COMPARISON OF SIMULATION CODES FOR MICROWAVE INSTABILITY IN BUNCHED BEAMS[*]

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INTRODUCTION

In accelerator design, there is often a need to evaluate the threshold to the (longitudinal) microwave instability for a bunched beam in an electron storage ring. Several computational tools are available that allow us, once given the wakefield representing a ring, to numerically find the threshold current and to simulate the development of the instability. In this work, we present results of computer simulations using two codes recently developed at the SLAC National Accelerator Laboratory: a Vlasov-Fokker-Planck (VFP) solver based on an algorithm by Warnock and Ellison [1], and a program that finds the threshold from the linearized Vlasov equation.

We apply the programs to find the instability threshold for three models of ring impedances: that of a $Q = 1$ resonator, of shielded coherent synchrotron radiation (CSR), and of a resistive wall. The first example is well-behaved, but the other two are singular wakes that need special care. Note that similar numerical studies of the threshold of a $Q = 1$ resonator wake have been performed by Oide and Yokoya [2], and others [3]–[5]. We compare the results of the two programs and discuss their respective capabilities and limitations. In this report we assume the slippage factor η is always positive. We work in Gaussian units.

VLASOV-FOKKER-PLANCK (VFP) CODE

Let us consider the longitudinal motion of a bunched beam in an electron storage ring. When there is a collective force induced by the bunch distribution $\lambda(q)$ through the wakefield $w(q)$, the evolution of the beam density distribution (in longitudinal phase space) $\psi(\theta, q, p)$ is governed by the Vlasov-Fokker-Planck (VFP) equation

$$\frac{\partial \psi}{\partial \theta} - \{H, \psi\} = 2\beta \frac{\partial}{\partial p}(p\psi + \frac{\partial \psi}{\partial p}), \qquad (1)$$

where $\{\}$ indicates a Poisson bracket. The Hamiltonian

$$H = \frac{1}{2}(q^2 + p^2) - I \int_{-\infty}^{q} dq'' \int_{-\infty}^{\infty} dq' \lambda(q') \, w(q'' - q'), \qquad (2)$$

with $\lambda(q) = \int_{-\infty}^{\infty} \psi(q, p) dp$. The independent variable $\theta = \omega_s t$, with ω_s the (nominal) synchrotron frequency and t the time. We use a normalized coordinate system: $q = z/\sigma_{z0}$, where z is longitudinal position, with the positive z-axis pointing to the front (the direction of motion), and σ_{z0} is nominal (zero current) rms bunch length;

$p = -\delta/\sigma_{\delta 0}$, where δ is relative energy deviation and $\sigma_{\delta 0}$ is nominal energy spread. We define normalized current as

$$I = \frac{r_e N_b}{2\pi \nu_s \gamma \sigma_{\delta 0}}, \qquad (3)$$

with r_e is the classical electron radius, N_b the bunch population, ν_s the nominal synchrotron tune, and γ the Lorentz energy factor. The terms on the right of Eq. 1 are the synchrotron radiation damping and quantum diffusion terms; here $\beta = 1/\omega_s \tau_d$, with τ_d the longitudinal damping time. Note that the wake $w(\Delta q)$—with dimension of inverse length—represents the effect of the entire ring, that argument $\Delta q > 0$ implies the test particle is ahead of the driving charge, and that $w > 0$ indicates energy loss.

To solve Eq. 1, Warnock and Ellison developed a robust algorithm based on the Perron-Frobenius operator, with the solution obtained on a regular grid in longitudinal phase space [1]. We have rewritten their code in C++, making some improvements to their grid interpolation scheme, in order to improve detection of the threshold to instability. In a typical run for this report, the maximum of $|q|$ and $|p|$ is 8, with 300 mesh points in each direction; the number of time steps per synchrotron period is 1024, with a total run lasting 400 synchrotron periods; the damping parameter is taken to be small, $\beta = 1.25 \times 10^{-3}$. The program initializes $\psi(q)$ with the solution to the Haïssinski equation.

