

Magnetic bion condensation: A new mechanism of confinement and mass gap in four dimensions

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ABSTRACT: In recent work, we derived the long distance confining dynamics of certain QCD-like gauge theories formulated on small $S^1 \times \mathbb{R}^3$ based on symmetries, an index theorem and abelian duality. Here, we give the microscopic derivation. The solution reveals a new mechanism of confinement in QCD(adj) in the regime where we have control over both perturbative and nonperturbative aspects. In particular, consider $SU(2)$ QCD(adj) theory with $1 \leq n_f \leq 4$ Majorana fermions, a theory which undergoes gauge symmetry breaking at small S^1 . If the magnetic charge of the BPS monopole is normalized to unity, we show that confinement occurs due to condensation of objects with magnetic charge 2, not 1. Due to index theorems, we know that such an object cannot be a two identical monopole configuration. Its net topological charge must vanish, and hence it must be topologically indistinguishable from the perturbative vacuum. We construct such objects, the magnetically charged, topologically null molecules of a BPS monopole and \overline{KK} antimonopole, which we refer as magnetic bions. An immediate puzzle with this proposal is the apparent Coulomb repulsion between BPS- \overline{KK} pair. An attraction which overcomes the Coulomb repulsion between the two is induced by $2n_f$ -fermion exchange. Bion condensation is also the mechanism of confinement in $\mathcal{N} = 1$ SYM on the same four-manifold. The $SU(N)$ generalization hints a possible hidden integrability behind nonsupersymmetric QCD of affine Toda type, and allows us to analytically compute the string tensions and thicknesses. We currently do not know the extension to \mathbb{R}^4 .

KEYWORDS: Nonperturbative QCD, Duality.

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1. Introduction

Probably, the most robust and important experimental and phenomenological observation about $SU(3)$ QCD is confinement, i.e., the absence of the free colored particles in isolation. Numerical lattice simulations unambiguously establish confinement in pure Yang-Mills theory and QCD. However, to date the analytical success had been limited. For reviews, see [1–3]

The QCD of Nature belongs to a subclass of asymptotically free and confining vector-like(QCD-like) gauge theories with no elementary scalars. This is a sufficiently good reason to warrant the study of the dynamics of such four dimensional QCD-like theories. In the last two decades, most theoretical efforts is concentrated into the dynamics of supersymmetric theories. It would be fair to say that despite many remarkable results obtained in such theories, its benefit to the QCD-like theories is still in its infancy. There is a very good reason for this. On \mathbb{R}^4 , there only exist one QCD-like supersymmetric theory, the pure $\mathcal{N} = 1$ SYM. All other supersymmetric theories have scalars, and hence non QCD-like by definition. In regimes where such theories are solved or understood quantitatively, such as mass deformation of $\mathcal{N} = 2$ SYM down to $\mathcal{N} = 1$ [4], the scalars never decouples from the dynamics. If they are forced to decouple by tuning certain parameters, one usually loses the theoretical control over the theory [5].

Our goal in this paper is more direct and motivated by the following question:

*Are there any asymptotically free and confining QCD-like theory in $d = 4$ dimensions (with no special properties such as supersymmetry) which we can understand its nonperturbative aspects exactly, and can derive the long distance (confining) dynamics starting with microscopic theory?*¹

On \mathbb{R}^4 , the answer seems to be out of reach currently. However, on locally four dimensional settings, such as spatial $S^1 \times \mathbb{R}^3$, the answer is yes. In particular, QCD with multiple adjoint representation fermions on small $S^1 \times \mathbb{R}^3$, ($S^1 \times \mathbb{R}^{2,1}$ in Minkowski setting) [8] becomes analytically tractable. Here, it is important that S^1 is not a thermal circle. It is a spatial circle along which fermions are endowed with periodic spin connection, and the resulting QCD-like theory is a zero temperature field theory on a space with one compact dimension. The benefits of considering this setup are, *i*) weak coupling, *ii*) unbroken (spatial) center symmetry. The latter is a consequence of the absence of thermal fluctuations and the fact that the quantum fluctuations favors the center symmetric vacuum.

In the small S^1 (weak coupling) limit of $SU(2)$ QCD(adj), the spatial Wilson line along the S^1 direction may be regarded as a *compact* adjoint Higgs field, and acquires a nontrivial (center symmetry respecting) vacuum expectation value, $U = \text{Diag}(e^{i\pi/2}, e^{-i\pi/2})$, due to radiatively induced Coleman-Weinberg potential. The photons and neutral fermions (A_μ, λ^I) parallel to U remains massless to all orders in perturbation theory, and all the other modes acquire masses and hence decouple from the infrared dynamics.

Nonperturbatively, there are topologically stable monopole configurations which are a consequence of gauge symmetry breaking. Since the adjoint Higgs field is compact, other than the BPS monopole, there is also a KK monopole. The magnetic and topological charges ($\int F, \int F\tilde{F}$) of these monopoles are normalized as

$$\text{BPS} : (+1, +\frac{1}{2}), \quad \overline{\text{BPS}} : (-1, -\frac{1}{2}) \quad \text{KK} : (-1, +\frac{1}{2}), \quad \overline{\text{KK}} : (+1, -\frac{1}{2}) \quad (1.1)$$

where bar denotes antimonopoles.

