Constructing the Tree-Level Yang-Mills S-matrix Using Complex Factorization

Philip C. Schuster[1], Natalia Toro[2]

[1] *Theory Group, SLAC National Accelerator Laboratory,*
Menlo Park, CA 94025, USA

[2] *Stanford Institute for Theoretical Physics, Stanford University,*
Stanford, CA 94305, USA

Abstract

A remarkable connection between BCFW recursion relations and constraints on the S-matrix was made by Benincasa and Cachazo in 0705.4305, who noted that mutual consistency of different BCFW constructions of four-particle amplitudes generates non-trivial (but familiar) constraints on three-particle coupling constants — these include gauge invariance, the equivalence principle, and the lack of non-trivial couplings for spins > 2. These constraints can also be derived with weaker assumptions, by demanding the existence of four-point amplitudes that factorize properly in all unitarity limits with complex momenta. From this starting point, we show that the BCFW prescription can be interpreted as an algorithm for *fully* constructing a tree-level S-matrix, and that complex factorization of general BCFW amplitudes follows from the factorization of four-particle amplitudes. The allowed set of BCFW deformations is identified, formulated entirely as a statement on the three-particle sector, and using *only* complex factorization as a guide. Consequently, our analysis based on the physical consistency of the S-matrix is entirely independent of field theory. We analyze the case of pure Yang-Mills, and outline a proof for gravity. For Yang-Mills, we also show that the well-known scaling behavior of BCFW-deformed amplitudes at large z is a simple consequence of factorization. For gravity, factorization in certain channels requires asymptotic behavior $\sim 1/z^2$.

1 Introduction

Gauge theories represent a highly constrained framework for describing interacting spin-1 particles. Weinberg's seminal papers of 1964-65 [1, 2, 3] demonstrated that many of these constraints can be seen as inevitable consequences of requiring scattering amplitudes to be both unitary and Lorentz-invariant. In particular, both charge conservation and Maxwell's equations follow from S-matrix arguments alone. Likewise, consistency of the spin-2 S-matrix requires the equivalence principle and Einstein's equations at tree-level.

A concrete and beautiful confirmation that much of the structure of gauge and gravity theories is contained in their S-matrix is the existence of purely on-shell recursion relations for gauge theory [4, 5] and gravity [6, 7, 8, 9], which allow the calculation of on-shell scattering amplitudes entirely in terms of lower-point on-shell amplitudes. However, these relations have always been derived from the underlying local field theories; their relation to S-matrix consistency arguments such as [1, 2, 3] were unclear.

A striking connection between BCFW recursion and consistency of the spin-1 and spin-2 S-matrix was uncovered by Benincasa and Cachazo [10]. These authors introduced a "four-particle test"— the requirement that two BCFW shifts generate the same answer for any four-point amplitude. This requirement could only be met when the "coupling constants" (coefficients of three-point amplitudes) satisfied non-trivial relations. For instance, it was shown in [10] that the four-particle test generates the Jacobi identity for spin-1 particles. However, the four-particle test only makes sense when each of the BCFW constructions involved is valid (i.e. field theory amplitudes vanish in the limit of large shifts). The constraints obtained are reminiscent of those identified by Weinberg as consequences of Lorentz invariance and unitarity. The physical origin of the constraints obtained in [10] is somewhat unclear, however, as the argument assumes the validity of BCFW constructions. An open question in [10] was whether new constraints would be found by applying the same criterion to higher-point amplitudes. This question is also considered in [11].

After a brief review of the spinor-helicity formalism and BCFW recursion in Section 2, we show in Section 3 that the conditions from the four-particle test can be understood as the result of demanding Lorentz invariance and an analytic continuation of unitarity to complex momenta, "complex factorization". The latter is simply the requirement that amplitudes factorize into sub-amplitudes when an intermediate complex momentum can go on-shell. A byproduct of this treatment is a simple criterion for BCFW shifts — shifts of the $[+-\rangle$ type (the only invalid shift in gauge theories) cannot possibly satisfy complex factorization in all limits.

BCFW recursion relations provide a formula for generating a set of arbitrary high-point amplitudes for spin-1 massless interacting particles. In Section 4 we show that amplitudes

generated by BCFW recursion are guaranteed to satisfy complex factorization, provided the four-particle consistency requirements on couplings and shifts are satisfied. We also provide an S-matrix derivation of the large-z scaling of BCFW amplitudes (as z^3 for $[+-\rangle$ deformations and $1/z$ for all others), which is used in our factorization argument. Thus, we have taken the programs described above fully on-shell: conditions on the structure of spin-1 amplitudes, and a construction of higher-point amplitudes from lower-point ones are justified with no reference to a local gauge theory.

We also outline the analogous result for gravity, and highlight the differences. One piece of the proof is missing: the argument relies on the scaling behavior of n-point spin-2 amplitudes at large BCF shift parameters z (i.e. growing as z^6 for $[+-\rangle$ shifts, and falling as z^{-2} for all others); an on-shell proof of this scaling result would complete the proof of factorization for BCFW gravity amplitudes.

2   Review of Formalism and Notation

In this section, we review two essential elements of the constructions in the remainder of this paper: the spinor-helicity formalism [12, 13, 14, 15] and BCFW recursion relations [5, 4, 16, 17]. The spinor-helicity formalism will be useful as a means of writing down manifestly Lorentz-invariant amplitudes for higher-spin massless particles, involving only the physical interacting degrees of freedom. BCFW recursion will be used as a means of generating amplitudes that satisfy unitarity in a subset of poles; however, the amplitudes thus constructed are not manifestly unitary in *all* limits — demanding unitarity in all remaining channels will give constraints on the fundamental three-particle couplings and on the set of allowed BCFW shifts, as discussed in Section 3.

2.1   Spinor-Helicity Formalism and Three-Point Amplitudes

Any four-vector can be related to a bispinor by $p_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha} p_\mu$; when $p^2 = 0$, $p_{\alpha\dot\alpha}$ has rank one, and we can write $p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$. The spinors λ_α and $\tilde\lambda_{\dot\alpha}$ are uniquely determined by p_μ, up to a complex rescaling $(\lambda, \tilde\lambda) \to (z\lambda, \frac{1}{z}\tilde\lambda)$, and they transform in the $(1/2, 0)$ and $(0, 1/2)$ representations of the Lorentz group; they also transform simply under helicity rotations, $(\lambda, \tilde\lambda) \to (e^{-i\theta/2}\lambda, e^{+i\theta/2}\tilde\lambda)$. In this notation, two-particle momentum invariants can be written as

$$p_i.p_j = \langle ij \rangle [ij]; \tag{1}$$

momentum conservation is the bi-spinor condition

$$\sum_i |i\rangle|i] = 0, \tag{2}$$

and the Schouten identity

$$[ij][k] + [jk][i] + [ki][j] = 0 \tag{3}$$

follows from the antisymmetry of spinor products and the fact that spinors live in a two-dimensional vector space.

The transformation properties of $\lambda, \tilde{\lambda}$ under both Lorentz transformations and little group helicity rotations make them very useful for formulating an on-shell theory of spin-s particles — all of the Lorentz transformation properties of states are neatly encoded in the spinors. The helicity rotation operator associated with any external momentum i is

$$H_i = -\frac{1}{2}\lambda\frac{d}{d\lambda} + \frac{1}{2}\tilde{\lambda}\frac{d}{d\tilde{\lambda}}; \tag{4}$$

Lorentz-invariant of an n-point amplitude in which the i'th particle has helicity h_i is guaranteed if

$$H_i A = h_i A \tag{5}$$

for all legs, and A has no free spinor indices. Amplitudes are then naturally expressed in terms of Lorentz invariant "holomorphic" spinor products $\langle\lambda\mu\rangle = \epsilon_{\alpha\beta}\lambda^\alpha\mu^\beta$ and "anti-holomorphic" products $[\tilde{\lambda}\tilde{\mu}] = \epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\lambda}^{\dot{\alpha}}\tilde{\mu}^{\dot{\beta}}$.

In fact, demanding the helicity transformation properties above fixes three-particle amplitudes completely up to coupling coefficients [10]. Note that the on-shell condition for three particles can be satisfied in two ways, either $[12] = [13] = [23] = 0$ or $\langle 12 \rangle = \langle 13 \rangle = \langle 23 \rangle = 0$; though all real momentum invariants vanish in this limit, the spinor products $\langle ij \rangle$ can be non-zero in the first case, as can $[ij]$ in the latter case. For example, a theory of several interacting massless spin-s $(s \geq 1)$ particles has three-particle amplitudes (fixed by Lorentz invariance),

$$
\begin{aligned}
A_3^{(a)}(i^{+,\alpha_i}, j^{+,\alpha_j}, k^{-,\alpha_k}) &= f_{\alpha_i\alpha_j\alpha_k}^{(a)} \left(\frac{[ij]^3}{[jk][ki]} \right)^s, \\
A_3^{(h)}(i^{-,\alpha_i}, j^{-,\alpha_j}, k^{+,\alpha_k}) &= f_{\alpha_i\alpha_j\alpha_k}^{(h)} \left(\frac{\langle ij \rangle^3}{\langle jk \rangle\langle ki \rangle} \right)^s,
\end{aligned} \tag{6}
$$

where $+, -$ refers to helicity, indexes $\alpha_i, \alpha_j, \alpha_k$ label species of particles, and i, j, k label four-momentum spinors. Demanding the amplitudes have crossing symmetry and are invariant under exchange of states requires the $f_{\alpha\beta\gamma}^{(a,h)}$ be completely anti-symmetric [10]. Under parity, $+$ and $-$ are exchanged. Demanding that pure spin-s interactions be invariant under parity requires $f^{(a)} = f^{(h)}$, an assumption that we'll make throughout this paper. To minimize the complexity of notation, we will often use the spinor labels i, j, k to refer to species labels. There is another set of three-particle amplitudes $A^{(a)}(i^+, j^+, k^+)$ and $A^{(h)}(i^-, j^-, k^-)$ consistent with Lorentz invariance; however we will assume their coefficients are zero.

For later use, we also note the three-particle amplitudes for one spin-s particles with a collection of scalars:

$$A_3(i^{a_i}, j^{a_j}; k^{+,\alpha_k}) = \kappa_{a_i a_j}^{\alpha_k} \left(\frac{[ki][jk]}{[ij]} \right)^s,$$

$$A_3(i^{a_i}, j^{a_j}; k^{-,\alpha_k}) = \kappa_{a_i a_j}^{\alpha_k} \left(\frac{\langle ki \rangle \langle jk \rangle}{\langle ij \rangle} \right)^s, \tag{7}$$

where we have again assumed a parity, and a_i labels species of scalars. In this case as well, the k_{ij}^a should be anti-symmetric in i, j in order for the amplitudes to be symmetric under interchange of scalars.

2.2 BCFW Recursion

The BCFW formalism [5, 4, 16, 17] can be expressed very concisely in spinor-helicity formalism, but we first summarize it directly in four-momentum space. Consider the amplitude $A(p_1, h_1, p_2, h_2, \ldots, p_n, h_n)$, where h_i labels helicity. The basic idea of the BCFW formalism is to deform amplitudes into a function of a single complex variable z, and then re-express the amplitude in terms of residues.

The simplest complex deformation of the amplitude that keeps all momenta on-shell is a deformation involving only two external legs. Consider the legs p_1 and p_2. Choose an arbitrary null four-vector q such that $q.p_1 = q.p_2 = 0$. Then we can deform by,

$$\hat{p}_1(z) = p_1 + zq, \ \hat{p}_2(z) = p_2 - zq, \tag{8}$$

which keeps $p_1(z)$ and $p_2(z)$ null. In the spinor-helicity language, this is satisfied by this isimplemented particularly easy to implement by $q = |1\rangle|2]$, so that the BCFW shift only deforms one of the two spinors associated with each leg:

$$|\hat{1}](z) \equiv |1] + z|2], \quad |\hat{2}\rangle(z) = |2\rangle - z|1\rangle, \quad (|1\rangle, |2] \text{ fixed}) \tag{9}$$

We will call this the $[1, 2\rangle$ shift. Since $q.p_1 = q.p_2 = q^2 = 0$, any kinematic invariant $(\sum p_i)^2$ is at most linear in z. These deformation naturally make the full on-shell amplitude a function of z,

$$A(z) \equiv A(\hat{p}_1(z), h_1, \hat{p}_2(z), h_2, \ldots, p_n, h_n). \tag{10}$$

At tree level, $A(z)$ is a rational function of z and so is fully determined by its poles and behavior as $z \to 0$.