LINEARIZED VLASOV (LV) CODE

The approach described in the previous section solves a full VFP problem and for an unstable equilibrium allows one to find the threshold to the instability as well as the nonlinear evolution of beam phase space above threshold. In cases where we only want to know the threshold current, a linearized Vlasov (LV) analysis can be used. We have developed a computer code that numerically solves the LV problem. The method for finding the threshold to instability in this case begins by finding the equilibrium density distribution, $\psi_0(q, p)$, through the solution of the Haïssinski equation. We linearize the Vlasov equation about this distribution, taking $\psi(\theta, q, p) = \psi_0(q, p) + \psi_1(\theta, q, p)$, and assuming that $|\psi_1| \ll |\psi_0|$. The linearized Vlasov equation takes the form

$$\frac{\partial \psi_1}{\partial \theta} - p\frac{\partial \psi_1}{\partial q} + K_0(q)\frac{\partial \psi_1}{\partial p} + K_1(\theta, q)\frac{\partial \psi_0}{\partial p} = 0,$$

$$K_0(q) = q - I \int_{-\infty}^{\infty} dq' dp \, \psi_0(q', p) \, w(q - q'), \qquad (4)$$

$$K_1(\theta, q) = -I \int_{-\infty}^{\infty} dq' dp \, \psi_1(\theta, q', p) \, w(q - q').$$

Eq. 4 is solved numerically on a mesh in q-p space, starting from a randomly generated initial distribution function ψ_1. Typically the mesh, in each direction, spans ± 6 and contains $\sim$ 250–300 points. If the system is unstable, after a sufficient time the evolution becomes dominated by the fastest exponentially growing mode. The growth rate Γ of the instability is found numerically by fitting $e^{\Gamma\theta}$ to the time evolution of the system. The code also computes the phase portrait of the unstable mode. Typically we take 100 time steps per synchrotron period.

The numerical algorithm is implemented in Mathematica. The wake function in the Vlasov equation can be input as an arbitrary Mathematica function, and includes predefined resistive, inductive, and CSR wakes. The code was tested on an SLC damping ring wake, and the results compared well with results of other programs.

CALCULATIONS

$Q = 1$ *Resonator Wake*

The $Q = 1$ resonator wake has often been used to model the impedance of a ring. The resonator wake is non-zero only for negative argument (the test particle behind the driving charge); it is given by

$$w(q) = W H(-q)\, e^{\Omega q/2}\Big(\cos \sqrt{3}\Omega q/2 + \frac{\sin \sqrt{3}\Omega q/2}{\sqrt{3}}\Big), \tag{5}$$

with parameters amplitude W and $\Omega \equiv \omega_r \sigma_{z0}/c$, with ω_r the resonator frequency; here $H(x)$ is the unit step function [$H(x) = 1$ for $x > 0$, $= 0$ for $x < 0$].

We have performed stability calculations for this model, with Ω over the range [.25, 2.0]. Analysis shows that the dynamics described by the Vlasov equation in this case, and hence the threshold of the instability, depend only on two dimensionless parameters, Ω and $S_{\mathrm{res}} = IW$. In Fig. 1 we plot the resulting threshold value of S_{res} *vs.* Ω, where the VFP results are given by blue symbols (that are joined by straight lines), those of LV are in red (an error bar indicates uncertainty in result). In general we see good agreement between the results of the two methods, and also reasonably good agreement when compared to the earlier results of Oide and Yokoya and others.

At low Ω the beam preferentially samples the capacitive part of the impedance, at high Ω the inductive part. Thus at *e.g.* $\Omega = 0.5$ the bunch length of the Haïssinski solution at threshold (of the main part of the beam; the bunch here has a long tail) is shorter than nominal, whereas at $\Omega = 2.0$ it is longer. For $\Omega = 2.0$ the bunch shape appears to be approaching a high-Ω asymptotic limit, one that is not the nominal Gaussian. If this assumption is correct, then the asymptotic shape is given by the Haïssinski solution for an inductive wake, *specif.* the solution—that has unit area—to $\lambda' = -q\lambda/(1 + S_{res}\lambda/\Omega^2)$, with S_{res}/Ω^2 a constant. Performing more VFP calculations for $\Omega = 2.25, 2.5$, we find that the threshold results (for $\Omega \gtrsim 2.0$) are reasonably consistent with this assumption when taking $S_{res}/\Omega^2 = 6.7$.
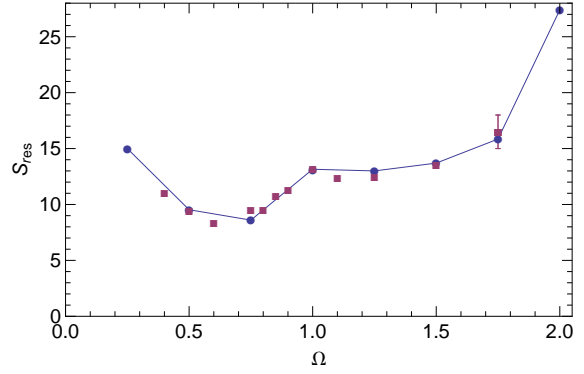

