f = n_f^*$. This implies that numerical studies on very small lattices can be used to determine the conformal window boundary n_f^* .

For finite N and sufficiently small L [small compared to $1/(N\Lambda)$], one can reliably analyze the theory using perturbative and semiclassical methods [22], as mentioned earlier. One finds unbroken continuous chiral symmetry, broken discrete chiral symmetry, a non-zero mass gap, and area-law behavior for large topologically trivial spatial Wilson loops.

The simplest, most plausible, scenario is that the chirally symmetric phase at small L (and any n_f) smoothly connects to the chirally symmetric conformal phase at $L = \infty$ and $n_f \geq n_f^*$. The chiral transition line separating these two phases must bend as L decreases, as shown, and approach an intercept at an “unconventional” scale of $1/(N\Lambda)$ [42]. This is the scale below which the long distance semiclassical analysis is valid. To the left of this line, one has a typical confining phase with spontaneously broken continuous chiral symmetry. The mass gap (inverse correlation length) vanishes due to the presence of Goldstone bosons. The spatial string tension (or area law coefficient for large spatial Wilson loops) will have finite, non-vanishing limits as $L \rightarrow \infty$. To the right of this line one has a phase with unbroken continuous chiral symmetry and finite, compactification-induced correlation length which diverges as $L \rightarrow \infty$. Similarly, this phase will have a non-zero spatial string tension for finite L , which vanishes as $L \rightarrow \infty$.

Massless adjoint fermions, with periodic boundary conditions on $\mathbb{R}^3 \times S^1$, prevent the spontaneous breaking of center symmetry [43, 6]. Consequently, at $N = \infty$ (illustrated in Fig. 2b), both the chirally

broken and chirally symmetric phases will exhibit volume independence. The chirally asymmetric phase will have a finite correlation length and non-zero spatial string tension, both independent of L . The chirally symmetric phase will have an infinite correlation length, irrespective of L . In this long-distance conformal phase, large Wilson loop expectation values will show Coulombic behavior. For rectangular $R \times T$ loops,

$$\lim_{T \rightarrow \infty} \frac{-1}{T} \ln \langle W(R \times T) \rangle \sim \begin{cases} \sigma R, & \text{confined phase;} \\ -g_*^2/R, & \text{IR conformal phase,} \end{cases} \quad (3.1)$$

with σ and g_*^2 having large N limits which are independent of the compactification size L . The phase transition line must extend straight down from $L = \infty$ and $n_f = n_f^*$. The value of n_f^* in the decompactified theory can be extracted from numerical studies on lattices with arbitrarily small L . In effect, large N volume independence allows one to trade extrapolations in lattice volume for extrapolations in the number of colors, N .

4. $\mathcal{N} = 4$ SYM

When a \mathbb{Z}_N center-symmetric theory is compactified on $\mathbb{R}^{d-1} \times S^1$, the dynamics of the theory may generate an effective potential for the Wilson line holonomy which causes its eigenvalues to attract, as in pure Yang-Mills theory, leading to spontaneous breaking of center symmetry. Alternatively, repulsive interactions between eigenvalues may be generated, as in massless QCD(adj) (with periodic fermion boundary conditions), or engineered (by the addition of explicit stabilizing terms), thereby preventing center symmetry breaking. But there is a third possibility, which is realized by maximally supersymmetric Yang-Mills ($\mathcal{N} = 4$ SYM) theory: a strictly vanishing effective potential for the Wilson line holonomy.

Consider $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ on \mathbb{R}^4 . The theory possess an $SU(4)_R$ R -symmetry group, and contains adjoint representation scalars $\Phi_{[IJ]}$ and fermions λ_I in the $\mathbf{6}$ and $\mathbf{4}$ of $SU(4)_R$, respectively. The renormalization group beta function vanishes identically, so the gauge coupling g^2 is a scale independent physical parameter of the theory. There is a continuous moduli space of vacua,

$$\mathcal{M}_{\mathbb{R}^4} = \mathbb{R}^{6N} / \mathcal{S}_N, \quad (4.1)$$

corresponding to mutually commuting scalar field expectation values with completely arbitrary values for their eigenvalues (modulo Weyl group permutations). The physical theory at the origin of moduli space is an interacting non-Abelian CFT. Generic points in moduli space correspond to Higgsing of the $SU(N)$ gauge group down to a maximal Abelian $U(1)^{N-1}$ subgroup.

When compactified on $\mathbb{R}^3 \times S^1$ with periodic spin connection (*i.e.*, periodic boundary conditions for fermions), the theory has a \mathbb{Z}_N center symmetry and remains supersymmetric. The behavior of the compactified theory was previously discussed by Seiberg [44]. He argued that the theory at the origin of the moduli space (4.1) flows to three-dimensional $\mathcal{N} = 8$ superconformal SYM theory at low energies, $E \ll g_3^2 \equiv g^2/L$, with an emergent $SO(8)_R$ symmetry. The compactified theory is manifestly not volume independent: results depend on the compactification scale L and long distance properties on $\mathbb{R}^3 \times S^1$ differ from the uncompactified theory on \mathbb{R}^4 .

However, the compactified theory on $\mathbb{R}^3 \times S^1$ has a *larger* moduli space than does the uncompactified theory. Due to the maximal $\mathcal{N} = 4$ supersymmetry, no superpotential for the Wilson line holonomy, either perturbative or non-perturbative, is generated. Consequently, compactification adds a new branch to the moduli space, with the holonomy Ω behaving as an adjoint Higgs field. The moduli space of $\mathcal{N} = 4$ SYM on $\mathbb{R}^3 \times S^1$ is

$$\mathcal{M}_{\mathbb{R}^3 \times S^1} = [\mathbb{R}^{6N} \times (\tilde{S}^1)^N] / \mathcal{S}_N, \quad (4.2)$$

where \tilde{S}^1 is the dual circle on which eigenvalues of the Wilson line holonomy Ω reside.⁷ In this compactified theory, the realization of the \mathbb{Z}_N center symmetry is entirely a matter of choice. Generic points in the moduli space (4.2) will involve a set of eigenvalues $\{e^{i\theta_a}\}$, $a = 1, \dots, N$, of Ω which are not invariant under any \mathbb{Z}_N transformation (other than the identity), and hence completely break the center symmetry. But since the choice of eigenvalues for Ω is arbitrary, the moduli space also includes points where the set of eigenvalues is completely \mathbb{Z}_N symmetric, as well as points where the set of eigenvalues is invariant only under some subgroup of \mathbb{Z}_N (if N is composite).