If we assume tree-level factorization, the only poles in the amplitude arise from propagators going on-shell, and their residues are fully determined by lower-point amplitudes.

If $A(z) \to 0$ as $z \to 0$, the amplitude is then fully determined by products of lower-point on-shell amplitudes. In this special case, we obtain the BCFW recursion relation [4, 5],

$$A_{BCF}(1, 2, \ldots, N) \equiv \sum_{L,R} \sum_{h_K} A(\hat{1}, L, -\hat{K}^{-h_K}) \frac{1}{K^2} A(\hat{K}^{h_K}, \hat{2}, R), \qquad (11)$$

where the sum is over partitions of $3, 4, \ldots, N$ into two sets L and R, $K = \sum_{i \in \{1, L\}} p_i$ is the (generically non-null) momentum flowing out of the right factor before the BCF shift, and $\hat{K} = \hat{p}_1(z) + \sum_{i \in L} p_i$ is the null momentum flowing out of the right graph *after* the BCF shift.

This expression transforms properly under all helicity rotations (4) $(H_i A^{(n)} = h_i A^n)$ so long as the same is true of the lower-point amplitudes from which it is generated. Thus, Lorentz invariance of BCFW amplitudes is manifest.

3   Consistency of Four-Point Amplitudes

In this section, we study the structure of consistent four-point amplitudes for four massless particles, satisfying two conditions — Lorentz invariance and a strong version of tree-level unitarity *in unconstrained complex momenta*, which we will refer to as complex factorization. We begin by explaining the conditions in some detail, then build amplitudes for specific examples. The general pattern that emerges is that, for high-spin theories, Lorentz invariance and factorization in a subset of channels fully constrains the leading behavior of the amplitude (in the sense of power-counting). This amplitude will only be able to satisfy unitarity in the remaining channel(s) if the coefficients of different three-point amplitudes are related. [1]

Our results are closely related to those of [10] (and very much motivated by that work), but we make a significantly weaker set of assumptions. Specifically, the authors of [10] find conditions that three-point amplitudes must satisfy *if four-point amplitudes can be constructed by a BCFW recursion*. Therefore, their argument relies on field-theoretic derivations of the validity of different BCFW shifts. Since our goal is to motivate the self-consistency of BCFW constructions independent of field theory, and find an S-matrix criterion for their validity, it is important that we *do not* assume this. However, it is easy to see that when the assumptions of [10] are satisfied, the two methods will give the same consistency conditions on three-point amplitudes.

3.1   Setup and Interacting Spin-1

We begin by setting up the constraints on four-particle scattering amplitudes from Lorentz invariance, and explaining the requirement of complex factorization. We will consider the

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[1]The authors thank N. Arkani-Hamed for suggesting this approach.

amplitude for scattering of four spin-1 particles as an explicit example, and derive the Jacobi identity.

We first demand that the four-particle scattering amplitude be a Lorentz scalar, and tranform as a product of one-particle states under independent helicity rotations of each. This condition is easily imposed in the spinor-helicity formalism — the only non-vanishing scalar invariants (under Lorentz invariance *and* individual helicity rotations) are

$$s = (p_1 + p_2)^2 = \langle 12 \rangle [12] = \langle 34 \rangle [34], \tag{12}$$

$$t = (p_1 + p_3)^2 = \langle 13 \rangle [13] = \langle 24 \rangle [24], \tag{13}$$

$$u = (p_1 + p_4)^2 = \langle 14 \rangle [14] = \langle 23 \rangle [23] = -s - t. \tag{14}$$

$$\tag{15}$$

If we define an arbitrary particular solution $\mathcal{H}(1, 2, 3, 4)$ that transforms correctly under helicity rotations, then a general four-point amplitude has the form

$$A(1^{h_1}, 2^{h_2}, 3^{h_3}, 4^{h_4}) = \mathcal{H}(1, 2, 3, 4) \times f(s, t, u) \tag{16}$$

for some function f. It will be most convenient to choose an $\mathcal{H}$ that is polynomial in the spinor-product invariants, but does not contain any Mandelstam scalar invariants. For example, for four spin-1 particles of helicities $+, +, -, -$,

$$\mathcal{H}(1-, 2-, 3+, 4+) = \langle 12 \rangle^2 [34]^2. \tag{17}$$

The requirement of complex factorization is the familiar tree-level unitarity — when any sum of momenta in a diagram goes on-shell, it gives rise to a single pole, associated with splitting the diagram in two, e.g.

$$\lim_{s \to 0} s \times A(1, 2, 3, 4) = \sum_{h, a} A(1, 2, -P_{12}^{-h}) A(3, 4, P_{12}^{h}), \tag{18}$$

where we sum over all allowed intermediate helicities, and a possible species index a (we will drop a in much of the discussion). We give this familiar criterion a new name, because we will require it to hold at arbitrary complex momenta. We have seen already that for particles of non-zero spin, there are two distinct three-point amplitudes in different on-shell limits — an $A^{(h)}(1, 2, 3)$ when $[ij] = 0$ and an $A^{(a)}(1, 2, 3)$ when $\langle ij \rangle = 0$. If $h_1 + h_2 + h_3 \neq 0$, then both $A^{(a)}$ and $A^{(h)}$ vanish in the real-momentum collinear limit, where $[ij]$ and $\langle ij \rangle$ both go to zero. This is a more general complexification of momenta than the usual analytic continuation of the Mandelstam variables, and we will see in a moment that there are cases where a real-momentum $s \to 0$ limit is trivial, but one of the two complex directions is not.

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Which three-point amplitudes appear in the unitarity condition 18 depends on how we take the limit $s \to 0$. We can take either

$$\lim_{[12],\langle 34\rangle \to 0} s \times A(1,2,3,4) = \sum_h A^{(h)}(1,2,-P_{12}^{-h})A^{(a)}(P_{12}^h,3,4) \tag{19}$$

or

$$\lim_{\langle 12\rangle,[34] \to 0} s \times A(1,2,3,4) = \sum_h A^{(a)}(1,2,-P_{12}^{-h})A^{(h)}(P_{12}^h,3,4). \tag{20}$$

The limit $[12] \to 0$ and $[34] \to 0$ also enforces either $[ij] \to 0$ for all i,j or a soft limit on one of the three legs; we will not demand unitarity in this limit. We can thus refer unambiguously to the two limits above as $[12] \to 0$ (19) or $\langle 12\rangle \to 0$ (20).

Having stated the requirement of complex factorization, let us apply it to the four-gauge-boson amplitude. One way of obtaining an expression that factorizes in the limits $[13] \to 0$ and $[14] \to 0$ is by using the BCFW formula, using a shift $|\hat{1}](z) \equiv |1] + z|2]$, $\qquad |\hat{2}\rangle(z) = |2\rangle - z|1\rangle$ (called a $[12\rangle$ shift). The two terms in the formula are shown in Figure 1 The result

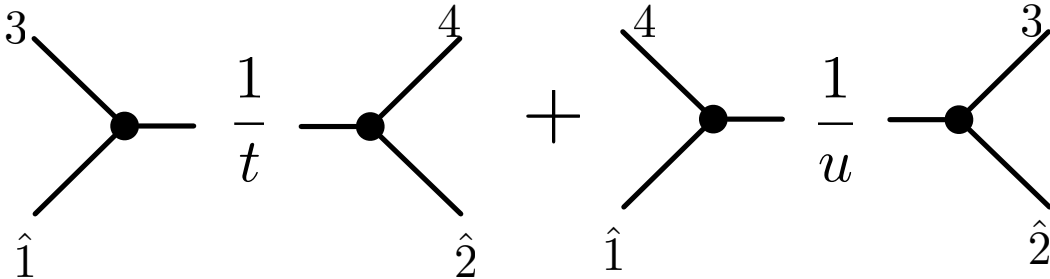

Figure 1: The two diagrams in the BCFW construction of a four-point amplitude $A(1,2,3,4)$ using a $[12\rangle$ shift. Both t- and u-channel poles are exposed in the left and right pieces respectively.

is

$$A_{BCF,[1,2\rangle}(1^-,2^-,3^+,4^+) = \langle 12\rangle^2[34]^2 \left[\frac{f_{\beta 13}f_{\beta 42}}{st} - \frac{f_{\alpha 14}f_{\alpha 32}}{su} \right], \tag{21}$$

where repeated indexes are summed. The fact that we have obtained this amplitude by a BCFW construction is quite incidental — it is the unique amplitude one can write down, involving only dimensionless coupling constants, that factorizes in the limits $[13] \to 0$ and $[14] \to 0$ and has no unphysical poles. For example, adding a term $\sim \frac{1}{(s+2t)(s+5t)}$ to the scalar function in brackets would introduce two single poles at unphysical locations, where no particle goes on-shell; adding a contribution $\sim 1/s^2$ would introduce an unphysical double singularity as $[12] \to 0$, and of course adding a term $\sim \frac{1}{st}$ or $\sim \frac{1}{ut}$ would change the t- and u-channel singularities.

One can check that the $\langle 13 \rangle$ and $\langle 14 \rangle \to 0$ singularities in (21) have the proper form, and they do. More interesting conditions come from the s-channel. We note that, because of the helicity factor, (21) has a pole as $[12] \to 0$ but not as $\langle 12 \rangle \to 0$. In fact, the real limit with $\langle 12 \rangle, [12]$ vanishing simultaneously *also* vanishes. But the limit $[12] \to 0$ is non-zero. Complex factorization (19) requires

$$\lim_{[12] \to 0} \langle 12 \rangle [12] A(1^-, 2^-, 3^+, 4^+) = A^{(h)}(1^-, 2^-, -P_{12}^{-h}) A^{(a)}(P_{12}^h, 3^+, 4^+), \qquad (22)$$

$$= \langle 12 \rangle^2 [34]^2 \left(\frac{f_{12\alpha} f_{34\alpha}}{t} \right), \qquad (23)$$

while A_{BCF} above has

$$\lim_{[12] \to 0} \langle 12 \rangle [12] A_{BCF}(1^-, 2^-, 3^+, 4^+) = \langle 12 \rangle^2 [34]^2 \frac{1}{t} (f_{\alpha 13} f_{\alpha 42} + f_{\alpha 14} f_{\alpha 32}) \qquad (24)$$

For the two expressions to agree, the couplings must satisfy a Jacobi Identity:

$$f_{\alpha_1 \alpha_4 \beta} f_{\beta \alpha_3 \alpha_2} + f_{\alpha_1 \alpha_3 \beta} f_{\beta \alpha_4 \alpha_2} + f_{12\beta} f_{\beta 34} = 0 \qquad (25)$$

(where we've restored the full species label for clarity). This result was also found in [10], by demanding agreement of two BCFW shifts involving different legs. It is clear why these give the same result: the two BCFW shifts are imposing factorization in different poles!

We have now obtained a four-point amplitude that can satisfy complex factorization in all channels; it is unique up to less singular terms, which by power-counting must have new, dimensionful couplings. Writing down *any* such amplitude required a relation between three-point coefficients. With this amplitude in hand, we can now check BCFW recursion relations explicitly, with different shifts. We have already seen that the $[1, 2 \rangle$ shift gives the correct amplitude; so do the $[1, 3 \rangle$ shift and all shifts of legs with helicities $[-- \rangle$, $[+- \rangle$, or $[++ \rangle$. However, if we try to build an amplitude with the $[3+, 1- \rangle$ shift, we find

$$A_{BCF;[31 \rangle}(1^-, 2^-, 3^+, 4^+) \propto \langle 12 \rangle^2 [34]^2 \frac{t^3}{s^4 u}. \qquad (26)$$

This satisfies factorization in the $[32]/\langle 14 \rangle$ and $[34]/\langle 12 \rangle$ limits ($A \sim 1/su$ as $u \to 0$ and is non-singular as $\langle 12 \rangle \to 0$), but has an unphysical fourth-power singularity as $[12] \to 0$ and no t-channel singularities. So we can see directly that this shift cannot produce physical amplitudes. Our finding that all shifts except $[+- \rangle$ are valid agrees with the gauge-theory result [4, 5].