In [8], we constructed the $d = 3$ dimensional long distance theory for QCD(adj) formulated on $\mathbb{R}^3 \times S^1$ by employing three tools: abelian duality, symmetries, and index theorem. The unique lagrangian to order e^{-2S_0} dictated by these considerations is

$$L^{\text{dQCD}} = \frac{1}{2}(\partial\sigma)^2 - b e^{-2S_0} \cos 2\sigma + i\bar{\psi}^I \gamma_\mu \partial_\mu \psi^I + c e^{-S_0} \cos \sigma (\det_{I,J} \psi^I \psi^J + \text{c.c.}) \quad (1.2)$$

where σ and ψ^I are (dimensionless) dual photon, fermion, and dimensionless coordinates (measured in units of compactification circumference L) are used. The mass gap for gauge bosons is manifest in this lagrangian. The inverse of the mass gap is the characteristic size of the chromoelectric flux tube, hence confinement is also manifest. (See [8] for other observables.)

¹Two archetypes of non QCD-like theory in which the long distance theory can be derived starting with the microscopic theory are Polyakov's treatment [6] of Georgi-Glashow model on \mathbb{R}^3 (a theory which confines), and Nekrasov's derivation [7] of the $\mathcal{N} = 2$ Seiberg-Witten prepotential (a theory which does not confine). Our goal is to find such quantitatively tractable examples among QCD-like theories.

In this work, we will derive this lagrangian by summing over all non-perturbative effects. Before doing so, note a simple but important feature of 1.2. It is clear that fermionic interaction terms arises due to the monopole effects. Any monopole carries a net topological charge. If massless fermions are present in the underlying theory, due to the index theorem, a monopole vertex must be associated with $2n_f$ fermion zero modes of one chirality and an antimonopole leads to $2n_f$ zero modes of the opposite chirality. Consequently, the terms involving fermion zero mode insertions is the sum of the vertices:

$$\begin{aligned} \text{BPS} &: e^{i\sigma} \det_{I,J} \psi^I \psi^J, & \text{KK} &: e^{-i\sigma} \det_{I,J} \psi^I \psi^J, \\ \overline{\text{BPS}} &: e^{-i\sigma} \det_{I,J} \bar{\psi}^I \bar{\psi}^J, & \overline{\text{KK}} &: e^{i\sigma} \det_{I,J} \bar{\psi}^I \bar{\psi}^J, \end{aligned} \quad (1.3)$$

where $e^{iq_m\sigma}$ is the monopole vertex operator and $q_m = \pm 1$ are magnetic charges of the corresponding (anti)monopole, and $\det_{I,J} \psi^I \psi^J$ are compulsory zero modes attached to it. Now, let us inspect the bosonic potential. It is

$$V(\sigma) \sim \cos 2\sigma \sim e^{i2\sigma} + e^{-2i\sigma} \quad (1.4)$$

Due to the index theorem, a bosonic potential cannot arise due to objects which carry a nonvanishing topological charge. Such objects, by construction, must have fermion zero mode insertions, and can not appear in the bosonic potential. It is easy to check that the functional integral $Z = \int D\sigma e^{-\int d^3x [\frac{1}{2}(\partial\sigma)^2 - b e^{-2S_0} \cos 2\sigma]}$ is equivalent to a plasma of magnetically charged particles with long range Coulomb interaction,

$$V(r) = \frac{2(\pm 2)}{4\pi r} \quad (1.5)$$

where charges are twice the one of the monopoles. In other words, the Debye phenomena (which renders the dual photon massive) is induced not due to excitations with magnetic and topological charge $(\pm 1, \pm \frac{1}{2})$, but rather with charges $(\pm 2, 0)$. Clearly, these are not elementary monopoles. The first question we want to answer is, what are these objects?

A fuller discussion of all pairs and their roles will be given in section 2.2. For now, let us observe that only a bound state of BPS monopole, and KK antimonopole, $\text{BPS}\overline{\text{KK}}$, and its antiparticle can induce the bosonic potential. Such an object has the correct quantum numbers $(1, \frac{1}{2}) + (1, -\frac{1}{2}) = (2, 0)$ and is the prime candidate for the magnetically charged object which leads to confinement in QCD(adj) in the $LA \ll 1$ regime.

There is an immediate puzzle with this proposal. The BPS and $\overline{\text{KK}}$ monopoles interact via Coulomb repulsion, hence in order to have a bound state, there must exist an attraction which may overcome the Coulomb repulsion.² In the QCD(adj) vacuum, a pairing mechanism arguably as strange as the BCS theory [9] takes place. An attraction which overwhelms the Coulomb repulsion between BPS and $\overline{\text{KK}}$ is generated via (even number of) fermion exchange.