Prior discussion [44] of compactification in $\mathcal{N} = 4$ SYM has focused on the case where $\Omega = 1$. With all eigenvalues of Ω clustered at a single point, volume independence fails, as noted above. Here, we instead wish to examine the situation at center-symmetric points in moduli space where eigenvalues are evenly distributed around the unit circle and⁸

$$\Omega = \text{diag} \left(1, e^{2\pi i/N}, e^{4\pi i/N}, \dots, e^{2\pi i(N-1)/N} \right). \quad (4.3)$$

For simplicity, focus on the case where scalar field expectations are small compared to $1/L$. For finite N , at such center symmetric points in the moduli space, the dynamics Abelianizes at large distances and reduces to an effective $U(1)^{N-1}$ gauge theory. This will be a valid description for distances large compared to the inverse of the (lightest) W -boson mass,

$$m_W = \frac{2\pi}{LN}. \quad (4.4)$$

Off-diagonal components of all fields acquire masses of order m_W or greater, due to their coupling to the component of the gauge field in the compactified direction. The surviving components are those aligned along the Cartan subalgebra of $SU(N)$. The effective theory describing physics at energies well below m_W is just a free $U(1)^{N-1}$ Abelian gauge theory with neutral massless fermions and scalars,

$$\mathcal{L} = \frac{1}{g_3^2} \sum_{a=1}^{N-1} \left[\frac{1}{4} (F_{ij}^a)^2 + \frac{1}{2} (\partial_i A_4^a)^2 + \frac{1}{2} \sum_{I < J=1}^4 (\partial_i \Phi_{IJ}^a)^2 + \sum_{I=1}^4 i \bar{\lambda}_I^a \sigma_i \partial_i \lambda_I^a \right]. \quad (4.5)$$

Here $g_3^2 \equiv g^2/L$ and A_4 is the component of the gauge field in the compactified direction.

One may wonder whether non-perturbative effects will generate mass terms for the photons or the neutral matter fields. This is precisely what happens in QCD(adj) due to the effects of topological

⁷More generally, when compactified on $\mathbb{R}^{4-k} \times (S^1)^k$, the quantum moduli space of $\mathcal{N} = 4$ SYM theory is $\mathcal{M}_{\mathbb{R}^{4-k} \times (S^1)^k} = [\mathbb{R}^6 \times (\tilde{S}^1)^k]^N / \mathcal{S}_N$, corresponding to arbitrary eigenvalues for the six scalar fields and k independent Wilson line holonomies, all mutually commuting, modulo Weyl group permutations.

⁸For N even, this should be multiplied by $e^{i\pi/N}$ so that $\det \Omega = 1$.

excitations. In $\mathcal{N}=4$ SYM with, for example, gauge group $SU(2)$ Higgsed down to $U(1)$, the index theorem dictates that a BPS monopole operator has the form $\mathcal{M}_1 = e^{-|\phi|+i\sigma} \det_{IJ} \lambda^I \lambda^J$ where σ is a scalar field dual to the $3d$ gauge field and $e^{\pm i\phi}$ are the eigenvalues of Ω . However, it is not hard to see that the Yukawa couplings present in $\mathcal{N}=4$ SYM, but absent in QCD(adj), lift the fermion zero modes in such a way that no non-perturbative superpotential can be generated. In other words, the non-renormalization theorems associated with the large amount of supersymmetry guarantee the masslessness of low energy excitations. Thus, the action (4.5), which describes the physics of the massless modes depicted in Fig. 1b, is not only valid in perturbation theory, it is also the correct non-perturbative description of long distance physics.

At face value, the above description seems to be at odds with volume independence. The theory at the origin of the moduli space on \mathbb{R}^4 is non-Abelian, but the finite N center-symmetric theory on $\mathbb{R}^3 \times S^1$ exhibits dynamical Abelianization. The resolution of this apparent puzzle reflects the interplay of the $N \rightarrow \infty$ limit and the compact topology of the \tilde{S}^1 portion of moduli space. Since \tilde{S}^1 is compact, as $N \rightarrow \infty$ the \mathbb{Z}_N symmetrically distributed eigenvalues become arbitrarily dense and form a continuum. The domain of validity of the effective $3d$ Abelian theory is restricted to energies small compared to $m_W = 2\pi/(NL)$ and shrinks to zero as $N \rightarrow \infty$.

To summarize, if one considers \mathbb{Z}_N center-symmetric vacua in $\mathcal{N}=4$ SYM theory, compactified on $\mathbb{R}^3 \times S^1$, then the appropriate long distance effective theory is not the $3d$ interacting superconformal theory with enhanced R -symmetry discussed by Seiberg [44]. With a center-symmetric compactification, this enhancement of R -symmetry does not take place and the original $SU(4)_R$ symmetry is apparent at all length scales. In contrast to the QCD-like theories discussed earlier, $\mathcal{N}=4$ SYM never develops a finite compactification-induced correlation length. Large N volume independence does apply to center-symmetric vacua in $\mathcal{N}=4$ SYM theory and implies that the compactified center-symmetric theory will, at $N = \infty$, exactly reproduce properties of the uncompactified CFT on \mathbb{R}^4 . But, just as in the discussion of confining theories in section 2, there is non-uniformity between the $N \rightarrow \infty$ and $L \rightarrow 0$ limits. This is reflected in the characteristic regimes:

$$E \ll 2\pi/(LN), \quad \text{low energy, Abelian free Coulomb}; \quad (4.6)$$

$$E \gg 2\pi/(LN), \quad \text{high energy, non-Abelian CFT}, \quad (4.7)$$

in complete accord with Fig. 1.⁹

5. Massive QCD(adj)

In $SU(N)$ Yang-Mills theory, compactified on $\mathbb{R}^3 \times S^1$ with compactification size $L \lesssim 1/\Lambda$, the one-loop contribution to the Wilson line effective potential produces eigenvalue attraction and spontaneous breaking of center symmetry. This symmetry realization is stable if very massive adjoint representation fermions are added to the theory, since their effects are negligible when the fermion mass m is large compared to $1/L$. But when massless adjoint representation fermions are present, with periodic boundary conditions, their contribution to the Wilson line effective potential favors eigenvalue

⁹In the strong coupling domain where $\lambda \equiv g^2 N \gg 1$, an AdS/CFT analysis [45] shows that the criterion for the high-energy non-Abelian CFT regime becomes $E \gg \lambda/(LN)$.

repulsion and overcomes the attractive pure gauge contribution, leading to unbroken center symmetry. Consequently, it is inevitable that QCD(adj) on $\mathbb{R}^3 \times S^1$, with periodic spin connection, exhibits one or more phase transitions as the fermion mass m is varied for fixed small compactification radius L .

In asymptotically free QCD(adj), one would naively expect the behavior of the massive theory to closely resemble the massless limit when the mass m is small compared to $1/L$. As we next discuss, this expectation is too simplistic. Quite a rich phase diagram emerges as the fermion mass and the compactification radius are varied.¹⁰

When $NL\Lambda \ll 1$, non-zero frequency Kaluza-Klein modes are weakly coupled and may be integrated out perturbatively.¹¹ With n_f adjoint fermions having a common mass m , the resulting one-loop potential may be conveniently expressed in the form

$$V[\Omega] = \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \left[-1 + \frac{1}{2} n_f (nLm)^2 K_2(nLm) \right] \frac{|\text{tr} \Omega^n|^2}{n^4}. \quad (5.1)$$

Here $K_2(z)$ is the modified Bessel function of the second kind, with asymptotic behavior

$$K_2(z) \sim \begin{cases} \frac{2}{z^2} - 2 + O(z^2), & z \ll 1; \\ \sqrt{\frac{\pi}{2z}} e^{-z}, & z \gg 1. \end{cases} \quad (5.2)$$

As the mass $m \rightarrow \infty$, the fermions decouple and the effective potential (5.1) reduces to the pure gauge result, $V_{\text{YM}}[\Omega] = -\frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} |\text{tr} \Omega^n|^2 / n^4$, up to exponentially small $O(e^{-Lm})$ corrections. In the opposite limit of massless fermions, the coefficient of the $|\text{tr} \Omega^n|^2$ in the series (5.1) reduces to $[-1 + n_f]/n^4$, previously found in [6], and $V[\Omega] \rightarrow -(n_f - 1) V_{\text{YM}}[\Omega]$.