3.2 Spin-1 With Matter

We can repeat this analysis for spin-1 interactions interacting with massless matter (for simplicity, we consider scalars) labelled by indexes a_i, with three-particle amplitudes given

by equation 7. Consider the four-particle amplitude $A(1^-, 2^+, 3, 4)$. We again use a $[12\rangle$ shift to build an amplitude, and again it is consistent with factorization in both t-channel limits and both u-channel limits (though the factorization as $\langle 1j \rangle \to 0$ does not follow from the BCFW construction, and must be checked explicitly):

$$A_{BCF}(1^-, 2^+, 3, 4) = \langle 13 \rangle \langle 14 \rangle [23][24] \left(\frac{\kappa_{3a}^1 \kappa_{a4}^2}{st} + \frac{\kappa_{4a}^1 \kappa_{a3}^2}{su} \right). \tag{27}$$

Again, any modification with the same energy-scaling would violate complex factorization in the t or u-channels, or introduce unphysical singularities elsewhere. As $[12] \to 0$, complex factorization requires

$$\lim_{[12] \to 0} \langle 12 \rangle [12] A(1^-, 2^+, 3, 4) = A^{(h)}(1^-, 2^+, K_{12}) A^{(a)}(-K_{12}, 3, 4) \tag{28}$$

$$= -\langle 13 \rangle \langle 14 \rangle [23][24] \left(\frac{f_{\alpha 12} \kappa_{34}^\alpha}{t} \right), \tag{29}$$

and in this case the $\langle 12 \rangle \to 0$ limit is identical. The limit of the BCFW construction is

$$\lim_{[12] \to 0} \langle 12 \rangle [12] A_{BCF}(1^-, 2^+, 3, 4) = \langle 13 \rangle \langle 14 \rangle [23][24] \tfrac{1}{t} (\kappa_{3a}^1 \kappa_{a4}^2 - \kappa_{4a}^1 \kappa_{a3}^2), \tag{30}$$

which satisfies factorization if

$$\kappa_{a_3 b}^{\alpha_1} \kappa_{b a_4}^{\alpha_2} - \kappa_{a_3 b}^{\alpha_2} \kappa_{b a_4}^{\alpha_1} = -f_{\beta \alpha_1 \alpha_2} \kappa_{a_3 a_4}^\beta \tag{31}$$

(where we have again restored full species labels). We recognize this as the requirement that the κ_{ij}^α furnish a representation of the Lie Algebra defined by the $f_{\alpha\beta\gamma}$.

For self-interacting scalars with a three-point amplitude $A(a_i, a_j, a_k) = \gamma_{a_i a_j a_k}$, we can apply s-channel factorization to the amplitude $A(1^-, 2, 3, 4)$ constructed with an $[12\rangle$ shift, and find that the scalar interaction must satisfy charge conservation. We do not show this explicitly here. One could also check using this amplitude or the previous one that at four-point, shifts $[-, 0\rangle$ and $[0, +\rangle$ are valid but shifts $[+, 0\rangle$ or $[0, -\rangle$ are not, consistent with the general field-theory result of [18].

3.3 Spin-2

We can repeat the above analysis for massless interacting spin-2 particles. First consider the amplitude $A(1^-, 2^-, 3^+, 4^+)$ constructed from a $[12\rangle$ shift ($+$ and $-$ labels $+2$ and -2 helicity states). Using the BCFW ansatz, we obtain,

$$A_{BCF}(1^-, 2^-, 3^+, 4^+) = [34]^4 \langle 12 \rangle^4 \left(\frac{f_{13\alpha} f_{24\alpha}}{s^2 t} + \frac{f_{14\alpha} f_{23\alpha}}{s^2 u} \right). \tag{32}$$

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Interestingly, each term separately has a double pole as $s \to 0$ (taking $[12] \to 0$). Complex factorization in this channel requires

$$\lim_{[12]\to 0} \langle 12 \rangle [12] A_{BCF}(1^-, 2^-, 3^+, 4^+) = \lim_{[12]\to 0} \langle 12 \rangle^4 [34]^4 \left(\frac{f_{13\alpha} f_{24\alpha} \cdot u + f_{14\alpha} f_{23\alpha} \cdot t}{stu} \right),$$

$$= \langle 12 \rangle^4 [34]^4 \frac{f_{12\alpha} f_{34\alpha}}{t^2}. \tag{33}$$

To make the double pole vanish, we require

$$f_{14\alpha} f_{23\alpha} = f_{13\alpha} f_{24\alpha}, \tag{34}$$

(so that $t + u = -s$ in the numerator cancels an s in the denominator), while getting the correct coefficient on the single pole requires additionally

$$f_{13\alpha} f_{24\alpha} = f_{12\alpha} f_{34\alpha}. \tag{35}$$

As pointed out in [10], these relations imply that the $f_{\alpha\beta\gamma}$ furnish an algebra that is commutative and associative, and therefore any multi-graviton theory can be reduced to a theory of self-interacting gravitons that decouple from one another. From here on, we will consider only one spin-2 species.

We next consider a single massless spin-2 particle coupled to a set of interacting scalars. As before, the fundamental three-particle scalar interactions are just constants $A(a_i, a_j, a_k) = \gamma_{a_i a_j a_k}$. The fundamental spin-2-scalar three-particle amplitudes are

$$A_3(i^{a_i}, j^{a_i}; k^+) = g_{a_i} \left(\frac{[ki][jk]}{[ij]} \right)^2,$$

$$A_3(i^{a_i}, j^{a_i}; k^-) = g_{a_i} \left(\frac{\langle ki \rangle \langle jk \rangle}{\langle ij \rangle} \right)^2, \tag{36}$$

(by symmetry, the spin-2-scalar couplings must be symmetric in the a_i, so we have diagonalized them). We now consider building the amplitude $A(1^-, 2, 3, 4)$ using a $[12\rangle$ shift. The BCFW construction gives,

$$A_{BCF}(1^-, 2, 3, 4) = \langle 14 \rangle^2 \langle 13 \rangle^2 [34]^2 \gamma_{234} \left(\frac{g_{33}u + g_{44}t}{s^2 tu} \right). \tag{37}$$

Demanding complex factorization in the $[12] \to 0$ channel requires,

$$\lim_{[12]\to 0} \langle 12 \rangle [12] A_{BCF}(1^-, 2, 3, 4) = \lim_{[12]\to 0} \langle 14 \rangle^2 \langle 13 \rangle^2 [34]^2 \gamma_{234} \left(\frac{g_{44} - g_{33}}{su} - \frac{g_{33}}{tu} \right),$$

$$= \langle 14 \rangle^2 \langle 13 \rangle^2 [34]^2 \gamma_{234} g_{22} \frac{1}{t^2}. \tag{38}$$

Again, there is a potential double pole. The double piece vanishes and the correct factorization limit is obtained provided $g_{22} = g_{33} = g_{44}$. This is a special case of the principle of equivalence for gravity –interacting matter couples to gravity with the same strength.

4 Factorization of Higher-Point Amplitudes

In this section, we will show that n-point amplitudes obtained by BCFW recursion satisfy the physical requirements of complex-momentum unitarity (factorization). We have already noted that Lorentz invariance is manifest in the construction, so long as the recursion begins from Lorentz-invariant primitive three-point amplitudes. We now describe the factorization requirement in more detail, for a theory of massless particles at tree-level.

In the limit that the summed momentum of a collection of legs, $P_I = \sum_{i \in I} p_i$ becomes null, unitarity requires that the amplitude have a single pole, whose residue is a product of two sub-amplitudes:

$$\lim_{P_I^2 \to 0} P_I^2 A(1, \ldots, n) = \sum_{h,a} A(I, -P_I^{-h,a}) A(P_I^{h,a}, \bar{I}), \tag{39}$$

where $\bar{I}$ is the set of legs *not* in I, which must contain at least two legs (we can label the same limit equivalently by either I or $\bar{I}$). When I contains only two legs,

$$(p_i + p_j)^2 = 2 p_i \cdot p_j = \lambda_i \lambda_j \, \tilde{\lambda}_i \tilde{\lambda}_j = \langle ij \rangle [ij] \tag{40}$$

goes to zero if either $\langle ij \rangle \to 0$ or $[ij] \to 0$; for complex momenta we can obtain limits in which one spinor product goes to zero while the other remains finite. This results in distinct factorization formulas, involving $A^{(a)}(i, j, -P_{ij})$ when $\langle ij \rangle \to 0$ and $A^{(h)}(i, j, -P_{ij})$ when $[ij] \to 0$.

Figure 2: A diagrammatic picture for the requirement of complex factorization.

The BCFW recursion singles out the two shifted legs, which we will label 1 and 2, and treats different poles differently. The BCFW formula (11) manifestly satisfies the factorization requirement (39) in each of the poles that appears in the sum, namely $P_{1I}^2 \to 0$ for some set of legs I. The explicit propagator $1/K^2$ in the term of (11) with $L = I$ becomes singular. Nothing else in the BCFW expression can be singular in this limit — the sub-amplitudes are evaluated at shifted momenta, at which no kinematic invariants inside the sub-amplitudes vanish. If either side of the pole has only two elements, we must specify which spinor product is exposed by the BCFW shift: if I has only one element i, the BCFW shift exposes the pole $[1i] \to 0$, and if $\bar{I}$ contains only the legs $2, j$, the BCFW shift exposes the pole $\langle 2j \rangle \to 0$.
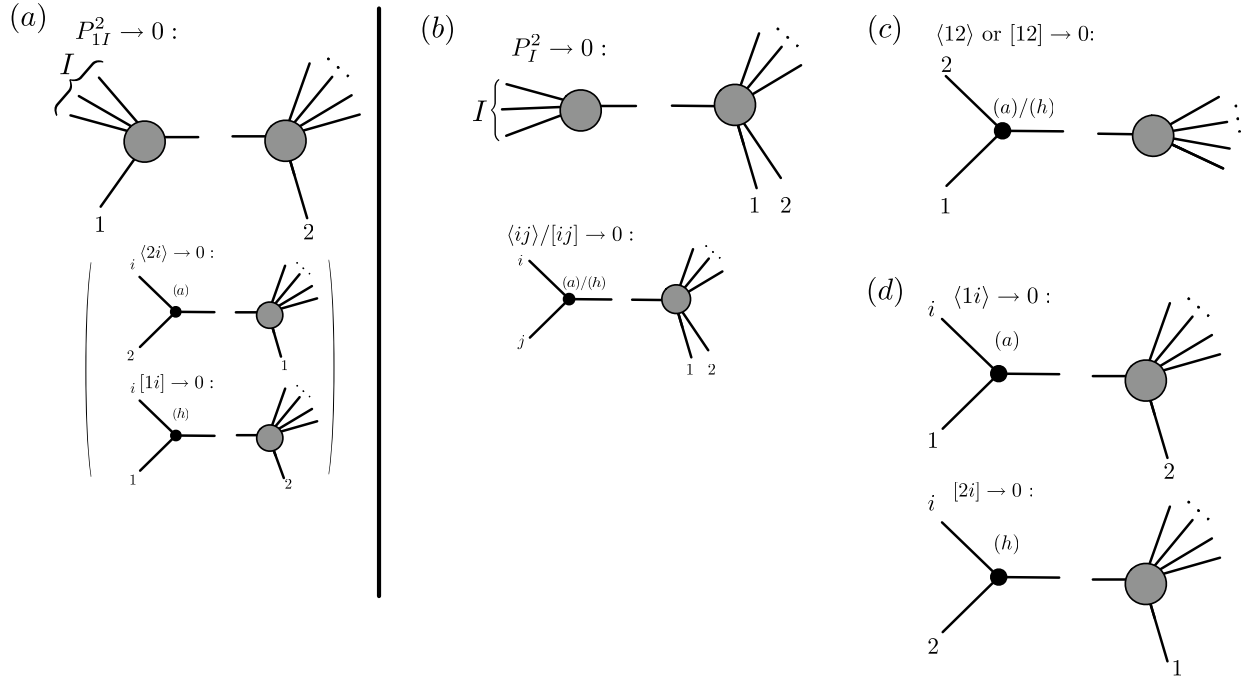
Figure 3: A graphical classification of poles in a general amplitude. The BCFW construction manifestly has proper factorization on the poles (a) to the left of the thick line. The three classes of diagrams to the right of the line must be checked explicitly. We discuss the "unshifted" poles (b) in Sec. 4.1 (the small diagram below is a special case). The remaining two-particle poles correspond to invariants that involve one or more BCF-shifted legs, but nonetheless are not altered by BCF shifts. The "unique diagram" poles shown in (c) are discussed in Sec. 4.2, and the "wrong-helicity" poles (d) in Sec. 4.3. Notable features of each pole are discussed in the text.