²This situation is analogous to the BCS theory of superconductivity. There must exist a net attraction between electron pairs which overcomes the shielded, yet repulsive Coulomb potential. Such an attractive force is provided through the exchange of phonons of the crystal lattice. A novel pairing mechanism is at work in QCD(adj) formulated on small $S^1 \times \mathbb{R}^3$.

In $n_f = 1$ QCD(adj) (i.e., SYM), this is a fermion pair exchange. In $n_f > 1$ QCD(adj), it is the exchange of $2n_f$ fermions. The attractive potential is a logarithmical one

$$V_{\text{eff}}(r) = 4n_f \log r + \frac{1}{4\pi r}, \quad r \gg 1 \quad (1.6)$$

and it easily overcomes the repulsive Coulomb interaction. This forces the BPS and $\overline{\text{KK}}$ monopoles to form a charged bound state. We refer to this molecule as a **magnetic bion**, and to the $\overline{\text{BPS}}$ -KK molecule as **anti-bion**. The important point that is worth repeating is that the net topological charge of the BPS- $\overline{\text{KK}}$ pair is identically zero: $\int_{\mathbb{R}^3 \times S^1} F\tilde{F} = 0$ even though for individual (isolated) BPS, it is $\int_{\mathbb{R}^3 \times S^1} F\tilde{F} = \frac{1}{2}$ and for $\overline{\text{KK}}$, it is $\int_{\mathbb{R}^3 \times S^1} F\tilde{F} = -\frac{1}{2}$. Consequently, bions do not have fermions zero modes attached to them, and they are the leading contribution to the effective bosonic potential for the dual photon.³

Considerations along these line also provides dynamical explanations for the absence of confinement in Yang-Mills *noncompact* Higgs system with adjoint fermions formulated on \mathbb{R}^3 . Affleck, Harvey and Witten in Ref. [11] showed that such systems do not confine despite the presence of magnetic monopoles. Their argument is based on symmetries and index theorems. Without much recourse to the microscopic theory, they showed that the photon arises as a Goldstone boson of spontaneously broken fermion number symmetry, hence remains massless nonperturbatively. Here, we give microscopic derivation of this beautiful symmetry argument based on the dynamics of monopoles (and bions). In one sentence, the absence of magnetically charged, but topologically null configurations (which may be the only source of a mass term for dual photon in the presence of fermions) implies the absence of confinement in the $SU(2)$ application. We also provide dynamical explanation for the absence of confinement in the $\mathcal{N} = 2$ SYM theory on \mathbb{R}^3 based on similar rationale⁴.

The discussion of nonsupersymmetric QCD(adj) can also be applied to $\mathcal{N} = 1$ SYM on $\mathbb{R}^3 \times S^1$ with only cosmetic changes. All one needs to be careful about is the extra massless scalar, and keep it in the effective theory. In fact, the long distance effective theory for SYM (which is a supersymmetric affine Toda theory) was derived far before our work on the subject [12–15].⁵ In spite of that, the fact that confinement was induced not due to monopole condensation, but rather via magnetic bion condensation was not appreciated before. Remarkably, the mechanism of confinement for $\mathcal{N} = 1$ SYM and nonsupersymmetric QCD(adj) is one and the same in the small S^1 regime.

The second part of the paper discusses the $SU(N)$ generalization of the nonsupersymmetric QCD(adj), and derive the long distance Lagrangian. The biggest surprise is that the

³In literature, the name BIon where BI stands for Born-Infeld is used previously [10]. We use bi- as two or having two. We hope that the two will not be confused. We did not use di-on due to its phonetic similitude to dyon.

⁴These theories (formulated on \mathbb{R}^3) are as important as QCD(adj) on $\mathbb{R}^3 \times S^1$. They exhibit that if massless fermions are present, having monopoles is not sufficient to have confinement.

⁵Our derivation of the bosonic potentials in SYM differs from earlier work, which was based on using supersymmetry as a completion device to obtain superpotential (hence bosonic potential) from the monopole induced fermionic terms. We instead chose to delineate on the microscopic origin of the bosonic potential, and obtained it directly without any recourse to supersymmetry. The final result is same with earlier work [12–15]. The real payoff of our approach is in its applicability to nonsupersymmetric theories.

bosonic sector of QCD(adj) maps into an integrable system, intimately related to possible integrable generalization of the affine Toda theories. We identify magnetic bions as bound states of magnetic monopoles with charge α_j and antimonopoles with charge $-\alpha_{j+1}$. The total net effects of bions can be encoded into a *prepotential*, out of which we may derive the potential. Interestingly enough, the relation between the prepotential and potential is same as the relation between the superpotential and potential in $\mathcal{N} = 1$ SYM, modulo the absence of Higgs scalar in the former (where it is massive). We give the analytic derivations of the string tensions T_k , and characteristic sizes of chromoelectric flux tubes m_k^{-1} , $k = 1, \dots, N-1$ in QCD(adj) in the small S^1 regime.