The n 'th term in the series (5.1) is evidently an effective mass term for the winding number n Wilson line $\text{tr}(\Omega^n)$. For arbitrary $SU(N)$ matrices, $\text{tr}(\Omega^n)$ is reducible to lower order traces when $|n| > N/2$, but traces with windings up to $|n| = \lfloor N/2 \rfloor$ may be regarded as independent. If the effective masses

$$m_n^2 \equiv \frac{4}{\pi^2 n^4} \left[-1 + \frac{1}{2} n_f z_n^2 K_2(z_n) \right], \quad z_n \equiv nLm, \quad (5.3)$$

are positive for all $n \leq \lfloor N/2 \rfloor$, then the minimum of the one-loop potential lies at the center symmetric point where $\text{tr} \Omega^n = 0$ for all non-zero $n \bmod N$. If some of these effective masses are negative, then the corresponding Wilson lines will develop non-zero expectation values, implying spontaneous breaking of center symmetry. Since $z^2 K_2(z)$ decreases monotonically for $z > 0$, if the first mass m_1^2 is negative then so are all higher masses. In this case, the \mathbb{Z}_N center symmetry is completely broken. If N is large (and composite), there can be a plethora of intermediate phases where the center symmetry partially breaks to different discrete subgroups. The effective mass squared (5.3) vanishes when z_n exceeds a threshold z_* whose value depends on n_f ,

$$z_*(n_f) = \{2.027, 2.709, 3.154, 3.484\}, \quad (5.4)$$

¹⁰Related analysis of QCD(adj) on $\mathbb{R}^3 \times S^1$ has been performed in Refs. [46, 47, 48]. Numerical simulations on a finite lattice mimicking this geometry are reported in Ref. [49].

¹¹When $\Omega \sim 1$ the relevant weak coupling criterion is $\Lambda L \ll 1$. But, as discussed in section 2, when the center symmetry is unbroken the KK-mode frequency spacing is smaller by a factor of $1/N$, so the weak coupling condition applies to the length scale NL instead of L .

for $n_f = 2, 3, 4, 5$.¹² Consequently, the Wilson line with winding number $n \leq \lfloor N/2 \rfloor$ becomes unstable when the compactification size L exceeds

$$L_n \equiv \frac{z_*}{nm}, \quad (5.5)$$

Since $L_{\lfloor N/2 \rfloor} < \dots < L_2 < L_1$, if L lies in the interval $[L_k, L_{k-1}]$ then $m_k^2 < 0$ and $m_{k-1}^2 > 0$, implying that Wilson lines wrapping fewer than k times are stable, while loops with k up to $\lfloor N/2 \rfloor$ windings are unstable. When $L < L_{\lfloor N/2 \rfloor}$, all independent Wilson lines are stabilized, and center symmetry is unbroken.

When $n_f > 1$, for any finite mass m , as one increases L from zero the initial phase is fully \mathbb{Z}_N symmetric. For sufficiently small L a perturbative analysis is reliable for any value of m . One may regard this regime as one in which the $SU(N)$ gauge group is Higgsed down to the maximal Abelian subgroup $U(1)^{N-1}$.¹³ When L exceeds $L_{\lfloor N/2 \rfloor}$, the highest (independent) winding mode becomes unstable and develops a non-zero expectation value — provided the weak coupling condition $N\Lambda \ll 1$ which underlies this analysis is valid. When $L \sim L_{\lfloor N/2 \rfloor} = O(1/Nm)$, this condition implies that the fermion must be heavy, $m \gg \Lambda$. As L continues to increase (with $m \gg \Lambda$), successively lower winding modes become unstable and will develop expectation values each time L passes a threshold L_k .¹⁴ All modes are locally unstable when L exceeds L_1 , so the center symmetry will be completely broken when $L > L_1$ (and $N\Lambda \ll 1$).

This analysis is valid in the small- L , large- m corner of the (L, m) -plane phase diagram, when $n_f > 1$. The behavior in other regions of the phase diagram, especially the small mass regime, depends on the number of fermion flavors. We will discuss separately three cases: multiple flavors below the conformal window, $2 \leq n_f < n_f^*$; a single flavor, $n_f = 1$; and multiple flavors within the conformal window, $n_f^* \leq n_f < n_f^{\text{AF}}$.

Multiple flavors: $2 \leq n_f < n_f^*$

With fewer than n_f^* flavors, the decompactified theory on \mathbb{R}^4 , for any value of fermion mass, has a typical confining phase with a strong scale of Λ . Consequently, for L large compared to $1/\Lambda$ the center symmetry will be unbroken. This absence of any center symmetry breaking for $L \gg 1/\Lambda$ should hold for all values of m .

Heavy fermions, with $m \gg \Lambda$, make a negligible contribution to dynamics on the scale of Λ , and so the difference between imposing periodic and antiperiodic boundary conditions on such fermions is also negligible. Consequently, for large m there will be a conventional confining/deconfining phase transition at $L = L_c = O(1/\Lambda)$, where $L_c = 1/T_c$ is the inverse transition temperature in pure Yang-Mills theory. Across this transition, the center symmetry changes from completely broken ($L < L_c$)

¹²The single flavor case $n_f = 1$ is discussed separately below. In QCD(adj), asymptotic freedom is lost at $n_f^{\text{AF}} = 5.5$.

¹³At large m and small L , confinement is the result of monopoles carrying a net magnetic charge and topological charge $1/N$. At $m = 0$ and small L , confinement results from magnetic “bions,” magnetically charged, topologically neutral combinations of monopoles and antimonopoles of differing types, which become bound due to fermion zero mode exchange [22]. Turning on a non-zero fermion mass lifts the fermion zero modes and allows the bions to unbind, smoothly converting the small m bion-induced confinement into monopole-induced confinement at large m .

¹⁴The slopes of the phase transition lines (5.5) for small L agree with Ref. [50], which studied the same class of theories on $S^3 \times S^1$. Our values (5.5) agree with the numerical values on the mL axis of Fig. 4b of Ref. [50].