Factorization in the remaining poles, which correspond to kinematic invariants that do not depend on z, must be verified explicitly. Checking that BCFW amplitudes factorize in these non-manifest limits is the main task of this section; we will demonstrate it by induction, assuming that $(n-1)$-point amplitudes factorize. The arguments are somewhat technical, but contain surprising structure; we will highlight the pieces of the BCFW expression that *do* contribute to each singularities, and the structural properties of the amplitudes that are required for factorization. We now classify these poles, and summarize the properties required for their factorization.

We first consider poles $P_I^2 \to 0$ where $1, 2 \in \bar{I}$ and $\bar{I}$ contains at least one additional leg. Terms in the BCF sum for which $I \subset L$ or $I \subset R$ are singular in this limit; because the BCFW shift does not affect the kinematics of the legs in I in any way, the inductive proof of Section 4.1 is straightforward, and independent of all details of the chosen amplitudes and BCFW shift. When I contains only two legs, one subtletly will appear that requires the same relations between coupling constants seen in the four-point amplitudes of Section 3.

BCFW shifts leave two other types of invariant unaffected: $[12]$ (and $\langle 12 \rangle$) and $\langle 1i \rangle$ ($[2i]$) for $i \neq 1, 2$. Unlike those discussed above, factorization in these limits depends on the spins and helicities of the particles involved. The collinear singularity as $[12] \to 0$ arises in the BCFW formula from amplitudes where a *soft* BCFW-shifted leg 1 is attached to each of the un-shifted legs (Sec. 4.2). This limit is closely related to the soft-photon and soft-graviton limits considered by Weinberg [2]. For spin-1, color-ordering reduces the factorization statement to the equality of one soft and one collinear diagram or of two soft diagrams. The spin-2 factorization receives contributions from soft singularities in $(n-2)$ terms, each of which produces a *double* pole in $\langle 12 \rangle$; the correct single pole of factorization is obtained using momentum conservation. Attempting to apply the BCFW recursion shifting legs with helicities $[+s, -s\rangle$, which is not a valid BCFW shift, we see that factorization cannot be satisfied.

The "wrong-helicity" factorization limit $\langle 1i \rangle \to 0$, which we discuss in Section 4.3, arises from multiple BCFW terms in both gauge theory and gravity. In gauge theory, their sum can be interpreted precisely as a BCFW construction of an $(n-1)$-point amplitude. However, the helicities of the shifted legs in the $(n-1)$-point amplitude can differ from those in the original n-point amplitude, so for example the proof of factorization for $[++\rangle$ shifts depends on the validity of $[-+\rangle$ shifts as well. The proof also depends on the large-z scaling properties of amplitudes, in particular the vanishing of amplitudes as $1/z^s$ at large z, and the growth of amplitudes under "invalid" shifts $[+-\rangle$ bounded by z^{3s}. A simple power-counting argument justifies this scaling for $s = 1$ (see Appendix A), but **we do not have a proof of this scaling for gravity, so the factorization argument in that case remains incomplete.** As for

the $\langle 12 \rangle$ poles, factorization is violated for the invalid shifts, in this case by non-vanishing boundary terms.

4.1   Factorization on Unshifted Multi-Leg Poles $P_I^2 \to 0$

Let I be a set of legs that excludes the shifted legs 1 and 2, and at least one other leg. Factorization requires that in the limit $P_I^2 \to 0$,

$$P_I^2 A(1, 2, \ldots, N) \to \sum_{h_I} A(\bar{I}, P_I^{h_I}) A(-P_I^{-h_I}, I), \tag{41}$$

where $\bar{I}$ is the complement of I in $1, 2, 3, 4, \ldots n$.

If I contains three or more legs, this result is readily obtained by considering the limit as $P_I^2 \to 0$ of the BCFW decomposition (11) of $A(1, 2, \ldots, n)$ (one subtlety will arise in the two-particle case, discussed below). The only terms that are singular in these limits are those with $I \subset L$ or $I \subset R$. Factorization of the left or right *sub-amplitudes* (the induction hypothesis) yields a simple limit:

$$K_I^2 A(\hat{1}, I, \tilde{L}, \hat{K}) \to \sum_{h_I} A(\hat{1}, \tilde{L}, \hat{K}, P_I^{h_I}) A(-P_I^{-h_I}, I), \tag{42}$$

where $\tilde{L}$ is the subset of L not explicitly written; terms with I in the right factor have analogous limiting behavior. In particular, all of these terms have a common factor $A(\hat{P}_I, I)$, which is one of the factors of the desired n-point factorization limit. Adding these terms, we find

$$P_I^2 A_{BCF} \to \sum_{h_K, h_I} A(I, -\hat{P}_I^{-h_I}) \sum_{L/R} \left(A(\hat{1}, \tilde{L}, P_I, \hat{K}) \frac{1}{K^2} A(-\hat{K}, \hat{2}, \tilde{R}) \right. \tag{43}$$

$$\left. + A(\hat{1}, \tilde{L}, \hat{K}) \frac{1}{K^2} A(-\hat{K}, \hat{2}, \tilde{R}, P_I) \right) \tag{44}$$

$$= \sum_{h_I} A(P_I - h_I, I) A(\bar{I}, P_I^{h_I}), \tag{45}$$

where in the last line, we have recognized terms (43) and (44) as a BCF formula for the lower-point amplitude $A(\bar{I}, P_I)$. This argument is illustrated diagramatically in Figure 4.

Note that the argument above fails completely if $\bar{I}$ contains only one and two — there are no diagrams in the BCFW sum for which $I \subset L$ or R, and in fact, as we will see, very different terms are singular in that case, which we consider in Section 4.2. We first elaborate on the special case that the set I contains only two particles (i, j); in this case, one additional class of diagrams plays a role.

Figure 4: A graphical summary of the argument for factorization of the BCFW amplitudes in unshifted poles. The top line identifies the subset of terms in the BCFW recursion relation that are singular as $P_I^2 \to 0$ (those in which all legs of I are on the same side). Their singularities are shown on the second line; pulling out a common factor $A(-P_I^{-h_I}, I)$, we recognize the sum as the BCFW-recursed expression for the second factor in (41).

4.1.1 Two-Particle Unshifted Poles

For definiteness, we consider the singularity $[34] \to 0$. In this case, one term in the BCFW expansion that we might expect to be singular is in fact non-singular, and two new terms are singular:

- One of the terms in the n-point BCFW sum (45) is

$$A(\hat{1}, 3, 4, \hat{K})\frac{1}{K_{134}^2}A(-\hat{K}, \ldots, \hat{2}). \tag{46}$$

  The singularity of the four-point sub-amplitude as $[34] \to 0$ is $A^{(h)}(3, 4, -P_{34}^{-h_{34}}) \times A^{(a)}(\hat{1}, P_{34}^{h_{34}}, \hat{K})$, but as $[34] \to 0$, $[\hat{1}K_{34}]$ also approaches zero. Therefore the $A^{(a)}$ factor vanishes for both gauge theory and gravity, and (46) does not contribute to the singularity in the factorization limit $[34] \to 0$.

- Two additional terms in the n-point amplitude, not included in (45), are singular — those in which the left factor contains only two legs: $\hat{1}$ and either 3 or 4. When legs $\hat{1}$ and 3 are alone in the left factor, for instance, the intermediate leg K has $|\hat{K}| \propto |3]$, so that as $[34] \to 0$ we also have $[\hat{K}3] \to 0$; therefore, this term can be singular even though legs 3 and 4 are split between factors. The singularity of this term is given by

$$\sum_{h_{13}, h^*} A^{(h)}(\hat{1}, 3, -\hat{K}_{13}^{-h_{13}})\frac{1}{K_{13}^2}\left\{\frac{\langle 34\rangle}{\langle \hat{K}_{13}4\rangle}A(\hat{K}_{13}^{h_{13}}, 4, P_{\hat{1}34}^{h^*})A(P_{\hat{1}34}^{-h^*}, \ldots, \hat{2}),\right\} \tag{47}$$

  where the factor in braces is obtained by factorization of the right BCFW factor. A similar term is obtained when legs 3 and 4 are exchanged.

We recall the argument for factorization in multi-particle poles: the factorization limits of every individual term in the n-point BCF amplitude can be interpreted as a contribution to the BCFW formula for the lower-point amplitude $A(P_I, \bar{I})$ appearing in (41). By the discussion above, there are two new terms in the BCFW ansatz that do not have this form. There is also a term that must appear in the BCFW formula (43-44) for the lower-point amplitude, but is not generated by factorizing any one term in the n-point BCFW ansatz. This missing term is

$$\sum_{h_{34}, h_K} A^{(h)}(3, 4, -P_{34}^{-h_{34}})A^{(h)}(\hat{1}, P_{34}^{h_{34}}, \hat{K}^{h_K})\frac{1}{K_{134}^2}A(-\hat{K}^{-h_K}, \ldots, \hat{2}) \tag{48}$$

(we maintain the conventions introduced earlier, that P's denote summed momenta that become null in singularity limits, K's unshifted intermediate momenta in the BCFW formula, and $\hat{K}$ the null shifted intermediate momenta in BCFW).

To recover the correct factorization result, the extra terms in the factorization of the n-point BCFW formula must compensate for the missing term — Eqn. (47) and its analogue with 3 and 4 exchanged must sum to (48). This is so, and follows from the Jacobi identity (spin-1) or equality of all couplings (spin-2). It is worth stressing that, although our result will be very reminiscent of the factorization of four-particle amplitudes in Section 3, it is not the same physical limit — indeed, the products $A^{(h)}A^{(h)}$ do not arise as factorization limits of four-point amplitudes. In fact, these products are non-zero only for external helicities $(-,-,-,+)$, for which the four-point amplitude vanishes.

To proceed, let us define

$$f(i,j;k,l) = \sum_{h_{int}} A^{(h)}(i,j,K_{int}) \times A^{(h)}(-K_{int},k,l) \tag{49}$$

The f's have a simple form:

$$f(i,j;k,l) = \left(\sum_a f_{ija} f_{akl} \right) \left(\frac{1}{\langle 1j \rangle \langle kl \rangle} H(i,j,k,l) \right)^s, \tag{50}$$

where

$$H(1^-,2^-,3^-,4^+) = \frac{[4\mu]^3}{[1\mu][2\mu][3\mu]} \tag{51}$$

depends only on the helicities of the particles (not on how they are paired), and is zero for all other combinations.