Let us complete the introduction by saying that closer and deeper inspection of non-supersymmetric QCD-like theories may also be used to build the relation between the inner goings-on of the supersymmetric and non-supersymmetric gauge theories. Suffice it to say that, the integrable systems which emerges in the QCD(adj) are variants of the affine Toda systems [16–18], which also appeared in the discussions of $\mathcal{N} = 1, 2$ SYM, and elliptic curves [19]. This direction will not be explored in this paper, but is potentially interesting.⁶

2. Dynamics of $SU(2)$ QCD(adj) on small $S^1 \times \mathbb{R}^3$

2.1 Perturbation theory

First, we wish to give the microscopic derivation of the dual theory 1.2. The action of $SU(N)$ QCD(adj) defined on $\mathbb{R}^3 \times S^1$ is

$$S = \int_{\mathbb{R}^3 \times S^1} \frac{1}{g^2} \text{tr} \left[\frac{1}{4} F_{MN}^2 + i \bar{\lambda}^I \bar{\sigma}^M D_M \lambda_I \right] \quad (2.1)$$

where $\lambda_I = \lambda_{I,at_a}$, $a = 1, \dots, N^2 - 1$ is Weyl fermion in adjoint representation, F_{MN} is the nonabelian gauge field strength, and I is the flavor index, and the generators are normalized as $\text{tr } t^a t^b = \delta^{ab}$. The classical theory possess a $U(n_f)$ flavor symmetry whose $U(1)_A$ part is anomalous. The symmetry of the quantum theory is

$$(SU(n_f) \times \mathbb{Z}_{2Nn_f}) / \mathbb{Z}_{n_f} \quad (2.2)$$

The quantum theory has the dynamical strong scale Λ , which arises via dimensional transmutation, and is given by $\Lambda^{b_0} = \mu^{b_0} e^{-8\pi^2/g^2(\mu)N}$ where μ is the renormalization group scale and $b_0 = (11 - 2n_f)/3$. We consider small n_f so that asymptotic freedom is preserved. The $n_f = 1$ case (SYM) will be discussed separately. We first discuss $N = 2$ QCD(adj), and $N \geq 3$ will be discussed in section 3.

At small S^1 ($LA \ll 1$), due to asymptotic freedom, the gauge coupling is small and a perturbative Coleman-Weinberg analysis is reliable [27]. Let $U(x) = P e^{i \int dx_4 A_4(x, x_4)}$ be the path ordered holonomy of the spatial Wilson line wrapping the S^1 , and sitting at the

⁶There are also recent, interesting works on the dynamics of four dimensional gauge theories, in particular for pure Yang-Mills, see [20–22], and for lattice works, see [1, 23] and references therein. Also, good reviews covering different aspects of monopoles and instantons can be found in [24–26].

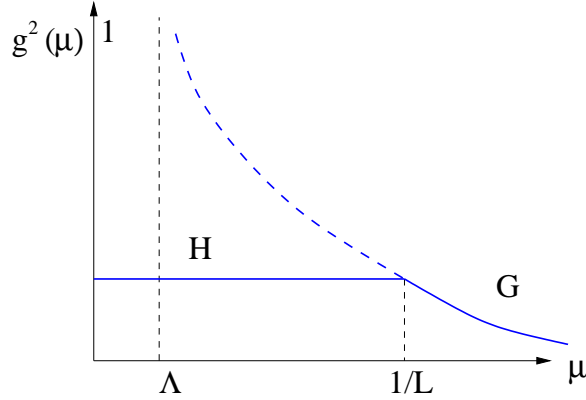


Figure 1: Summary of perturbative analysis: Solid line indicates the running of the gauge coupling in QCD(adj) compactified on a circle S^1 with circumference L , and dashed line is the usual running on \mathbb{R}^4 . In the regime $1/L \gg \Lambda$ perturbative Coleman-Weinberg analysis is reliable, and leads to a radiatively induced gauge symmetry breaking $G \rightarrow H$ where $G = SU(2)$ and $H = U(1)$. Since the heavy W bosons and their fermionic partners which are charged under H decouple from the long distance theory, the coupling constant of the IR theory does not flow. The reader familiar with the $\mathcal{N} = 2$ SYM theory on \mathbb{R}^4 will realize certain similarities. Unlike pure $\mathcal{N} = 2$ SYM, QCD(adj) confines.

point $x \in \mathbb{R}^3$. Integrating out the heavy KK-modes along the S^1 circle, $|\omega_n| \geq \omega_1$ where $\omega_n = \frac{2\pi}{L}n, n \in \mathbb{Z}$, induce a nontrivial effective potential for $U(x)$ [28]. The action for the classical zero modes reduce to

$$S = \int_{\mathbb{R}^3} \frac{L}{g^2} \text{tr} \left[\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu \Phi)^2 + g^2 V(|\Phi|) + i \bar{\lambda}^I (\bar{\sigma}^\mu D_\mu + \bar{\sigma}_4 [\Phi, \cdot]) \lambda_I \right] \quad (2.3)$$