Figure 1: For the $Q = 1$ resonator wake, threshold value of S_{res} *vs.* resonator frequency Ω. Symbols give results of the VFP solver (blue) and the LV code (red).

(Incidentally, note that the Boussard criterion [6], for the resonator in the high-Ω regime, reads $S_{res}/\Omega^2 = \sqrt{2\pi}$.)

CSR Wake

We consider the CSR wakefield generated by an electron moving on a circular orbit with bending radius ρ in the middle of two parallel plates [7]. In the case of no shielding the wake is non-zero only for positive q (*i.e.* the test particle ahead of the driving charge); it is given by

$$w_0(q) = -\frac{4\pi}{3^{4/3}} H(q) \frac{\rho^{1/3}}{(q\sigma_{z0})^{4/3}}. \tag{6}$$

This wake is singular and requires special care. In the simulations, we obtain the bunch wake v_{ind} by convoluting with the bunch shape λ. For such a singular wake, however, we integrate by parts and discard the boundary term; *i.e.* we let the bunch wake $v_{\mathrm{ind}}(q) = \int s(q')\lambda'(q - q')\, dq'$, where $s(q) = \int_{-\infty}^q w(q')\, dq'$ and λ' is the derivative of the bunch distribution (for a justification, see *e.g.* Ref. [8]). Because of the λ' in the integral, simulation with such a wake is more sensitive to numerical errors or noise, and obtaining reliable results becomes more challenging.

With shielding, the wake $w(q) = w_0(q) + w_1(q)$, with

$$w_1(q) = -\rho^{1/3}\Big(\frac{\Pi}{\sigma_{z0}}\Big)^{4/3} G(\Pi q), \tag{7}$$

where the shielding parameter $\Pi = \sigma_{z0}\rho^{1/2}/h^{3/2}$, and $2h$ is the separation between the two plates. The term $w_1(q)$ is the contribution to the wake of the image charges generated by the metal plates; note that it is in general non-zero for both signs of argument. The function G is given by

$$G(\zeta) = 8\pi \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \frac{Y_k(\zeta)[3 - Y_k(\zeta)]}{[1 + Y_k(\zeta)]^3}, \tag{8}$$

where Y_k is a root of the equation

$$Y_k - \frac{3\zeta}{k^{3/2}} Y_k^{1/4} - 3 = 0. \tag{9}$$

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This equation has two real, positive roots and two complex roots. We choose the smaller real root when $\zeta < 0$, the larger real root when $\zeta > 0$. Normally, in the simulations, we sum k up to 25.

For the CSR wake, the beam dynamics depend only on two dimensionless parameters, Π and $S_{\text{csr}} = I\rho^{1/3}/\sigma_{z0}^{4/3}$.