Terms (47) and (48) all have the limiting form

$$\sum_{h_{int},h_P} (\text{kinematics}) \times f(i,j;k,P^{h_P}) \times A^{(n-2)}(-P^{-h_P},\hat{2},\dots), \tag{52}$$

where i,j,k is some ordering of $\hat{1},3,4$, where $\hat{1}$ is the limiting BCF-shifted momentum of leg 1 in (48), and $P = K_{34} + \hat{p}_1$ (all three terms approach this uniform kinematics as $[34] \to 0$). Factorization requires the coefficients of $A^{(n-2)}$ in (47) ($+\ 3 \leftrightarrow 4$) to reproduce the coefficients in (48), i.e.

$$-\frac{1}{K_{134}^2} f(3,4;\hat{1},P) + \frac{1}{K_{13}^2} \frac{\langle 34 \rangle}{\langle \hat{K}_{134} \rangle} f(\hat{1},3;4,P) + \frac{1}{K_{14}^2} \frac{\langle 43 \rangle}{\langle \hat{K}_{143} \rangle} f(\hat{1},4;3,P). \tag{53}$$

By judicious use of kinematic identities, this can be rewritten as

$$\frac{\langle 34 \rangle}{[2P]} H(1,3,4,P)^s \left\{ (\langle 1P \rangle 34)^{s-1} f_{1Pa} f_{a34} + (\langle 13 \rangle 4P)^{s-1} f_{13a} f_{a4P} + (\langle 14 \rangle P3)^{s-1} f_{14a} f_{aP3} \right\}, \tag{54}$$

which vanishes by the Jacobi identity for $s = 1$ and the Schouten identity (setting all f equal) for $s = 2$.

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Thus, the non-standard terms (47) are equal to the term (48) that was missing from the naive sum, and factorization on poles like $[34] \to 0$ is guaranteed by the BCFW construction for both spin-1 and spin-2. An analogous result would hold when the opposite-helicity invariants $\langle ij \rangle \to 0$, except that in that case, the roles of legs 1 and 2 are interchanged, and the identity involves anti-holomorphic 3-point amplitudes instead of holomorphic ones.

4.2 Factorization on "Unique Diagram" Poles ($[12]$)

The BCFW formula does not contain any sub-amplitudes with "propagator" singularities as $[12] \to 0$ because, by construction, legs 1 and 2 are in separate factors and the remaining $(n-2)$ legs are split between the factors. Instead, the collinear singularities $[12] \to 0$ arises remarkably in the BCFW formula through a *soft* singularity in one of the factors. Specifically, whenever the left-hand factor in the BCFW decomposition (11) is a three-point amplitude $A^{(h)}(\hat{1}, i, -\hat{K})$ for some leg i, we find $|\hat{1}] = \frac{[21]}{[2i]}|i] \to 0$ (analogously, in the $\langle 12 \rangle \to 0$ limit, $|\hat{2}\rangle$ becomes soft when the right-hand factor is a three-point amplitude). The right factor, an $(n-1)$-point amplitude, approaches a uniform kinematic limit for all i, which is in fact exactly the kinematics that appears in the factorization formula:

$$\lim_{[12] \to 0} \langle 12 \rangle [12] A(1, \ldots, n) = \sum_{h'} A^{(h)}(1, 2, -P_{12}^{-h'}) A(P_{12}^{h'}, 3, \ldots, n). \tag{55}$$

The behavior of the three-point amplitude $A^{(h)}(\hat{1}, i, -\hat{K})$ as $|\hat{1}] \to 0$ depends on the helicity h_1 of the soft leg. If $h_1 = +s$, all three-point amplitudes $A^{(h)}$ in (55) vanish as $|\hat{1}] \to 0$, so the BCFW sum is non-singular. This makes the factorization limit either trivial (when $h_2 = +1$, the three-point amplitude in (55) is also zero) or manifestly violated (when $h_2 = -1$, (55) is non-zero, but the BCFW formula cannot reproduce this singularity). The latter case corresponds to the shift $[+, -\rangle$, which we have already seen does not produce consistent four-point amplitudes.
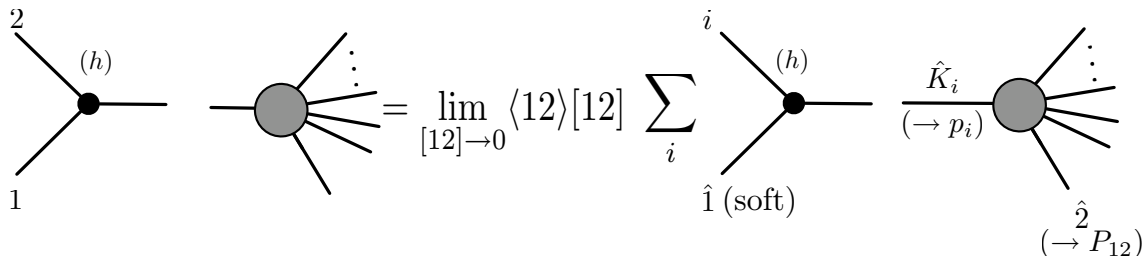

Figure 5: Left: factorization limit for the "unique diagram" pole $[12] \to 0$. Right: in each of the BCFW terms that contribute to this singularity, $\hat{1}$ is a *soft* line attached to one of the unshifted legs.

If $h_1 = -s$, the three-point amplitudes in (55) do have a soft singularity (they scale as $[12]^{-s}$), and the factorization condition is non-trivial. In this case, the singularity of the BCFW amplitude is given by

$$\lim_{[12]\to 0} \langle 12\rangle[12]A_{BCF}(1,\ldots,n) = \lim \sum_{i,h_{1i}} \quad \left(\frac{\langle 12\rangle[12]}{\langle 1i\rangle[1i]} A(\hat{1}^{-s,a_1}, i^{h_i,a_i}, -\hat{K}_{1i}^{-h_{1i},b}) \right)$$

$$\times A(\hat{2},\ldots,i-1,\hat{K}_{1i}^{h_{1i},b}, i+1,\ldots,n), \qquad (56)$$

where we have written helicity and species indices explicitly in the three-point amplitude and for the modified leg K_i in the $(n-1)$-point amplitude.

We begin by describing the kinematics of (56) more explicitly to illustrate the soft limit, then consider the sums separately for the spin-1 and spin-2 cases. The shifted spinor $|\hat{1}] = |1] + z_*|2]$ in the i'th term is proportional to $|i]$; the constant of proportionality and the value of z_* can be found by taking the inner product of $|1] + z_*|2] = c|i]$ with $|2]$ or $|i]$, respectively:

$$|\hat{1}] = \frac{[12]}{[i2]}|i] \quad \text{at } z_* = -\frac{[1i]}{[2i]}. \qquad (57)$$

We note that $|\hat{1}]$ is indeed becoming soft (one can understand the limit as follows: as $|1]$ and $|2]$ become nearly proportional, a fine-tuned subtraction of nearly equal spinors is required to obtain a spinor $|\hat{1}] \propto |i]$). Since $|\hat{1}]$ is soft, the momentum $\hat{K}_i$ leaving the $(n-1)$-point factor must approaches p_i:

$$\hat{K}_i = (|i\rangle + \frac{[12]}{[i2]}|1\rangle)|i], \qquad (58)$$

and the shifted momentum of leg 2 is,

$$\left(|2\rangle + \frac{[1i]}{[2i]}|1\rangle \right) |2] \to P_{12} = (p_1 + p_2). \qquad (59)$$

As we have noted, the limiting kinematics of the $(n-1)$-point amplitudes as $[12] \to 0$ is independent of which leg i appears in the left factor! In fact, it is the same kinematics that appears in the $(n-1)$-point amplitude of (55). However, the $\hat{K}_{1i}$ line leaving the $(n-1)$=point diagram in each case can have different internal species quantum numbers than those of the original outgoing line i.

One might also expect the line K_i to have different helicity than h_i, but in fact these diagrams need never be considered. We have already noted that, if $h_1 = +s$, the factorization requirement is either trivial (when $h_2 = +s$) or impossible to satisfy ($h_2 = +-$); the non-trivial case is $h_1 - s$. But the gauge/gravity amplitude $A^{(h)}(-s,-s,+s)$ is only non-zero if its two remaining legs have opposite helicities, which in our notation is $h(K_i) = h_i$.

Before considering the singular three-point terms in more detail, we clarify why the generic case — in which the left factor is a higher-point amplitude, does not contribute. In this case,

as $[12] \to 0$, the shifted leg $|1]$ approaches a non-singular limit

$$|\hat{1}] = -\frac{K_J^2}{2p_1.K_J}|1],\tag{60}$$

where K_J is the (generically non-null) sum of momenta of the legs in the left factor. As no momenta or invariants within either sub-amplitude vanish at this point, we expect no singularities. Likewise, no singularity occurs when the right-hand factor is three-point, because $|2\rangle$ and the spinor $|1\rangle$ by which it is shifted are not orthogonal.

4.2.1 Spin-1 Factorization

We now consider the three-point amplitudes in somewhat more detail for the theories of pure spin-1 "gauge-theory-like" $(++-$ and $-+)$ interactions. The kinematic factor times three-point amplitude in the i'th term (in parentheses in Eqn. (56)) is given by

$$\frac{\langle 12\rangle[12]}{\langle 1i\rangle[1i]}\left(\langle 1i\rangle\frac{[i2]}{[12]}\right)f_{a_1,a_i,b} = -\langle 12\rangle\frac{[2i]}{[1i]}f_{a_1,a_i,b}.\tag{61}$$

Note that $\frac{[2i]}{[1i]}$ approaches an i-independent, finite limit as $[12] \to 0$, so different terms in the sum (56) differ only in the replacement of the i'th particle's species index by a dummy label b, and contraction into a coefficient $f_{a_1,a_i,b}$. One can see that this identity holds by separately considering every contraction of many f's that could appear in the $(n-1)$-point amplitudes in (56), and repeatedly using the Jacobi identity.

It is much less cumbersome, however, to switch to the color-ordered formalism [19, 20, 21]. Because the cyclic ordering of leg indices is significant in color-ordered amplitudes, we will call the BCFW-shifted legs $[x, y\rangle$ in this discussion rather than $[1, 2\rangle$, to avoid suggesting that they are color-adjacent when they need not be. We have already seen that, for four-point spin-1 amplitudes to factorize, three-point couplings f must satisfy a Jacobi identity. We can then associate each "species index" a with an element T_a in the adjoint representation of a Lie algebra; the relation on species indices described above is a trivial consequence of this structure. The only color structure ever generated in tree amplitudes is a single trace, allowing us to express any amplitude as a sum of color traces times colorless primitive (or color-ordered) amplitudes,

$$A(1^{a_1}, 2^{a_2}, \ldots, n^{a_n}) = \sum_{P(2,\ldots,n)} Tr(T^{a_1}T^{a_{i_2}}\ldots T^{a_{i_n}}) \times A_{c.o.}(i_1, i_2, \ldots, i_n).\tag{62}$$

The color-ordered amplitude receives contributions only from diagrams in which the legs are cyclically ordered from 1 to n clockwise on a plane, and all coefficients $f_{a,b,c}$ can be replaced with a uniform coupling constant f. Aside from this, color-ordered BCFW amplitudes have

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the same structure as unordered ones. The color-ordering is, for our purpose, simply a book-keeping device to focus on one color-structure (i.e. one set of contracted f's) at a time using their known algebraic structure. This bookkeeping simplifies the discussion considerably: instead of n diagrams, for a given color-ordering only two diagrams contribute to this discussion.

We must consider two cases separately: when the legs x and y are color-adjacent, and when they are not. If legs x and y are not color-adjacent, then no diagrams in which x and y connect at a three-point vertex have the correct color ordering, so there is no singularity as $[xy] \to 0$; there are, however, two non-vanishing terms in the BCFW expression (56) (namely, $i = x \pm 1$). These appear with opposite signs, since the color-ordering includes $A(x-1, x, -K) = -A(x, x-1, -K)$ and $A(x, x+1, -K)$ (with no sign flip). Thus the two diagrams cancel, correctly giving no singularity.