The minimum of the potential V_{eff} is located at $|\Phi| \equiv \phi = \frac{\pi}{2}$, hence $U = \text{Diag}(e^{i\pi/2}, e^{-i\pi/2})$. Since $\text{tr} U = 0$, the \mathbb{Z}_2 center symmetry is preserved. By the Higgs mechanism, the gauge symmetry is broken down as

$$SU(2) \rightarrow U(1) \quad (2.4)$$

Due to the gauge symmetry breaking via an ‘adjoint Higgs field’, the neutral fields aligned with U along the Cartan subalgebra $(A_{3,\mu}, \lambda_3^I)$ remain massless, and off-diagonal components acquire mass, given by the separation between the eigenvalues of the Wilson line

$$m_{W^\pm} = m_{\lambda^{I,\pm}} = \pi/L \quad (2.5)$$

where \pm refers to the charges under unbroken $U(1)$. Therefore, in perturbation theory, the low energy theory is a $d = 3$ dimensional abelian $U(1)$ gauge theory with n_f massless fermions with a free action

$$S = \int_{\mathbb{R}^3} \frac{L}{g^2} \left[\frac{1}{4} F_{3,\mu\nu}^2 + i \bar{\lambda}_3^I \bar{\sigma}^\mu \partial_\mu \lambda_{3,I} \right] \quad (2.6)$$

At distances shorter than L , the coupling constant flows according to the four dimensional renormalization group. Since the heavy $W^\pm, \lambda^{I,\pm}$ which are charged under $U(1)$ decouple

from the long distance physics at scale L and above, the coupling constant ceases to run at $1/L \gg \Lambda$ much before the strong coupling sets in, see fig.2.1. In perturbation theory, this is the whole story.

2.2 Nonperturbative effects and abelian duality

Nonperturbatively, the perturbatively free infrared fixed point is unstable. This follows from the existence of monopoles, at the cores of which the $U(1)$ symmetry of the free theory enhances to the the whole $SU(2)$.⁷

Due to gauge symmetry breaking via a *compact* adjoint Higgs field, there are two types of monopoles, BPS and KK, as well as their antimonopoles $\overline{\text{BPS}}$, $\overline{\text{KK}}$ [30, 31]⁸ These four types of monopoles are distinguished by their quantized magnetic and topological charges $(\int F, \int F\tilde{F})$ normalized as

$$\begin{aligned} \text{BPS} &: (+1, \frac{1}{2}), & \overline{\text{BPS}} &: (-1, -\frac{1}{2}), \\ \text{KK} &: (-1, \frac{1}{2}), & \overline{\text{KK}} &: (+1, -\frac{1}{2}). \end{aligned} \quad (2.7)$$

Due to the chiral anomaly relation [32],

$$\partial_\mu J^{\mu 5} = \frac{g^2(2Nn_f)}{32\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \quad (2.8)$$

each object with a vanishing topological charge is associated with a certain number of fermionic zero modes. Integrating both sides over the space, we find

$$\Delta Q_5 = n_\lambda - n_{\bar{\lambda}} = 4n_f \int \frac{g^2}{32\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} = \begin{cases} 4n_f(\frac{1}{2}) = 2n_f & \text{for BPS or KK} \\ 4n_f(-\frac{1}{2}) = -2n_f & \text{for } \overline{\text{BPS}} \text{ or } \overline{\text{KK}} \end{cases} \quad (2.9)$$

where the term inside the parenthesis is the topological charge. As it should be clear, $4n_f$ is the number of fermionic zero modes associated with a four dimensional instanton, whose topological charge is $+1$. Since the topological charges of monopoles are a fraction of the one of the instanton, they are sometimes referred as fractional instantons. Clearly, a BPS-KK pair has the correct quantum numbers to be the constituents of the instanton [30, 31].

By abelian duality [6, 29], we know that the functional integral in a gauge theory in the presence of a single monopole with charge ± 1 located at the position x is equivalent to the insertion of an operator $e^{\pm i\sigma(x)}$ in the path integral of the dual theory. However, the index version of the chiral anomaly relations 2.9 tells us that a monopole acts as if it contains a source for every fermion flavor, and an antimonopole acts as if it contains a sink for all

⁷There is a semantic problem here which I could not avoid. The objects that I refer as monopoles are essentially instantons from the $d = 3$ dimensional point of view (See for example, pages 1232- 1240 of ref. [29]). However, the theory I am working with is locally four dimensional, and there are in fact four dimensional instantons as well. Hence, I will stick with the four dimensional language. Technically however, the quantum numbers are unambiguous. I thank Misha Shifman and David Tong for their explanations on this matter.