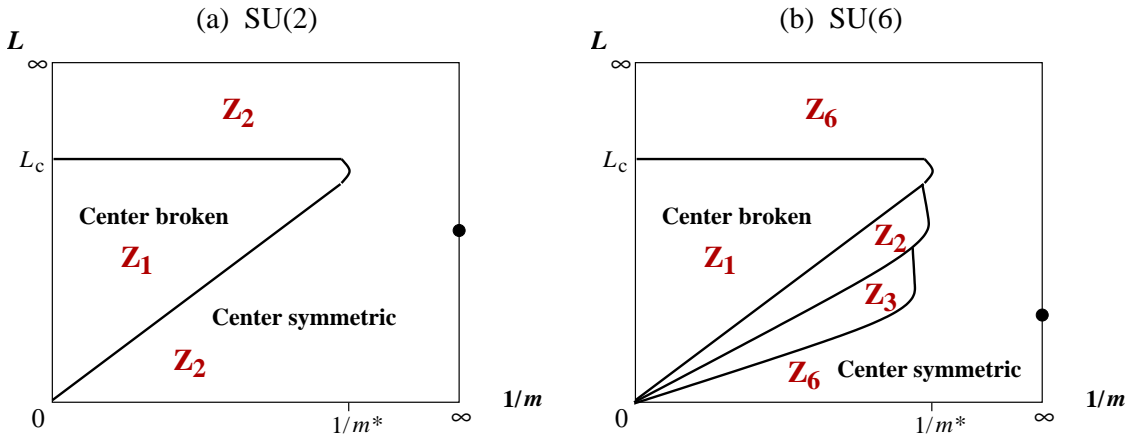


Figure 3: Center symmetry realization of QCD(adj) on $\mathbb{R}^3 \times S^1$ with $2 \leq n_f < n_f^*$ and periodic spin connection, as a function of compactification size L and inverse fermion mass $1/m$. Illustrated are the cases $N = 2$ (left) and $N = 6$ (right). The left hand axis where $m = \infty$ corresponds to pure Yang-Mills theory, where a “confining/deconfining” phase transition occurs at $L_c \sim \Lambda^{-1}$. Phases are labeled according to the *unbroken* subgroup of \mathbb{Z}_N center symmetry; the “ Z_1 ” region is the phase with totally broken center symmetry. The slopes of the transition line(s) approaching the origin are perturbatively computable. Within the center symmetric phase, the limit of vanishing size L (bottom axis) is expected to be continuously connected to the large L , large m domain (upper left corner) which corresponds to decompactified Yang-Mills theory. The dot on the right-hand boundary indicates the chiral symmetry transition point of the massless theory, expected to occur at an $O(1/(N\Lambda))$ value of L .

to fully restored ($L > L_c$). The completely broken phase just below L_c presumably connects directly to the completely broken phase at $L > L_1$ identified in the small L analysis.

As noted above, the conventional understanding of confinement in QCD-like theories (on \mathbb{R}^4) implies that a center-symmetric phase will be present at vanishingly small m and sufficiently large L . The perturbative analysis valid when $\Lambda \ll 1/(NL)$ shows that a center-symmetric phase is also present when the fermion mass is sufficiently small, $m \lesssim 1/(NL)$. The most plausible scenario is that these center symmetric regions are part of a single connected center symmetric phase which exists for all L when the fermions are sufficiently light, $m < m^* = O(\Lambda)$.¹⁵ The resulting phase diagram, as a function L and $1/m$, is illustrated in Fig. 3 for two representative values of N .

Single flavor: $n_f = 1$

Single flavor QCD(adj), in the massless limit, is $\mathcal{N} = 1$ supersymmetric Yang-Mill theory. Turning on a small but non-vanishing mass corresponds to a soft breaking of supersymmetry. Exact supersymmetry implies that the one loop potential vanishes at $m = 0$, as one may easily confirm after substituting $n_f = 1$ into the result (5.1) and sending $m \rightarrow 0$. For small but non-zero fermion mass, the sub-leading

¹⁵The chiral limit of QCD(adj) with periodic boundary conditions on $S^3 \times S^1$ was studied in Ref. [51], where it was argued that there is no center symmetry changing phase transition, consistent with expectations in the partially decompactified $\mathbb{R}^3 \times S^1$ limit.

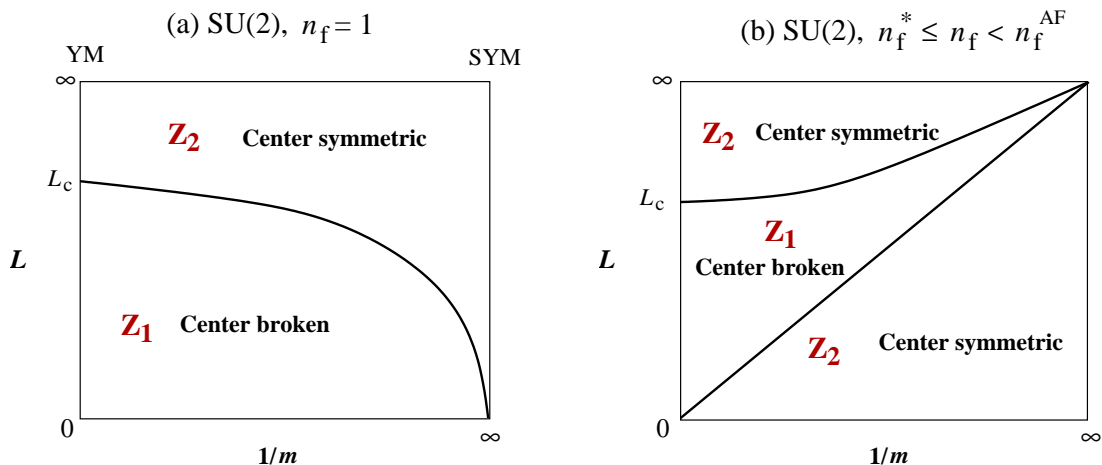


Figure 4: Center symmetry realization of QCD(adj) on $\mathbb{R}^3 \times S^1$ with periodic spin connection, as a function of compactification size L and inverse fermion mass $1/m$, for the case of a single flavor (left), or multiple flavors within the conformal window (right), $n_f^* \leq n_f < n_f^{\text{AF}}$. For $n_f = 1$, the right-hand $m = 0$ axis corresponds to $\mathcal{N} = 1$ SYM, which does not break center symmetry for any L . With non-zero fermion mass m , a single adjoint fermion is insufficient to stabilize the center symmetry at arbitrarily small L . For $n_f^* \leq n_f < n_f^{\text{AF}}$ (right), the phase boundaries separating center symmetry broken and unbroken phases are expected to extend to $L = \infty$ as $m \rightarrow 0$, with no continuous connection between the center-symmetric phases at large and small L .

term in the small z asymptotic behavior (5.2) contributes and the $n_f = 1$ effective potential becomes

$$V[\Omega] = -\frac{2m^2}{\pi^2 L^2} \sum_{n=1}^{\infty} \frac{1}{n^2} |\text{tr } \Omega^n|^2 + O(m^4). \quad (5.6)$$

Within the domain of validity of the perturbative analysis, this shows that Wilson lines with all winding numbers are unstable when the fermion mass is non-zero, despite the periodic boundary condition for fermions. Consequently, the $n_f = 1$ theory at any non-zero mass m and sufficiently small L will have completely broken center symmetry.

When $m = 0$ there is no perturbative contribution (at any order) to the Wilson line effective potential, but there is a non-perturbatively induced effective potential which ensures unbroken center symmetry in the supersymmetric theory on $\mathbb{R}^3 \times S^1$ [43]. For small but non-zero m , and small L , there will be competition between the one-loop $O(m^2)$ soft supersymmetry breaking potential and the non-perturbatively induced superpotential, leading to non-uniformity in the $m \rightarrow 0$ and $L \rightarrow 0$ limits. The transition line separating center-symmetric and completely broken phases must emerge from the $L = m = 0$ corner of the phase diagram, as illustrated for an $SU(2)$ theory on the left side of Fig. 4. Unlike the previous multi-flavor case, for larger values of N there is no reason to expect the phase diagram to contain any region with partially broken center symmetry.