If x and y are color-adjacent (for definiteness, say $x = 1$, $y = 2$), the factorization limit (55) is non-zero:

$$A^{(h)}(1^{-1}, 2^{h_2}, -P_{12}^{-h_2})A(P_{12}^{h_2}, 3, \ldots, n) = \langle 12 \rangle \frac{[2\mu]}{[1\mu]} A(K_{12}, 3, \ldots, n), \tag{63}$$

and is reproduced exactly by the one non-zero term in the color-ordered BCFW expression (using (56) and antisymmetry of the three-point amplitude),

$$\frac{\langle 12 \rangle [12]}{\langle 1i \rangle [1i]} A^{(h)}(n, \hat{1}, -\hat{K}_n)A(P_{12}, 3, \ldots, \hat{K}_n). \tag{64}$$

In each case, then, the color structure established at four-point and properties of three-point amplitudes suffice to guarantee the factorization of n-point BCFW amplitudes as $[12] \to 0$ (the analysis for $\langle 12 \rangle \to 0$ is analogous, with the roles of legs 1 and 2 exchanged).

4.2.2 Spin-2 Factorization

To study spin-2 interactions, we use the fact derived from spin-2 four-point amplitudes in [10] that interactions among spin-2 particles can always be written as self-interactions of a single species. Therefore, there are no species indices in (56), and the $(n-1)$-point amplitudes approach truly identical limits. Unlike the case of spin-1, however, the kinematic $\times$ three-point factors of (56) do depend on the i'th particle's momentum — they are given by,

$$\frac{\langle 12 \rangle [12]}{\langle 1i \rangle [1i]} \left(\langle 1i \rangle \frac{[i2]}{[12]} \right)^2 = \frac{1}{[12]} \frac{\langle 12 \rangle [2i]^2 \langle 1i \rangle}{[1i]}. \tag{65}$$

Moreover, the individual terms become singular as $[12] \to 0$ (i.e., since we are attempting to evaluate a residue, each BCFW term in fact has a *double* soft singularity as $[12] \to 0$).

Replacing each of the $(n-1)$-point amplitudes in (56) with its limiting kinematics, we find,

$$\lim_{[12]\to0}\langle12\rangle[12]A_{BCF}=A^{(n-1)}\times\lim_{[12]\to0}\left(\sum_i\frac{1}{[12]}\frac{\langle12\rangle[2i]^2\langle1i\rangle}{[1i]}\right).\tag{66}$$

Writing,

$$\frac{[2i]^2}{[1i]}=\frac{[2\mu]}{[1\mu]}\left([2i]+\frac{[12]}{[1\mu]}[\mu i]\right)+O([12]^2),\tag{67}$$

the limit of (66) becomes,

$$\frac{[2\mu]}{[1\mu]}\left[\lim\frac{\langle12\rangle}{[12]}\left(\sum_i[2i]\langle1i\rangle\right)+\frac{\langle12\rangle}{[1\mu]}\left(\sum_i\langle1i\rangle[\mu i]\right)\right].\tag{68}$$

By momentum conservation, the first sum in parentheses is identically zero and the second is equal to $\langle12\rangle[2\mu]$; thus we recover the singularity limit,

$$A^{(n-1)}\times\langle12\rangle^2\left(\frac{[2\mu]}{[1\mu]}\right)^2.\tag{69}$$

In fact the argument above is a bit too quick: for small but finite $[12]$, each $(n-1)$-point amplitude in (56) is evaluated at slightly different kinematics; because the prefactors in (56) are themselves growing as $\frac{1}{[12]}$, this displacement could change the final result by a non-zero amount. As we show explicitly in Appendix B, this correction has no effect at all on the limit –it is proportional to a sum over all legs i of $[\mu i][\mu|\frac{dA^{(n-1)}}{d|i]}$, which vanishes because $|i]dA/d|i]$ is antisymmetric.

4.3   Factorization on Wrong-Factor Poles ($\langle1j\rangle$)

The final class of factorization limits we must check is the limit $\langle1j\rangle\to0$ for $j\neq2$ (the case $[2j]\to0$ for $j\neq1$ is analogous). We will consider concretely the limit $\langle13\rangle\to0$ (again, the sequence of labels is arbitrary). In this limit, we require

$$\lim_{\langle13\rangle\to0}[13]\langle13\rangle A(1,3,\ldots,n)=\sum_{h_{13}}A^{(a)}(1,3,-P_{13}^{-h_{13}})A(P_{13}^{h_{13}},2,\ldots,n).\tag{70}$$

As before, we will prove this for n-point amplitudes by induction, assuming that all lower-point amplitudes factorize appropriately and can be generated by the BCFW construction.

We will classify terms of the BCFW sum as in Figure 6 (we will mention only terms that have potential singularities). We will first consider terms (a), in which the left factor contains legs 1 and 3, and at least one additional leg — these will combine into an expression like (70), in which the $(n-1)$-point amplitude has been expressed using BCFW recursion. Terms (b) and (c) will not contribute to the factorization limit, but to show this we will need to use the

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fact proved in Appendix A for spin-1, that at large shifts z, *n-point amplitudes generated by BCFW have the same z-scaling as 3-point amplitudes* (i.e. z^{-1} for all "legal" BCFW shifts and no faster than z^3 for the "illegal" shift $[+,-\rangle$).

The outline of this proof holds for gravity as well, but the sum of terms (a) will contain additional contributions, that only vanish if the $(n-1)$-point amplitudes fall as z^{-2} under valid BCFW shifts. Likewise, the absence of singularities from terms (c) only vanish if multi-point gravity amplitudes have the same z-scaling as 3-point amplitudes. We know this to be true of spin-2 BCFW amplitudes from analysis of the gauge-theory amplitudes [8, 9], but do not have an S-matrix argument for why it must be so.
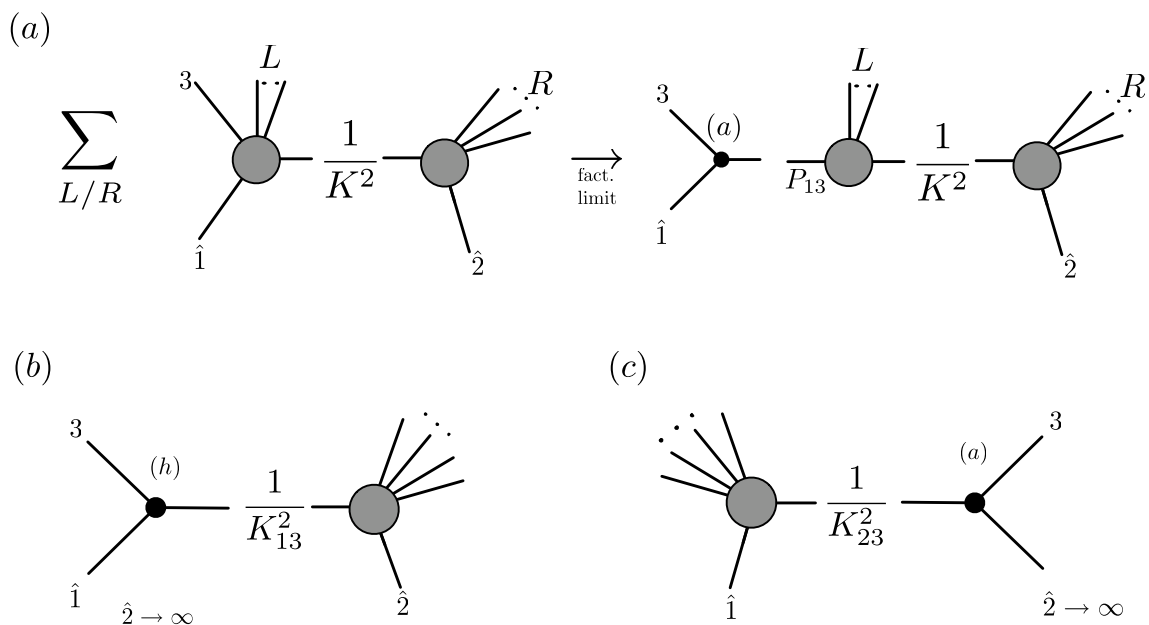

Figure 6: Potentially singular terms in the limit $\langle 1i \rangle \to 0$: Terms of the form (a) reproduce the correct factorization limit. Terms (b) have no singularity in theories of spin ≥ 1. To see that terms (c) are not singular, one must study the large-z scaling of lower-point amplitudes.

4.3.1 Terms (a)

Terms of the form (a) in Figure 6 are each evaluated at a different $z = z_*(L)$, but the shift does not change $|1\rangle$ or $\langle 13 \rangle$. By the induction hypothesis, the left BCFW factor in each such term should factorize on the $\langle 13 \rangle$ pole as,

$$\lim_{\langle 13 \rangle \to 0} [\hat{1}3]\langle 13 \rangle A(\hat{1}, 3, \tilde{L}, -\hat{K}) = \sum_{h_{13}} A^{(a)}(\hat{1}, 3, -P_{\hat{1}3}^{-h_{13}}) A(P_{\hat{1}3}^{h_{13}}, L, -\hat{K}). \tag{71}$$

The three-point amplitude in (71) vanishes if $h_1 = h_3 = -s$; otherwise, it is non-zero. Writing $P_{\hat{1}3} \to |1\rangle(|\hat{1}] + \frac{\langle 3\mu \rangle}{\langle 1\mu \rangle}|3])$ as $\langle 13 \rangle \to 0$, we find $A^{(a)} \propto [\hat{1}3]^s \left(\frac{\langle 3\mu \rangle}{\langle 1\mu \rangle} \right)^p$. The power p depends on the helicities h_1, h_3, h but is the same for all terms; it is convenient to express all amplitudes in terms of the unhatted 3-point amplitude,

$$A^{(a)}(\hat{1}, 3, -P_{\hat{1}3}^{-h_{13}}) = \left(\frac{[\hat{1}3]}{[13]} \right)^s A^{(a)}(1, 3, -P_{13}^{-h_{13}}). \tag{72}$$

Substituting this form into (71) and summing over all such contributions to the BCFW formula, we obtain

$$\lim_{\langle 13 \rangle \to 0} [13]\langle 13 \rangle \sum_{L,h} A(\hat{1}, 3, L, -\hat{K}_{13L}^{-h}) \frac{1}{K_{13L}^2} A(\hat{K}_{13L}^h, R, \hat{2}) \tag{73}$$

$$= A^{(a)}(1, 3, -P_{13}^{-h_{13}}) \sum_{L,h} \left(\frac{[\hat{1}3]}{[13]} \right)^{s-1} A(P_{\hat{1}3}^{h_{13}}, L, -\hat{K}_{13L}^{-h}) \frac{1}{K_{13L}^2} A(\hat{K}_{13L}^h, R, \hat{2}). \tag{74}$$

If $s=1$, we recognize the sum in (74) as a BCFW formula for the $(n-1)$-point amplitude $A(P_{13}^{h_{13}}, 2, 4, \ldots, n)$ in (70), obtained by a $[P_{13}, 2\rangle$ shift (note $|\hat{P}_{13}] = |P_{13}] + z|2]$ is the same as $P_{\hat{1}3}$ appearing above). Thus (74) reproduces the desired factorization limit (70), provided $[P_{13}^{h_{13}}, 2\rangle$ is a valid shift.

In general, the helicity of P_{13} can differ from h_1 (for instance, if $h_1 = +1$ and $h_3 = -1$ then the only non-vanishing contribution in (74) comes from $h_{13} = -1$). So the validity of factorization on $\langle 1j \rangle$ poles for shifts $[+, +\rangle$ depends on the validity of a lower-point BCFW formula using shifts $[-, +\rangle$. It is important to check that the above procedure never introduces the BCFW construction using a $[P_{13}^+, 2^-\rangle$ shift. Indeed, this would require an anti-holomorphic amplitude $A^{(a)}(1^-, 3^h, P_{13}^-)$, which is absent from the theory. Thus the shifts valid at 4-point are a closed set.