⁸Were the gauge symmetry broken by a noncompact Higgs field, the KK monopole would not be there. As we will discuss, this is the case in the extension of the Polyakov model in the presence of adjoint fermions, a theory which does not confine.

fermion flavors. Adapting a combination of techniques developed by 't Hooft [33] and by Polyakov [6] to our problem, we can sum up all the monopole effects. The functional integral (with a source) in the presence of a monopole

$$\int DA_\mu D\psi^I D\bar{\psi}^I e^{-S_{\text{one mon.}}(A,\psi,\bar{\psi})+J \det \psi^I \psi^J + \bar{J} \det \bar{\psi}^I \bar{\psi}^J} \quad (2.10)$$

is same as having

$$e^{-S_0} \int D\sigma D\psi^I D\bar{\psi}^I e^{-S_{d,0}(\sigma,\psi,\bar{\psi})+J \det \psi^I \psi^J + \bar{J} \det \bar{\psi}^I \bar{\psi}^J} e^{i\sigma(x)} \det_{I,J} \psi^I \psi^J \quad (2.11)$$

where $S_{d,0}(\sigma,\psi,\bar{\psi}) = \int_{\mathbb{R}^3} [\frac{1}{2}(\partial\sigma)^2 + i\bar{\psi}^I \gamma_\mu \partial_\mu \psi^I]$ is the free kinetic term. Hence, a functional integral in the presence of a monopole can be translated into having a monopole vertex $e^{i\sigma(x)}$ with accompanying fermionic zero modes. We can insert the monopole at any $x \in \mathbb{R}^3$, and we can consider an arbitrary number of them. The sum over all possible monopole configurations is

$$\sum_{n_{\text{BPS}}=0}^{\infty} \sum_{n_{\overline{\text{BPS}}}=0}^{\infty} \sum_{n_{\text{KK}}=0}^{\infty} \sum_{n_{\overline{\text{KK}}}=0}^{\infty} \frac{e^{-(n_{\text{BPS}}+n_{\overline{\text{BPS}}}+n_{\text{KK}}+n_{\overline{\text{KK}}})S_0}}{n_{\text{BPS}}! n_{\overline{\text{BPS}}}! n_{\text{KK}}! n_{\overline{\text{KK}}}!} \left[\int d^3x e^{i\sigma(x)} \det_{I,J} \psi^I \psi^J \right]^{n_{\text{BPS}}} \left[\int d^3x e^{-i\sigma(x)} \det_{I,J} \bar{\psi}^I \bar{\psi}^J \right]^{n_{\overline{\text{BPS}}}} \left[\int d^3x e^{-i\sigma(x)} \det_{I,J} \psi^I \psi^J \right]^{n_{\text{KK}}} \left[\int d^3x e^{i\sigma(x)} \det_{I,J} \bar{\psi}^I \bar{\psi}^J \right]^{n_{\overline{\text{KK}}}} \quad (2.12)$$

Performing the summation yields monopole induced terms of order e^{-S_0} in our effective lagrangian

$$\exp \left[\int d^3x e^{-S_0} (e^{i\sigma} + e^{-i\sigma}) (\det_{I,J} \psi^I \psi^J + \det_{I,J} \bar{\psi}^I \bar{\psi}^J) \right] \quad (2.13)$$

Therefore, the combined effect of BPS and KK monopoles is $\cos \sigma \det \psi^I \psi^J$. This vertex is manifestly invariant under continuous $SU(n_f)$ flavor symmetry, acting as $\psi \rightarrow U\psi$ where $U \in SU(n_f)$. The microscopic theory also possesses a \mathbb{Z}_{4n_f} discrete chiral symmetry.⁹ The effective theory, in order to respect the \mathbb{Z}_{4n_f} discrete chiral symmetry, intertwines it with a discrete shift symmetry of the dual photon:

$$\psi^I \rightarrow e^{i2\pi/(4n_f)} \psi^I, \quad \sigma \rightarrow \sigma + \pi \quad (2.14)$$

both of which acts as negation on the determinantal fermion vertex and cosine combinations

$$\det_{I,J} \psi^I \psi^J \rightarrow -\det_{I,J} \psi^I \psi^J, \quad \cos \sigma \rightarrow -\cos \sigma \quad (2.15)$$

respectively, so that the effective theory respects the real symmetries of the underlying theory.

⁹More generally, consider $SU(N)$ QCD(adj) with n_f flavors. The chiral symmetry is $[SU(n_f) \times \mathbb{Z}_{2Nn_f}] / \mathbb{Z}_{n_f}$, where the common \mathbb{Z}_{n_f} is factored out to prevent double counting. The \mathbb{Z}_2 subgroup of the \mathbb{Z}_{2N} is $(-1)^F$ fermion number modulo 2, which cannot be spontaneously broken so long as Lorentz symmetry is unbroken. Thus, the only genuine discrete chiral symmetry of $SU(N)$ QCD(adj) which may potentially be broken is the remaining \mathbb{Z}_N , irrespective of the number of flavors. In small S^1 , we explicitly demonstrate the existence of N vacua, and spontaneous breaking of chiral \mathbb{Z}_N symmetry (which is intertwined with the discrete shift symmetry of photon). This \mathbb{Z}_N symmetry should not be confused with the spatial center symmetry, $\mathcal{G}_s = \mathbb{Z}_N$ which is unbroken in spatial compactification of QCD(adj).