Multiple flavors: $n_f^* \leq n_f < n_f^{\text{AF}}$

For this range of flavors, the chiral limit of the theory is in the conformal window, with long distance behavior (on \mathbb{R}^4) described by a non-trivial renormalization group (RG) fixed point. When the theory

is compactified on $\mathbb{R}^3 \times S^1$ and a non-zero fermion mass is introduced the scale invariant long distance behavior is cut-off, either by m or $1/L$. The resulting behavior in the (L, m) phase diagram will be qualitatively identical to the case $2 \leq n_f < n_f^*$ discussed above — except in the corner where $m \rightarrow 0$ and $L \rightarrow \infty$. Specifically, nothing changes in the small L perturbative analysis. For all finite values of m , a center symmetric phase will be present for sufficiently small compactification size. One or more transition lines (depending on the value of N) emerge from the $L = 1/m = 0$ corner of the phase diagram with slopes given by Eq. (5.5). In the large mass, large L portion of the phase diagram, $m \gg \Lambda \gtrsim 1/L$, the theory reduces to pure Yang-Mills theory and exhibits a confinement/deconfinement transition at $L = L_c = O(1/\Lambda)$.

In the massless limit, on \mathbb{R}^4 , the gauge coupling $g^2(\mu)$ ceases to run logarithmically below the scale Λ and asymptotes to a non-zero fixed-point value g_*^2 . Ref. [52] argues that the fixed point coupling will be relatively weak, and that the massless theory on $\mathbb{R}^3 \times S^1$ will have unbroken center symmetry for all L . Turning on a small fermion mass $m \ll \Lambda$ will have negligible effect on the RG flow for $\mu \gg m$, but will eliminate the fermion contribution to the RG evolution at scales small compared to m . Since the massless fermion contribution to the beta function was essential for producing a fixed point, the effective gauge coupling will only remain (nearly) constant for scales between Λ and m . Below the scale m the coupling will resume its increase, growing just as it does in pure Yang-Mills theory and eventually becoming strong and driving confinement at a scale $\Lambda_{\text{YM}} \sim m e^{-8\pi^2/(b_0 g_*^2)}$.

When the mass-deformed theory is compactified on $\mathbb{R}^3 \times S^1$, the confining phase will be stable, and center symmetry unbroken, provided $1/L \ll \Lambda_{\text{YM}}$. If the prediction of Ref. [52] of a rather small value of the fixed point coupling is correct, then Λ_{YM} will be substantially smaller than m (even though there is no arbitrarily large parametric separation). Assuming so, a conventional confining/deconfining phase transition, just as in pure Yang-Mills theory, must occur when L decreases to $L_c = O(1/\Lambda_{\text{YM}})$, since the fermions have negligible effect on physics at scales well below m . When the compactification size decreases further to $O(1/m)$ or smaller, the fermions can influence the resulting symmetry realization. When the fermions (with non-thermal periodic boundary conditions) are sufficiently light compared to $1/L$ they will stabilize the center symmetric phase, just as in the massless limit.

The simplest possibility for the resulting phase diagram is sketched on the right hand side of Fig. 4, for the case of $N = 2$. The novel feature is the “sliver” of totally broken center symmetry phase extending all the way to $L = \infty$ and $m = 0$, surrounded on either side by a phase with unbroken center symmetry. For $N > 3$, there will also be phases with partially broken center symmetry emerging from $L = 1/m = 0$ and lying in between the totally broken and small- L symmetric phases, just as in Fig. 3(b). We expect these partially broken phases to terminate at a non-zero $O(\Lambda)$ value of m , but cannot exclude the logical possibility that they also persist in slivers extending to the upper right $m = 1/L = 0$ corner.

Large N limit of massive QCD(adj)

As $N \rightarrow \infty$, \mathbb{Z}_N center symmetry transformations become a dense set within the continuous $U(1)$ group. At $N = \infty$, any discrete cyclic group \mathbb{Z}_p may be regarded as a subgroup of the center symmetry. For $n_f > 1$, when the compactification size L lies in the interval $[L_p, L_{p-1}]$, then the effective potential (5.1) is minimized when the eigenvalues of Ω form p clumps equi-spaced around the unit circle, so that $\text{tr} \Omega^p \neq 0$ while all traces of Ω to lower powers vanish. At these minima, the center symmetry is

broken to \mathbb{Z}_p . For fixed $m \gg \Lambda$, as L decreases from $O(1/\Lambda)$ to zero, passing through each $L_p \propto 1/p$, there is an infinite sequence of phase transitions with an accumulation point at $L = 0$:

$$\begin{array}{l} \text{Compactification size :} \quad L_1 > L_2 > L_3 > \cdots > L_{p-1} > L_p > \cdots \\ \text{Residual center symmetry :} \quad \mathbb{Z}_1 \quad | \quad \mathbb{Z}_2 \quad | \quad \mathbb{Z}_3 \quad | \quad \cdots \quad | \quad \mathbb{Z}_p \quad | \quad \cdots \end{array} \quad (5.7)$$

For very large but finite values of N , not all of these intervals will correspond to distinct symmetry realizations. Which transitions are associated with changes in symmetry realization will depend on the prime factorization of N . But within each interval $[L_p, L_{p-1}]$, the eigenvalues of Ω will form p clumps — some having $\lfloor N/p \rfloor$ eigenvalues and others having $\lceil N/p \rceil$. In other words, the finite N eigenvalue distribution will approximate p equi-distributed clumps as closely as possible. The Wilson line expectation value $\langle \frac{1}{N} \text{tr} \Omega^k \rangle$ will be $O(1)$ when the winding number k is an integer multiple of p , and $O(1/N)$ when the winding k is not a multiple of p .

As discussed above, for arbitrarily large fermion mass, an unbroken center symmetry phase will exist both at large compactification size, $L \gtrsim O(1/\Lambda)$, and sufficiently small size, $L < L_{\lfloor N/2 \rfloor}$ when $n_f > 1$. Consequently, one might hope that large N volume independence would relate massive QCD(adj) on \mathbb{R}^4 to the theory in the small L center symmetric domain. Since properties of QCD(adj) with $m \gg \Lambda$ differ negligibly from pure Yang-Mills theory, this suggests that massive QCD(adj) with sufficiently small L and large m could serve as a reduced model exactly reproducing properties of large- N Yang-Mills theory. This, however, is not the case on $\mathbb{R}^3 \times S^1$. The problem is that the condition for unbroken center symmetry in the reduced theory, $L < L_{N/2} = 2z_*/(Nm)$, combined with the *strong coupling* condition (2.4) for large N volume independence, $NLA \gg 1$, together imply that $m \ll \Lambda$. In other words, when taking $N \rightarrow \infty$ for fixed mass $m \gg \Lambda$, staying within the center symmetric small- L phase forces one to send L to zero, $L \lesssim 1/(Nm)$, and this makes it impossible to remain within the $NLA \gg 1$ regime required for large N volume independence. This means that massive QCD(adj) on small $S^1 \times \mathbb{R}^3$ cannot be used as a large N reduced model for (continuum) Yang-Mills theory on \mathbb{R}^4 .

Nevertheless, provided the phase diagram sketched in Fig. 3 is qualitatively correct as $N \rightarrow \infty$, large N volume independence is valid in massive QCD(adj) with $2 \leq n_f < n_f^*$, for all L , when the fermion mass is below the $O(\Lambda)$ lower limit m^* of any broken center symmetry phase. And likewise for $n_f^* \leq n_f < n_f^{\text{AF}}$ and $m < O(\Lambda)$, large N volume independence will relate the center-symmetric phases at large and small L , despite the presence of an intervening broken symmetry phase.