We have performed stability calculations for this model, for shielding parameter Π up to 15. The threshold results are given in Fig. 2. We find good agreement between the VFP and LV results. With no shielding $S_{\text{csr}} = 0.50$; there is a deep dip in the curve in the vicinity of $\Pi = 0.7$ where $S_{\text{csr}} = 0.17$; then most of the results follow closely the straight line $S_{\text{csr}} = 0.5 + 0.12\Pi$ (the dashes). In Fig. 3 we plot the Haïssinski solution at threshold and the wake induced voltage v_{ind} for selected values of shielding parameter, Π. With no shielding the bunch shape is markedly triangular; with increasing shielding it moves gradually toward that of the unperturbed Gaussian. We see that v_{ind}, in amplitude, drops quickly as Π increases from zero; by $\Pi \gtrsim 1.5$ this function, in addition, has become largely inductive.
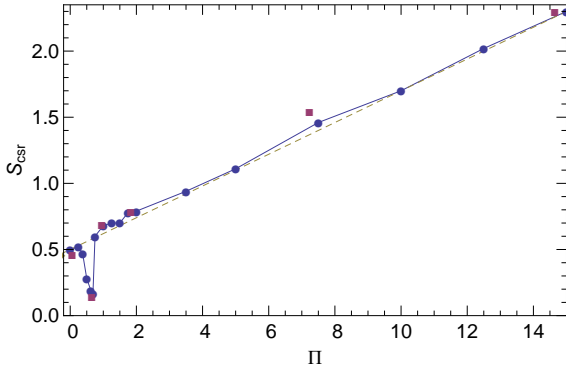

Figure 2: For the CSR wake, threshold value of S_{csr} *vs.* shielding parameter, $\Pi = \rho^{1/2}\sigma_{z0}/h^{3/2}$. Symbols give results of the VFP solver (blue) and the LV code (red).

Resistive Wall Wake

Here we consider the ring impedance to be what one finds on the axis of a round, metallic beam pipe of radius a, conductivity σ_c, and length $\mathcal{C}$:

$$w(q) = H(-q)\frac{\mathcal{C}}{2\pi a}\sqrt{\frac{c}{\sigma_c}}\frac{1}{(-q\sigma_{z0})^{3/2}}. \qquad (10)$$

We see that this wake is also singular.

For the resistive wall wake the instability threshold is determined by the dimensionless parameter $S_{\text{rw}} = Iw(-1)$. Simulations with the VFP code give a threshold at $S_{\text{rw}} = 9.15$. Unfortunately, the LV code could not reproduce this result. We believe that the reason for the discrepancy between the VFP and LV codes is due to the singular nature of the wake (10), and plan to investigate it in the future.
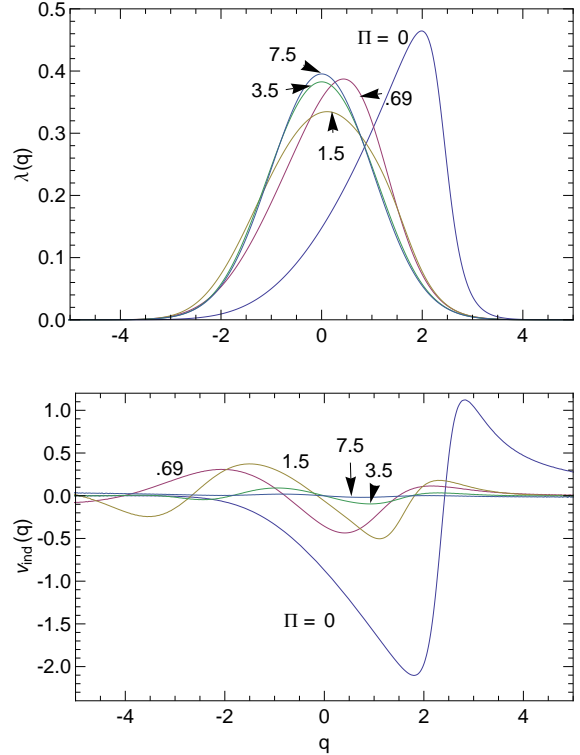

Figure 3: Haïssinski solution at threshold and wake induced voltage for the cases $\Pi = 0.0, 0.69, 1.5, 3.5, 7.5$. Note that bunch head is to the right.

CONCLUSION

For the task of finding the microwave threshold, we have shown that our Vlasov-Fokker-Planck and Linearized Vlasov solvers agree quite well when applied to impedance models of (1) a $Q = 1$ resonator and (2) shielded CSR. For shielded CSR we have shown that only two dimensionless parameters, the shielding parameter Π and the strength parameter S_{csr} are needed to describe the system; we have, in addition, shown that the threshold dependence on Π, except for a deep dip in the vicinity of $\Pi = 0.7$, is to good approximation linear and given by $S_{\text{csr}} = 0.5 + 0.12\Pi$.

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