Something surprising happens for the spin-2 ($s = 2$) case: each term in the would-be BCFW sum is multiplied by $\frac{[\hat{1}3]}{[13]} = 1 + z_*(L)\frac{[23]}{[13]}$. The 1's combine into a BCFW-shifted amplitude, yielding the correct factorization limit as in the spin-1 case. However, the sum of terms proportional to $z_*(L)$ remains. These are, however, simply the residues of $A(z)$ (recall that the terms in the BCFW sum are residues of $A(z)/z$, and we have multiplied them by the z's at which the poles occur)! therefore, the limit of our partial sum is given by

$$A^{(a)}(1, 3, -P_{13}^{-h_{13}}) \left[A(P_{13}^{h_{13}}, 2, 4, \ldots, n) + \oint_\infty dz A(P_{13}(z)^{h_{13}}, 2(z), 4, \ldots, n) \right]. \tag{75}$$

The contour integral at infinity vanishes when $A(z) \to 1/z^2$ — this is a known property of gravity amplitudes [8, 9], but it is surprising from both their Lagrangian or recursive definitions (we will not prove it here). The appearance of this formula in the factorization

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requirement is mysterious, and suggestive that the $1/z^2$ scaling of gravity amplitudes is crucial for self-consistency of the theory.

4.3.2 Terms (b) and (c)

These are not the only terms in the BCFW sum that can be singular in the $\langle 13 \rangle \to 0$ limit. We now consider two more potentially singular terms in the BCFW sum (which we expect to vanish, since we recovered the correct answer already from terms (a)):

$$\sum_{h_K} A^{(h)}(\hat{1}, 3, -\hat{K}^{-h_K}) \frac{1}{P_{13}^2} A(\hat{K}^{h_K}, \hat{2}, \ldots, n), \tag{76}$$

and,

$$\sum_{h_K} A(\hat{1}, \ldots, -\hat{K}^{-h_K}) \frac{1}{P_{23}^2} A^{(a)}(\hat{K}^{h_K}, \hat{2}, 3). \tag{77}$$

The contribution (76) has a net $\frac{1}{\langle 13 \rangle}$ in the explicit propagator, but the 3-point amplitude is proportional to $\langle 13 \rangle^s$ so (76) is not singular for $s \geq 1$. The term (77) has potentially singular behavior because the shifted momenta,

$$|\hat{2}\rangle = \frac{\langle 12 \rangle}{\langle 13 \rangle} |\hat{3}\rangle \quad , |\hat{K}\rangle |\hat{K}] = \left(-\frac{\langle 12 \rangle}{\langle 13 \rangle} |3\rangle \right) (|2] + \langle 13 \rangle \langle 12 \rangle |3]), \tag{78}$$

go to infinity as $\langle 13 \rangle \to 0$. The growing momentum is suggestive of a large-z BCFW shift – a limit in which we expect lower-point amplitudes to approach zero. To make this explicit, we write the momenta $\hat{1}$ and $-\hat{K}$ in a somewhat unusual way. We first define "initial" momenta,

$$|1_0\rangle |1_0] = |1\rangle \left(|1] - \frac{\langle 23 \rangle}{\langle 12 \rangle} |3] \right) \quad |K_0\rangle |K_0] = |2\rangle \left(|2] + \frac{\langle 13 \rangle}{\langle 12 \rangle} |3] \right); \tag{79}$$

note that as $\langle 13 \rangle \to 0$, these momenta approach limits $|1_0\rangle |1_0] \to |1_*\rangle |1_*] = K_{13}$ and $|K_0\rangle |K_0] \to |K_*\rangle |K_*] = p_2$. The hatted momenta in (77) can be written as a BCFW shift of the "initial" momenta:

$$\hat{p}_1 = |1\rangle (|1_0] - z_*|K_0]), \quad -\hat{K} = (|K_0\rangle - z_*|1_0\rangle)|K_0], \quad \text{where } z_* = \frac{\langle 23 \rangle}{\langle 13 \rangle}. \tag{80}$$

All invariants involving "starred" momenta are generically finite, and differ from invariants of "naught" only by terms supressed by $1/\langle 13 \rangle$. Therefore, the scaling of the $(n-1)$-point amplitude in (77) as $\langle 13 \rangle \to 0$ is dictated by the large-z scaling of

$$A_{BCF,[1_*,K_*\rangle}(1_*, K_*, 4, \ldots, n), \tag{81}$$

which we derive for spin-1 in Appendix A. Using this result and the explicit kinematics of (78) for the three-point amplitude, we find that, as long as $[1, 2\rangle$ is a "valid" BCFW shift, (77) is non-singular as $\langle 13 \rangle \to 0$. This is summarized in Table 1. We highlight three examples:

- If $h_1 = h_2 = +1$ and $h_3 = +1$, then the only non-zero term comes from $h_K = -1$. The three-point amplitude diverges as $1/\langle 13 \rangle$, but the scaling of the $(n-1)$-point amplitude under a large $[+, +\rangle$ shift is $1/z_* \sim \langle 13 \rangle$, so the product has a finite limit.

- If instead $h_3 = -1$, then the 3-point amplitude vanishes as $\langle 13 \rangle^3$, but the $(n-1)$-point amplitude grows as z_*^3 under a $[+, -\rangle$ shift; again the product approaches a finite limit as $\langle 13 \rangle \to 0$.

- For "invalid" BCFW shifts $[+, -\rangle$, term (c) does contribute a singularity (in fact, an unphysical multiple pole!). For example, we consider the contribution from $h_K = +1$ when $h_1 = h_3 = +1$ and $h_2 = -1$. The three-point amplitude in (77) scales as $1/\langle 13 \rangle$ and the $(n-1)$-point amplitude scales as z_*^3 under the $[1^+, (-K)^-\rangle$ shift – as $\langle 13 \rangle \to 0$, the product can have an unphysical $\langle 13 \rangle^{-4}$ singularity!

h_1	h_2	h_3	h_K	$(n-1)$-point	Three-point	Total scaling
+	+	+	-	ϵ	$1/\epsilon$	1
+	+	-	+	$1/\epsilon^3$	ϵ^3	1
-	+	+	-	ϵ	$1/\epsilon$	1
-	+	-	+	ϵ	ϵ^3	ϵ^4
-	-	+	+	ϵ	$1/\epsilon$	1
-	-	-	X	(c) vanishes identically		
+	-	+	+	$1/\epsilon^3$	$1/\epsilon$	ϵ^{-4}
+	-	-	X	(c) vanishes identically		

Table 1: Scaling of the term (77) with $\epsilon \equiv \langle 13 \rangle$ as $\langle 13 \rangle \to 0$. For given h_2 and h_3, at most one h_K has a non-vanishing contribution to this diagram. An "X" denotes helicity combinations in which the 3-point amplitude $A^{(a)}(2, 3, K)$ vanishes for all h_K. We evaluate the 3-point scaling explicitly using the kinematics of (78) and the $(n-1)$-point scaling using $z \sim 1/\epsilon$ and the general scaling found in Appendix A. The bottom two lines correspond to the "invalid" shift.

The scaling of gravity amplitudes is known to be the square of the gauge theory scaling described above – this scaling suffices to guarantee that the term (77) does not contribute to the factorization limit in gravity, either. However, unlike gauge theory, we have not found an S-matrix argument for why this must be so. The pivotal role of $1/z^2$ scaling in showing that gravity amplitudes factorize is striking and unexpected — this is a faster scaling than is required to prove BCFW in field theory, for example! It is possible that in some cases, the boundary-term of (75) cancels the one arising from (77). But such a cancellation is unlikely to be universal. For example, if $h_1 = h_2 = +1$ and $h_3 = -1$, the boundary term of (75) is associated with a coefficient of $1/z$ in a falling amplitude, while (77) is the product of a *growing* $(n-1)$-point amplitude with a 3-point amplitude that falls as $\langle 13 \rangle^6$!

5 Summary and Remarks

The BCFW recursion relation is a remarkable formula, demonstrating that the entire structure of Yang-Mills and gravity amplitudes at tree-level can be derived recursively from 3-point amplitudes, which are fully specified by their Lorentz structure. The coefficients of these 3-point amplitudes must satisfy additional consistency conditions, identified in [10], for the BCFW four-point amplitude to be well-defined.

We have shown that in fact, when these consistency conditions are violated, no four-particle amplitude with correct Lorentz transformation *and* factorization properties can be defined. Thus, the consistency requirements are quite strong — their violation truly indicates an inconsistency of the set of three-point amplitudes, not merely of a BCFW construction. We have used it to reproduce several standard results (Jacobi identity, charge conservation, and equivalence principle) for spins 1 and 2 coupled to other particles of the same spin, or to scalar matter.

Related to this four-particle factorization requirement is a necessary criterion for BCFW constructions: when a four-point amplitude exists, we can ask whether a given BCFW construction reproduces it. We find that the constructions known to be valid from gauge theory (shifts $[--\rangle$, $[-+\rangle$, and $[++\rangle$) do, while attempts to generate amplitudes using a wrong-helicity BCF shift ($[+-\rangle$) result in "amplitudes" with unphysical multiple poles. Thus, S-matrix consistency alone shows that these shifts are invalid.

The self-consistency of BCFW amplitudes beyond four-point is not obviously guaranteed by these four-particle conditions. We have demonstrated that the four-particle conditions are in fact sufficient, for interacting spin-1 fields, by showing inductively that any BCFW construction using a valid shift has the correct factorization properties in all complex-momentum factorization limits. The equivalence of different BCFW constructions of the same amplitude (the generalization of the four-particle test of [10]) follows from this fact and power-counting. The factorization arguments fail for the invalid $[+-\rangle$ shifts. We have also outlined a proof of the analogous statement for gravity. The argument requires that n-point amplitudes scale at large-z BCFW shifts in the same way as the fundamental three-point amplitudes, i.e. as z^{3s} for $[+-\rangle$ shifts and $1/z^s$ for all others. We have proved this result for spin-1 (but not yet for spin-2), again using only S-matrix arguments.

One may ask why BCFW amplitudes should include all possible poles, given that they are only explicitly constructed by factorizing diagrams on a subset of poles. Formally, the sufficiency of considering this subset is guaranteed by the large-z behavior of amplitudes; the physical intuition suggested by the study of four-particle amplitudes is that BCFW works precisely when the helicity structure of amplitudes requires them to have simultaneous singularities in multiple invariants (e.g. $1/(stu)$ in gravity four-point amplitudes).

These two very different explanations of BCFW may be more related than they appear — we have argued that the large-z scaling of spin-1 amplitudes can be derived from their factorization properties, mass dimension, and transformation under helicity rotations of the two shifted legs. Requiring that amplitudes factorize drastically constrains the functional form of any part of an amplitude that grows faster (or falls slower) with z than the three-particle amplitudes. In fact, one cannot write down any amplitude that grows faster with z than three-particle amplitudes, factorizes, and transforms properly under helicity rotations of the two shifted legs. This argument may generalize to spin-2, and is the closest analogue we are aware of in the spinor-helicity language to the "spin Lorentz" symmetries found in the background-field approach of [9].

The results of this paper are completely unsurprising — indeed, the factorization of gauge and gravity amplitudes is a consequence of their equivalence to well-known field theories. However, the technical mechanisms for achieving factorization are somewhat remarkable, and suggestive. First, it is striking that the conditions appear as factorization conditions *only at complex momenta* (e.g. the constraints on four-particle gauge theory amplitudes can only be exposed by considering the singularity as $[12] \to 0$ with $\langle 12 \rangle$ finite – if both go to zero simultaneously, the amplitude is non-singular). Moreover, the BCFW terms that are singular as $[12] \to 0$ are precisely the *soft photon/graviton* singularities that appear in Weinberg's classic derivations of charge conservation and the equivalence principle — even though the limit we consider need only be collinear (in one spinor factor of the momenta).

Particularly noteworthy is the connection between the requirement of factorization in "wrong-helicity" two-particle poles that are not exposed by BCF (e.g. $\langle 13 \rangle \to 0$ when $|1]$ is BCF-shifted) and the $1/z^2$ scaling of gravity amplitudes. The sensitivity of this factorization limit to the $1/z$ coefficient of gravity amplitudes is so striking because the field theory argument for BCFW does not require this coefficient to vanish – in that construction, it is a seemingly irrelevant accident. In contrast, the appearance of $1/z^2$ scaling in the factorization requirement suggests a connection between the (hard) large-z behavior of gravity amplitudes and factorization in collinear limits. We should, however, point out that the derivation in Section 4.3 includes **two** appeearances of the $1/z$ coefficient — it is conceivable that in some consistent theories, these coefficients do not vanish but one cancels the other.