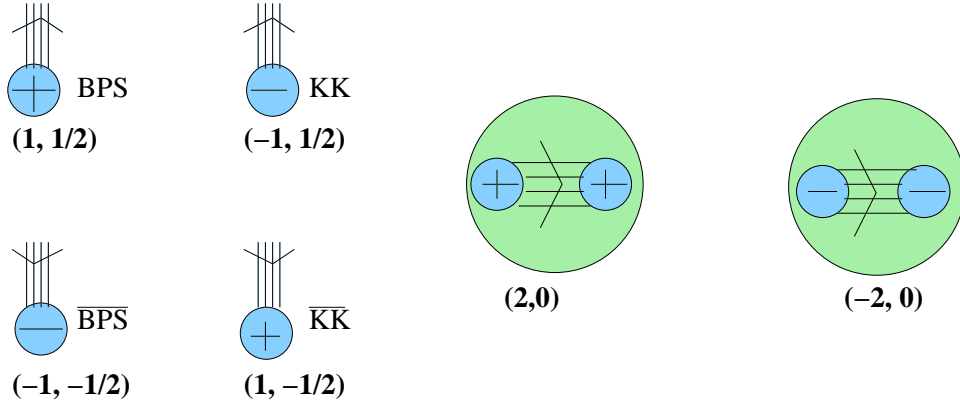


Figure 2: (Left) Magnetically and topologically charged monopoles carries compulsory fermion zero modes. Consequently, they can not induce a bosonic potential for dual photon. (Right) Topologically null, magnetically charged bions has no external fermionic legs. Hence, they induce the leading bosonic potential, which implies mass for the dual photon and confinement. The figure is for $SU(2)$ with $n_f = 2$. The combination of the BPSKK monopoles (which is not depicted) is an instanton (or caloron). It is present in confined phase, but is not the source of the dual photon mass term.

In the effective Lagrangian, this is the set of all nonperturbative effects at order e^{-S_0} in the e^{-S_0} expansion. However, the discrete \mathbb{Z}_2 shift symmetry $\sigma \rightarrow \sigma + \pi$, unlike a continuous shift symmetry, cannot prohibit a mass term for the scalar σ . Clearly, a term $e^{-S_0} \cos \sigma$ is forbidden by \mathbb{Z}_2 . But its square is an allowed operator. If fermions were not present,

$$e^{-S_0} \cos \sigma \sim e^{-S_0} (e^{i\sigma} + e^{-i\sigma}) \quad (2.16)$$

would be an allowed term as in the Polyakov's discussion of the Georgi-Glashow model, and would induce a mass term of order $e^{-S_0/2}$ for dual photon. However, because of the index theorem 2.9, a monopole must come with fermion zero modes, and a term such as $e^{i\sigma}$ can not appear on its own, but must appear in combination $e^{i\sigma} \det_{I,J} \psi^I \psi^J$.

Symmetry principles also tells us that, at the e^{-2S_0} order, we can write

$$[e^{-S_0} \cos \sigma]^2 \sim e^{-2S_0} (1 + 1 + e^{2i\sigma} + e^{-2i\sigma}) \quad (2.17)$$

and this would generate a mass term for the dual photon, hence leading to confinement. We wish to understand the dynamical origin of this potential.

Let us first forget about the issues about fermion zero modes, and decide on the basis of quantum numbers, which objects may contribute to the nonperturbative potential. Since we know that, due to index theorem, such an object can not be a monopole, let us enlist all possible pairs of monopoles, the magnetic and topological charges of constituents and pairs, and the types of the long range Coulomb interactions, repulsive or attractive. In nonsupersymmetric QCD(adj) with $2 \leq n_f \leq 4$, the list of all Coulomb interaction channels

for monopoles is given by

Type	Type	σ -int.	$(\int F, \int F \widetilde{F})$
BPS $- e^{+i\sigma}$	BPS $- e^{+i\sigma}$	rep.	$(+1, +\frac{1}{2}) + (+1, +\frac{1}{2}) = (2, 1)$
BPS	$\overline{\text{BPS}} - e^{-i\sigma}$	att.	$(+1, +\frac{1}{2}) + (-1, -\frac{1}{2}) = (0, 0)$
BPS	KK $- e^{-i\sigma}$	att.	$(+1, +\frac{1}{2}) + (-1, +\frac{1}{2}) = (0, 1)$
BPS	$\overline{\text{KK}} - e^{+i\sigma}$	rep.	$(+1, +\frac{1}{2}) + (+1, -\frac{1}{2}) = (2, 0)$
$\overline{\text{BPS}}$	$\overline{\text{BPS}}$	rep.	$(-1, -\frac{1}{2}) + (-1, -\frac{1}{2}) = (-2, -1)$
$\overline{\text{BPS}}$	KK	rep.	$(-1, -\frac{1}{2}) + (-1, +\frac{1}{2}) = (-2, 0)$
$\overline{\text{BPS}}$	$\overline{\text{KK}}$	att.	$(-1, -\frac{1}{2}) + (+1, -\frac{1}{2}) = (0, -1)$
KK	KK	rep.	$(-1, +\frac{1}{2}) + (-1, +\frac{1}{2}) = (-2, 1)$
KK	$\overline{\text{KK}}$	att.	$(-1, +\frac{1}{2}) + (+1, -\frac{1}{2}) = (0, 0)$
$\overline{\text{KK}}$	$\overline{\text{KK}}$	rep.	$(+1, -\frac{1}{2}) + (+1, -\frac{1}{2}) = (2, -1)$

(2.18)

In the presence of the fermion zero modes, the (bosonic) potential must arise due to the sector of the theory with zero topological charge so that there will not be any fermion zero mode insertions in it. In other words, the objects which may contribute to the potential must be topologically indistinguishable from the perturbative vacuum.