6. Multiple compactified dimensions

We now turn to a consideration of QCD(adj) on $\mathbb{R}^2 \times (S^1)^2$. The extension to compactifications on higher dimensional torii will be discussed below.

Let Ω_1 and Ω_2 denote the holonomy of the gauge field around the two independent cycles of the 2-torus $T^2 = (S^1)^2$. The set of vacua of the classical gauge theory is the space of flat connections, $F_{\mu\nu} = 0$, which implies commuting covariantly constant holonomies,

$$[\Omega_1, \Omega_2] = 0. \quad (6.1)$$

Hence, for vacuum configurations the two holonomies can be simultaneously diagonalized by a suitable gauge transformation. In other words, the classical vacua may be parametrized by two sets of N

eigenvalues,

$$\Omega_1 = \text{diag} (e^{i\alpha_1}, e^{i\alpha_2}, \dots, e^{i\alpha_N}), \quad (6.2a)$$

$$\Omega_2 = \text{diag} (e^{i\beta_1}, e^{i\beta_2}, \dots, e^{i\beta_N}), \quad (6.2b)$$

up to a global gauge transformation (which simultaneously conjugates Ω_1 and Ω_2). Each diagonalized holonomy Ω_i may be regarded as taking values in the maximal torus of $SU(N)$, which we label as \mathbf{T}_N . For generic configurations with distinct eigenvalues, the subgroup of global gauge transformations which preserve the diagonalized form (6.2) is the Weyl group \mathcal{W} of $SU(N)$, whose elements *simultaneously* permute the eigenvalues of the two holonomies, $(\alpha_i, \beta_i) \rightarrow (\alpha_{\sigma(i)}, \beta_{\sigma(i)})$, where $\sigma \in \mathcal{S}_N$ (with \mathcal{S}_N the N -element permutation group). Consequently, the classical moduli space of the theory can be described as

$$\mathcal{M} = (\mathbf{T}_N)^2 / \mathcal{S}_N. \quad (6.3)$$

Another useful way to describe \mathcal{M} is as follows. Let \tilde{T}^2 denote the torus dual to the base space 2-torus. If $\vec{L} \equiv (L_1, L_2)$ are the periods of T^2 , then the dual torus \tilde{T}^2 has periods $\tilde{L}_1 = 2\pi/L_1$ and $\tilde{L}_2 = 2\pi/L_2$. Each pair of simultaneous eigenvalues $(e^{i\alpha}, e^{i\beta})$ may be written as $(e^{ik^1 L_1}, e^{ik^2 L_2})$ where $\vec{k} \equiv (k^1, k^2) \in \tilde{T}^2$. Consequently, the moduli space \mathcal{M} may also be viewed as the configuration space of N identical ‘‘eigenparticles’’ which are placed on the dual torus \tilde{T}^2 . (Clearly, this description generalizes immediately to compactifications on higher dimensional torii.)

Integrating out fluctuations will produce an effective potential which depends on the commuting holonomies Ω_i . In QCD(adj) with n_f fermions (having a common mass m and periodic spin connection) the one-loop result may be expressed as a sum of pairwise interactions between eigenparticles,

$$V[\Omega_1, \Omega_2] = \frac{1}{2} \sum_{i,j=1}^N v[\vec{\alpha}_i - \vec{\alpha}_j], \quad (6.4)$$

with

$$v[\Delta\vec{\alpha}] \equiv \sum'_{\vec{n} \in \mathbb{Z}^2} m_{\vec{n}}^2 \cos[\vec{n} \cdot \Delta\vec{\alpha}], \quad (6.5)$$

and

$$m_{\vec{n}}^2 \equiv \frac{2}{\pi^2 |\vec{L}\vec{n}|^4} \left[-1 + \frac{1}{2} n_f (m |\vec{L}\vec{n}|)^2 K_2(m |\vec{L}\vec{n}|) \right]. \quad (6.6)$$

Here $\vec{\alpha}_i \equiv (\alpha_i, \beta_i)$, $\vec{n} = (n_1, n_2)$, $\vec{L}\vec{n} \equiv (L_1 n_1, L_2 n_2)$, and the prime on the sum indicates omission of the term with $\vec{n} = 0$. Alternatively, one may express this potential as a sum of squares of Wilson lines,

$$V[\Omega_1, \Omega_2] = \frac{1}{2} \sum'_{\vec{n} \in \mathbb{Z}^2} m_{\vec{n}}^2 |\text{tr}(\Omega_1^{n_1} \Omega_2^{n_2})|^2. \quad (6.7)$$

Unbroken $(\mathbb{Z}_N)^2$ center symmetry implies vanishing of the traces $\text{tr}(\Omega_1^{n_1} \Omega_2^{n_2})$ for all winding numbers n_1 and n_2 which are not multiples of N . The fermion contribution (proportional to n_f) to the effective mass squared $m_{\vec{n}}^2$ is positive. One may see from the pairwise potential (6.5) that this means fermions generate repulsive interactions between every pair of eigenparticles. The equivalent form (6.7) shows that the positive fermion contribution to $m_{\vec{n}}^2$ works to stabilize the center symmetric configuration with vanishing traces. In contrast, the (-1) pure gauge contribution to $m_{\vec{n}}^2$ produces attractive interactions

between eigenparticles which work to destabilize center symmetry. This exactly parallels the situation on $\mathbb{R}^3 \times S^1$, except that eigenvalue pairs on \tilde{T}^2 take the place of single eigenvalues on S^1 .

Naturally, the competition between fermion and gauge field contributions leads to different types of minima depending on the values of the periods \vec{L} , the fermion mass m , and the number of flavors n_f . For sufficiently large fermion mass (within the small L domain of validity of the perturbative analysis), the gauge field contribution will dominate, causing all N eigenparticles to coalesce and producing complete spontaneous breaking of the $(\mathbb{Z}_N)^2$ center symmetry. For sufficiently light and numerous fermions, the repulsive fermion-induced interaction between eigenparticles will dominate. This will cause the eigenparticles to disperse throughout the dual torus. Finding the arrangement which minimizes the effective potential is analogous to a sphere-packing problem; the precise result will depend sensitively on N and the aspect ratio L_1/L_2 . When, for example, $L_1 = L_2 \equiv L$ and N is a perfect square, the minimum of $V[\Omega_1, \Omega_2]$ for sufficiently light mass and $n_f > 1$ should correspond to “crystallization” of the N eigenparticles into a $\sqrt{N} \times \sqrt{N}$ square array.¹⁶ Such a configuration produces vanishing order parameters, $\text{tr}(\Omega_1^{n_1} \Omega_2^{n_2}) = 0$, for windings n_1 and n_2 which are non-zero modulo \sqrt{N} , corresponding to spontaneous breaking of the $(\mathbb{Z}_N)^2$ center symmetry down to a $(\mathbb{Z}_{\sqrt{N}})^2$ subgroup. For these configurations, the evaluation of the effective potential is reliable provided $\sqrt{N}L\Lambda \ll 1$; this is the condition that the coupling be weak on the $O[1/(\sqrt{N}L)]$ scale of the lightest off-diagonal degrees of freedom (*i.e.*, charged W 's) which were integrated out to obtain the effective potential.