The structure found in these familiar theories suggests three related directions at tree-level for further investigation. First, it is possible that the "constructive" approach used here could be used to find new BCFW recursive constructions (presumably equivalent to known Lagrangian theories). There are sets of three-point amplitudes for which a consistent four-particle amplitude exists, but cannot be obtained by any BCFW recursion relation (e.g. ϕ^3 theory, or interacting spin-1 particles with non-zero $(+,+,+)$ and $(-,-,-)$ vertices). It seems

likely that a more general recursion relation exists, that allows us to generate an S-matrix for any such set of primitive amplitudes. We also expect that analogous structural constraints (such as anomalies) appear at one-loop. Finally, in higher dimensions, there are theories with no known Lorentz- and gauge-invariant action, and it is possible that generalizations of our construction to higher dimensions would permit the study of an S-matrix for such theories.

Acknowledgments

A   Large-z Scaling of n-point Gauge Theory Amplitudes

The large-z scaling behavior of gauge theory amplitudes $A^{(n)}(p_1(z), p_2(z), \ldots, n)$, when two legs are deformed by a BCFW shift $[1, 2\rangle$ parametrized by z

$$|1] \rightarrow |1] + z|2], \quad |2\rangle \rightarrow |2\rangle - z|1\rangle, \tag{82}$$

is known to be determined by the helicities of legs 1 and 2. For $h_1 = +1$ and $h_2 = -1$, $A^{(n)}$ grows with z^3 at large z, and for other helicity choices (+/+, -/-, and -+) it falls as $1/z$.

In field theory, the former growth follows from naive power-counting, whereas the $1/z$ behavior at large z is not obvious from diagrammatic arguments. For BCFW amplitudes, these scalings are guaranteed by the factorization and Lorentz structure of the amplitudes, as we will now show. We use the $(n-1)$-point scaling result (which depends only on factorization of $(n-1)$-point amplitudes) in the n-point factorization argument of 4.3.

We begin with the $1/z$ scalings under "valid" BCF shifts. Of course, this scaling is guaranteed by the BCFW construction (in which every term has a pole $\sim 1/(z - z_*)$), if $A^{(n)}$ was generated by the same BCFW shift $[1, 2\rangle$.

In fact, this is all we need, because all valid BCFW shifts must generate the same n-point amplitude. We have shown in Section 4 that any two BCFW amplitudes have identical singularity structure at every kinematic singularity; therefore, they can only differ by completely non-singular terms. However, they can depend only on the dimensionless coupling constants of the three-point amplitudes so the total mass dimension of any such term would be non-negative. Power-counting alone suffices to rule out such terms in 5-point amplitudes and higher (even at four-point, power-counting and correct helicity transformation properties prohibit such new terms).

Therefore, all BCFW amplitudes agree, and must fall as $1/z$ in any limit that corresponds to a "valid" shift.

A Pure Scaling/Factorization Argument We could have obtained the same result by a more general inductive argument that does not rely on the functional form of the BCFW amplitudes but *only* on factorization and power-counting. Instead, we assume that lower-point amplitudes have the correct scaling as the z-parameter of (82) approaches infinity, and proceed by induction. This argument will also apply to the z^3 scaling of "wrong" shifts.

Suppose there is some component $f^0(p_1(z), p_2(z), \ldots, p_n)$ of an n-point amplitude that scales as z^0. It can necessarily be built out of z-independent invariants

$$\langle 12 \rangle, \, [12], \, [XY], \, P^2_{XY..Z} \tag{83}$$

and

$$\langle 1X \rangle, \, [2X], \tag{84}$$

or combinations of invariants

$$\frac{[1X]}{[1X]}, \, \frac{[1X]}{\langle 2X \rangle}, \tag{85}$$

where X, Y denotes any legs besides 1 and 2 (in expressions with multiple X's, they should be regarded as distinct, arbitrary legs).

However, the form of f^0 is tightly constrained by complex factorization — it cannot have poles in the invariants of (83). For example, at small $[XY]$ we have the factorization:

$$\lim_{[XY] \to 0} [XY] \langle XY \rangle A^{(n)}(p_1(z), p_2(z), \ldots, n) = A^{(n-1)}(p_1(z), p_2(z), ..., K) A^{(h)}(K, X, Y). \tag{86}$$

On the right-hand side, the $(n-1)$-point amplitude scales as $1/z$ by induction and the three-point amplitude is manifestly z-independent; therefore all terms in $A^{(n)}$ singular as $[XY] \to 0$ must also scale as $1/z$. Identical logic applies to the other poles in (83).

We now wish to show that any function f^0 that satisfies the 1 and 2 helicity transformation properties for an amplitude, is z-independent, and has no poles in the invariants of (83) must have non-negative mass dimension. To begin, we write down particular solutions with proper helicity transformations:

$$a_{-,-} = \frac{\langle 1X \rangle^2}{[2X]^2} \, (d = 0), \quad a_{+,+} = \frac{[2X]^2}{\langle 1X \rangle^2} \, (d = 0), \quad a_{-,+} = \langle 1X \rangle^2 [2X]^2 (d = 4). \tag{87}$$

General solutions can be obtained by multiplying these by z-independent, helicity-scalar invariants that have no forbidden poles:

$$\frac{[1X]}{[1Y]}, \quad \frac{\langle 1X \rangle}{\langle 1Y \rangle}, \quad \frac{[2X]}{[2Y]}, \quad \frac{\langle 2X \rangle}{\langle 2Y \rangle}, \quad \frac{\langle 2X \rangle [2X]}{\langle 1X' \rangle [1Y']}, \quad \langle 12 \rangle [12], \quad \frac{[12]\langle 1X \rangle}{[2Y]}, \quad \frac{\langle 12 \rangle [2X]}{\langle 1Y \rangle} \quad [XY], \langle XY \rangle. \tag{88}$$

but all such terms have non-negative mass dimension.

Five-point and higher amplitudes must have negative mass dimension, which we can only obtain by re-introducing "forbidden" poles or putting a negative power of z in the denominator. For $[+,+\rangle$ and $[-,-\rangle$ shifts, pure power-counting is not sufficient to rule out amplitudes that scale as z^0, but we have already constructed the unique four-particle gauge amplitudes, and they scale as $1/z$ (the additional constraint in this case comes from helicity transformations under the two unshifted legs).

To summarize: if k-point amplitudes scale as $1/z$ under large BCF shifts for k¡n, any term in an n-point amplitude that violates this scaling must not have singularities on which it factorizes into a z-independent amplitude and a lower-point amplitude with z-scaling determined by the induction hypothesis. But we cannot write any function consistent with factorization and helicity transformation that has the correct mass dimension to appear in an n-point amplitude but has none of the forbidden singularities. Therefore, the n-point gauge amplitudes must fall as fast as their lower-point counterparts, namely as $1/z$.

z^3 **Growth**   For helicities 1^+ and 2^-, we proceed as in the previous argument, but in this case we attempt to construct a function f^4 that scales as z^4, transforms correctly under helicity rotations of 1 and 2, and has negative mass dimension but no forbidden singularities.

A particular solution to the first two requirements is

$$a_{+,-} = [1X]^2 \langle 2Y \rangle^2 \, (d = 4). \tag{89}$$

Again, however, we cannot obtain any term of dimension 0 or lower without introducing forbidden poles. Therefore, the leading behavior of n-point amplitudes must be, as in the 3-point case, z^3.

B   Absence of Derivative Contributions to $[12]$ Singularity in Gravity

In this appendix, we revisit the expression (56) for the singularity as $[12] \to 0$ in gravity. The three-point amplitudes in (56) diverge as $[12]^{-2}$ for gravity, so $O([12])$ corrections to the $(n-1)$-point amplitudes in (56) will contribute to the singularity unless they cancel when summed over all legs. We consider these contributions here (all other terms of order 1 or $[12]$ were considered in Sec. 4.2).

We point out a property of (56) that we will use repeatedly: the $[12]^{-2}$ singularity in the i'th term is proportional to $[2i]\langle 1i \rangle$. Therefore, any contribution of $O([12])$ that is independent of i will appear as

$$c \left(\sum_i [2i] \langle 1i \rangle \right) = 0 \tag{90}$$

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by momentum conservation. Here, we are considering $O([12])$ effects, so only those that are i-dependent can contribute.

To study the differences in the amplitude at finite $[12]$, we should construct a sufficiently general explicit path in n-particle kinematics, parameterized by ϵ, with $[12] \propto \epsilon$ at small ϵ. On any such path, at least one leg besides 1 and 2 will have to shift momentum by $O(\epsilon)$ (the only momentum-conserving deformations that affect only legs 1 and 2, but keep these legs null and conserve momentum are BCF shifts, and these do not change $[12]$). The amplitudes will depend on this deformation, *but the terms for different i will all depend in the same way*, so when summed, they do not contribute to the singularity by (90).

This allows us to calculate using a particularly simple path, for instance one in which only leg n compensates for the deformations of legs 1 and 2:

$$|1\rangle = (1-\alpha)|\lambda\rangle, \quad |2\rangle = \alpha|\lambda\rangle - \epsilon|\mu\rangle, \, |2] = |n], \quad \text{and}\, |n(\epsilon)\rangle = |n\rangle - \epsilon|\mu\rangle, \qquad (91)$$

for arbitrary $|\mu\rangle$, $|\lambda\rangle$, and α. The BCFW shift in the i'th term will deform $|1]$, and replace $|i]$ with a slightly shifted momentum $\hat{K}_i$ — specifically, it will involve an $(n-1)$-point amplitude

$$A\left(|1\rangle\left(|1] + \epsilon\frac{\langle i\mu\rangle}{\langle i1\rangle}|n]\right), 3, \dots, |i\rangle\left(|i] + \epsilon\frac{\langle 1\mu\rangle}{\langle 1i\rangle}|n]\right), \dots, (|n\rangle - \epsilon|\mu\rangle)|n]\right). \qquad (92)$$

Then

$$\frac{dA_i}{d\epsilon} = \frac{\langle i\mu\rangle}{\langle i1\rangle}\left(\tilde{\lambda}_n \cdot \frac{\partial A}{\partial\tilde{\lambda}_1}\right) + \frac{\langle 1\mu\rangle}{\langle 1i\rangle}\left(\tilde{\lambda}_n \cdot \frac{\partial A}{\partial\tilde{\lambda}_i}\right) + \ i\text{-indep. terms.} \qquad (93)$$

The sum in (56) contains a term

$$\frac{[1n]}{\alpha\langle 1\mu\rangle}\sum_{i\neq 1,n}[ni]\langle 1i\rangle\frac{dA_i}{d\epsilon}. \qquad (94)$$

Inserting just the first term of (93) into this sum, we recover

$$-\left(\tilde{\lambda}_n \cdot \frac{\partial A}{\partial\tilde{\lambda}_1}\right)\sum_{i\neq 1,n}\langle i\mu\rangle[ni] = \left(\tilde{\lambda}_n \cdot \frac{\partial A}{\partial\tilde{\lambda}_1}\right) \times \langle 1\mu\rangle[n1], \qquad (95)$$

which has the same form as the second term, but for $i = 1$. Thus the total is a sum (which can now be written over all legs, since the $i = n$ term vanishes and the $i = 1$ term comes from (95)):

$$\langle 1\mu\rangle\sum_i[ni]\left(\tilde{\lambda}_n \cdot \frac{\partial A}{\partial\tilde{\lambda}_j}\right). \qquad (96)$$

This is identically zero on any bi-spinor $[xy]$ involving two of the legs that are summed over, since it receives opposite contributions from $i = x$ and $i = y$. Hence it also vanishes on any arbitrary function of kinematic invariants built out of the legs. Thus, we are justified in ignoring this source of $[12]$-dependence in the factorization argument of Sec. 4.2.2.

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