More generally, for $O(1)$ values of L_1/L_2 and large values of N , the minimum of the effective potential will correspond to configurations where the eigenparticles are distributed with approximately uniform density on the dual torus. The typical nearest-neighbor separation between eigenparticles on the dual torus will be $\sim 2\pi/\sqrt{N}L$, where L is the geometric mean of L_1 and L_2 , and the appropriate weak coupling condition will again be $\sqrt{N}L\Lambda \ll 1$.

For a highly anisotropic torus, $L_2 \gg L_1$, the eigenparticles will form a one-dimensional array. When, for example, $L_2 = pL_1$ for some large integer p and $M \equiv N/p$ is an integer factor of N , then $\vec{a}_j = (2\pi j/N, 2\pi j/M)$ is a minimum energy configuration. This configuration is invariant (up to Weyl permutations) under a \mathbb{Z}_N subgroup obliquely embedded within the $\mathbb{Z}_N \times \mathbb{Z}_N$ center symmetry group and corresponds to a partial breaking of the $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry down to a single \mathbb{Z}_N .

In contrast to the situation on $\mathbb{R}^3 \times S^1$, no configuration of eigenparticles on the dual torus can be invariant under the entire $(\mathbb{Z}_N)^2$ center symmetry. The unbroken subgroup of the center symmetry will be some $\mathbb{Z}_p \times \mathbb{Z}_q$ with $pq \leq N$. Consequently, there will be (at least) $O(N)$ gauge-inequivalent degenerate minima, related by center transformations. But before one can conclude that this truly implies spontaneous breaking of center symmetry, the effects of fluctuations must be considered. On $\mathbb{R}^2 \times T^2$, fluctuations of eigenvalues away from the minimum energy configurations behave like two-dimensional scalar fields. These fluctuations, for $n_f > 1$, acquire a non-zero mass of order $\sqrt{\lambda}/L$ from the one-loop effective potential, with $\lambda \equiv g^2 N$ the 't Hooft coupling. When L is small and the perturbative evaluation of the effective potential is reliable, the probability of such fluctuations overcoming the action barriers separating different minima is negligible. Consequently,

¹⁶More explicitly, this means $\Omega_1 \propto C_{\sqrt{N}} \otimes 1_{\sqrt{N}}$ and $\Omega_2 \propto 1_{\sqrt{N}} \otimes C_{\sqrt{N}}$, where $1_{\sqrt{N}}$ is a \sqrt{N} -dimensional identity matrix and $C_{\sqrt{N}} = \text{diag}(1, \omega, \omega^2, \dots, \omega^{\sqrt{N}-1})$, with $\omega \equiv \exp(2\pi i/\sqrt{N})$ a $\mathbb{Z}_{\sqrt{N}}$ generator. However, we have not checked to see if, for example, a bcc lattice is preferred over a simple cubic lattice. For suitable values of N the minimum energy configuration of eigenvalues will be some regular lattice; exactly which lattice is inessential to our discussion.

on $\mathbb{R}^2 \times T^2$, multiple light adjoint representation fermions cannot prevent at least partial breaking of center symmetry in the limit of small compactification radii. A rich pattern of phase boundaries will emerge from the $L_1 = L_2 = 1/m = 0$ corner of the phase diagram, just as for the $\mathbb{R}^3 \times S^1$ case, when $n_f > 1$.

Once again, $n_f = 1$, or single flavor QCD(adj), is a special case. With a non-zero mass m , the center-symmetry stabilizing fermion contribution can never dominate the destabilizing gauge field contribution in the coefficients (6.6). Consequently, as the compactification sizes $L_1, L_2 \rightarrow 0$ for fixed mass $m > 0$, the center symmetry will be completely broken, just like the situation on $\mathbb{R}^3 \times S^1$.

In the massless limit, the $n_f = 1$ theory becomes $\mathcal{N} = 1$ supersymmetric Yang-Mills theory. Center symmetry is surely unbroken when the compactification size is sufficiently large. In supersymmetric theories with supersymmetry preserving boundary conditions, it is believed that there are no phase transitions as a function of volume. The singularities in the holomorphic coupling are of (real) codimension two [53, 54], so even if a singularity was encountered as the compactification size decreased, it is possible to analytically continue around it. Hence, center symmetry in $\mathcal{N} = 1$ SYM (compactified on $\mathbb{R}^2 \times T^2$) should be unbroken for any value of L_1 and L_2 .

It is instructive to see how this conclusion can emerge from a small L analysis. The perturbative effective potential vanishes identically in this supersymmetric theory but, as noted above, a non-perturbative superpotential is generated on $\mathbb{R}^3 \times S^1$ [43], and hence necessarily also on $\mathbb{R}^2 \times T^2$. (This follows from considering the regime $L_1 \ll L_2$, where physics must approach the $\mathbb{R}^3 \times S^1$ case.) The parametric size of the resulting bosonic potential is $\Lambda^6 L^2$ where $L = \min(L_1, L_2)$. This potential lifts the classical moduli space (6.3) but, for $\Lambda L \ll 1$, it is a very weak “pinning” potential for the eigenparticles and generates an effective mass μ for fluctuations of eigenvalues which is tiny, $\mu \sim O(\sqrt{\lambda} \Lambda^3 L^2)$. The resulting fluctuations, on scales which are large compared to L but small compared to $1/\mu$, have amplitudes large enough to wash out the distinction between neighboring degenerate minima related by center transformations.¹⁷ Hence, for $\mathcal{N} = 1$ SYM, a classical analysis of the Wilson line effective potential is invalid; the effects of nearly massless two-dimensional eigenvalue fluctuations cannot be neglected. Integrating out the effects of these fluctuations on length scales between L and μ^{-1} will modify the effective potential relevant for even longer scales, leading to merging of the multiple minima and restoration of the full center symmetry.

Returning to the case of $n_f > 1$, we noted above that for arbitrarily small L there will be at least partial breaking of the center symmetry. The implications of this for large N volume independence is, perhaps surprisingly, negligible. With multiple fermion flavors and sufficiently small compactification size, all coefficients m_n^2 in the Wilson line effective potential (6.7) with up to N windings are positive, favoring unbroken center symmetry. But no configuration of eigenvalues, for finite N , can force all order parameters $\text{tr}(\Omega_1^{n_1} \Omega_2^{n_2})$ to vanish (for windings n_1 and n_2 non-zero modulo N). Nevertheless when, for

¹⁷Consider, for example, the specific case of $N = M^2$ and $L_1 = L_2 = L$ where the minimum energy configuration of eigenparticles should be a square lattice. Eigenvalue fluctuations behave like massless $2d$ fields, with $\langle \delta\alpha(x) \delta\alpha(0) \rangle \sim (\lambda/4\pi N) \ln 1/(\mu^2 x^2)$, for $|x| \lesssim 1/\mu$. Examining, *e.g.*, the order parameter $\mathcal{O} \equiv \frac{1}{N} \text{tr}(\Omega_1^M)$, the difference in $\langle \mathcal{O} \rangle$ between neighboring minima is $|e^{2\pi i/M} - 1| = O(1/M)$. If \mathcal{O}_d denotes \mathcal{O} averaged over a region of size d , one finds $\langle (\delta\mathcal{O}_d)^2 \rangle \sim \frac{1}{N} [\frac{\lambda}{4\pi} \ln \frac{1}{\mu^2 d^2}]^2$. So the rms fluctuation in \mathcal{O}_d is $O(1/M)$ or larger when $d \leq \bar{d} \sim \mu^{-1} e^{-2\pi/\lambda}$. Inserting $\mu^{-1} \sim (L/\sqrt{\lambda})(\Lambda L)^{-3} \sim (L/\sqrt{\lambda}) e^{8\pi^2/\lambda}$ shows that $\bar{d} \sim (L/\sqrt{\lambda}) e^{(8\pi^2 - 2\pi)/\lambda}$, where $\lambda = \lambda(1/L)$. So for small L , there is a parametrically large hierarchy, $L \ll \bar{d} \ll 1/\mu$.

example, $N = M^2$ and $L_1 = L_2$, the unbroken $\mathbb{Z}_{\sqrt{N}} \times \mathbb{Z}_{\sqrt{N}}$ subgroup of center symmetry still forces all loops with winding numbers less than \sqrt{N} to vanish. As $N \rightarrow \infty$, this (plus unbroken translation symmetry) is sufficient to guarantee that the loop equations of the compactified and decompactified theories coincide. For other values of N or L_1/L_2 , the minimum energy configurations of eigenparticles on the dual torus \tilde{T}^2 will lead to non-zero values of Wilson lines even when the winding numbers are small. But repulsion of eigenparticles on the dual torus will cause the order parameters $\frac{1}{N} \text{tr}(\Omega_1^{n_1} \Omega_2^{n_2})$ to be $O(1/N)$ — not $O(1)$ — provided $n_1, n_2 \ll N$. This means that the large N limit of the expectation value of any topologically non-trivial Wilson loop vanishes, showing that the full center symmetry is effectively restored as $N = \infty$. Consequently, large N volume independence will relate $n_f \geq 2$ QCD(adj) on $\mathbb{R}^2 \times T^2$ with sufficiently small compactification size to the decompactified theory on \mathbb{R}^4 .

Most aspects of the above discussion of QCD(adj) on $\mathbb{R}^2 \times T^2$ generalize immediately to the case of $\mathbb{R} \times T^3$ or T^4 . The only difference in expressions (6.4)–(6.7) is that $\vec{\alpha}$ and \vec{n} change from two-component to three- or four-component vectors.¹⁸ However, fluctuations of eigenvalues become progressively more important as the number of uncompactified dimensions decreases. For finite values of N , the discrete center symmetry cannot break spontaneously when the theory is compactified on $\mathbb{R} \times T^3$ or T^4 . However, as is well known, spontaneous symmetry breaking can occur at $N = \infty$ even in finite volume theories. Consequently, one must carefully examine the effects of eigenvalue fluctuations as $N \rightarrow \infty$. When repulsive interactions dominate and eigenparticles are distributed throughout the dual torus, one may show from the expression (6.7) that the potential barrier which separates degenerate minima related by a center transformation is $O(N^0)$, and does not grow as N becomes large. On T^4 , where there is no infinite volume of uncompactified dimensions, this implies that fluctuations sampling all center-symmetry related minima will have non-vanishing probabilities as $N \rightarrow \infty$, thereby ensuring unbroken center symmetry at $N = \infty$. The situation is different when attractive interactions between eigenparticles dominate and the eigenparticles clump. In this case, the potential barrier between different minima is $O(N)$, and the probability of symmetry-restoring fluctuations vanishes at $N = \infty$.

On $\mathbb{R} \times T^3$, the relevant fluctuations are tunneling transitions which are localized in the uncompactified dimension, but the basic conclusion is the same: when the repulsive fermion-induced interaction between eigenparticles dominates the attractive gauge field contribution, fluctuations sampling all center-symmetry related configurations should retain non-zero probabilities as $N \rightarrow \infty$, so that center symmetry remains unbroken at $N = \infty$.

The basic interaction between eigenparticles switches between attractive and repulsive at an $O(1)$ value of mL (for $n_f > 1$). This is consistent with the numerical work of Bringoltz and Sharpe [55]¹⁹ who investigated a single-site model of QCD(adj) and found that the center symmetry is intact for a

¹⁸There is one noteworthy change for T^4 compactifications: the sum over $\vec{n} \in \mathbb{Z}^4$ which now appears in the coefficients (6.5) is only conditionally convergent for $\Delta\vec{\alpha} \neq 0$, and has a logarithmic divergence at $\Delta\vec{\alpha} = 0$. This reflects the fact that coinciding eigenparticle positions on the dual torus represent special configurations where some of the off-diagonal fluctuations which were integrated out to produce the effective potential (6.4) become massless. On T^4 with coinciding eigenvalues, these are off-diagonal zero-modes for whom quartic interactions remain relevant. Inappropriately applying a Gaussian approximation to these zero modes leads to the unphysical logarithmic singularity in the effective potential.

¹⁹For related work, also see Ref. [56, 57].

wide range of fermion mass, up to an $O(1)$ value of ma (with a the lattice spacing), for reasonably large values of N and values of the bare 't Hooft coupling in the range typically used in lattice QCD studies. For compactifications on $\mathbb{R} \times T^3$ or T^4 the bottom line, once again, is that large N volume independence will be valid when the fermions are sufficiently light and numerous. But, due to enhanced fluctuations in lower dimensions, the upper limit on the range of fermion masses for which unbroken center symmetry persists down to $L \rightarrow 0$, at infinite N , should be the larger of $(O(1/L), O(\Lambda))$ and not just $O(\Lambda)$ as in the earlier cases of $\mathbb{R}^2 \times T^2$ and $\mathbb{R}^3 \times S^1$. See Refs. [58] for more discussion of this issue.

7. Prospects

Volume independence is an exact property of certain large- N gauge theories. Although the idea is old, it was widely believed that, for four dimensional gauge theories, only a partial reduction was possible down to a minimal compactification size $L = L_c \sim 1/\Lambda$, and it has often been asserted that the small volume theory is weakly coupled whenever $L\Lambda \ll 1$. This proves not to be the case. The understanding that emerges from valid examples of volume independence (as $L \rightarrow 0$) leads to a modified picture. When center symmetry is unbroken, a weak coupling description is only possible when $L\Lambda$ is small compared to a positive power of $1/N$. It is possible for a QCD-like large N gauge theory, formulated in a box much smaller than the inverse strong scale, to reproduce infinite volume results.

There is an ongoing effort in the lattice gauge theory community to determine the lower boundary of the conformal window for various QCD-like theories. One of the technical issues in all lattice gauge theory simulations is controlling finite-volume effects and performing reliable infinite volume extrapolations. The $1/N$ suppression of finite-volume effects in large- N center symmetric theories allows one to trade a large volume extrapolation for a large N extrapolation, and should be helpful for studies of conformal windows in large N theories.

In a lattice formulation of gauge theories, large- N volume independence implies a non-perturbative equivalence between four-dimensional field theories and zero dimensional matrix models or one dimensional quantum mechanics of large- N matrices [6, 17, 55]. It may be possible to develop useful techniques directly in zero or one dimension (which are not applicable to higher dimensional field theories) in order to gain insight into four-dimensional gauge dynamics. We believe that these applications of large- N volume independence, and undoubtedly others, merit further consideration.

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