This immediately rules out the four possible monopoles, and six of the ten pairs in our list from contributing to the bosonic potential. In particular, the two identical monopole configurations such as BPSBPS with $(1, \frac{1}{2}) + (1, \frac{1}{2}) = (2, 1)$, have the correct magnetic charge, but its topological charge does not permit it to contribute to the bosonic potential. Another interesting combination which does not lead to the confining potential is a BPSKK pair. The BPSKK pair in fact constitute an instanton (sometimes called a caloron, [30,31]) with charge $(1, \frac{1}{2}) + (-1, \frac{1}{2}) = (0, 1)$ and does not induce mass term for the dual photon.

The monopole and antimonopole pairs such as BPS- $\overline{\text{BPS}}$ are topologically null, but also magnetically neutral. Their contribution to the effective potential can only be an uninteresting constant. There remains a single option. A bound state of BPS monopole, and KK antimonopole, BPS- $\overline{\text{KK}}$, and its conjugate. Such an object has the correct quantum numbers $(\pm 1, \frac{1}{2}) + (\pm 1, -\frac{1}{2}) = (\pm 2, 0)$. We referred to this object as a magnetic bion. Consequently, the bion is the prime candidate which may lead to confinement in QCD(adj) in the $LA \ll 1$ regime.

However, there is an immediate puzzle with this proposal. There is a long range Coulomb repulsion between BPS- $\overline{\text{KK}}$ constituents of the bion. If we wish to have a bound state, there must exist an attractive interaction which overcomes the repulsive Coulomb force. Indeed, there is!

2.3 Pairings and attractive multi-fermion exchanges

The presence of fermion zero modes changes things drastically. We will demonstrate that for the pairs with net topological charge zero, there exists an attractive $V_{\text{eff}} \sim \log r$ interaction between the constituents due to fermion pair exchanges. For the pairs with a nonvanishing topological charge, the constituents do not interact at all due to chirality at leading order.

Let us first show the first assertion: Consider a BPS and $\overline{\text{KK}}$ monopoles located at $x, y \in \mathbb{R}^3$, where $|x - y| \gg 1$. (x, y are dimensionless coordinates in units of L .) We can extract their interactions from the connected correlator of the BPS vertex $V_{\text{BPS}}(x)$, and $\overline{\text{KK}}$ vertex $V_{\overline{\text{KK}}}(y)$ in the free dual theory with action $S_{d,0}(\sigma, \psi, \bar{\psi})$

$$\begin{aligned} \langle V_{\text{BPS}}(x) V_{\overline{\text{KK}}}(y) \rangle_0 &= \langle e^{i\sigma(x)} \det_{IJ} \psi^I \psi^J(x) e^{+i\sigma(y)} \det_{I'J'} \bar{\psi}^{I'} \bar{\psi}^{J'}(y) \rangle_0 \\ &= \langle e^{i\sigma(x)} e^{i\sigma(y)} \rangle_0 \langle \det_{IJ} \psi^I \psi^J(x) \det_{I'J'} \bar{\psi}^{I'} \bar{\psi}^{J'}(y) \rangle_0 \\ &\sim e^{-G(x-y)} [S_F(x-y)]^{2n_f} \end{aligned} \quad (2.19)$$

where $G(x-y) = \frac{1}{4\pi|x-y|}$ is the Coulomb potential, which is the position space propagator of the σ field, $G(x) = \int \frac{d^3p}{(2\pi)^3} e^{ipx} \frac{1}{p^2}$ and $S(x) = \frac{\sigma^\mu x^\mu}{4\pi|x|^3}$ is the $d=3$ dimensional free fermion propagator $S(x) = \sigma^\mu \frac{\partial}{\partial x^\mu} G(x)$. The static interaction potential between the BPS and $\overline{\text{KK}}$ pair is

$$V_{\text{eff}}(x-y) = -\log \langle V_{\text{BPS}}(x) V_{\overline{\text{KK}}}(y) \rangle_0 = \frac{1}{4\pi|x-y|} + 4n_f \log|x-y| \quad (2.20)$$

Asymptotically, $4n_f \log|x-y|$ is the dominant attractive interaction term, and it easily overcomes the Coulomb repulsion. Therefore, there exist a stable bion bound state with the total magnetic and topological charge $(+2, 0)$, and antibion with charge $(-2, 0)$.

Analogously, the net interaction between a BPS- $\overline{\text{BPS}}$ pair is attractive in both interaction channels, either Coulomb, or fermion exchange interactions. The long distance attraction has the form $-\log \langle V_{\text{BPS}}(x) V_{\overline{\text{BPS}}}(y) \rangle_0 = -\frac{1}{4\pi|x-y|} + 4n_f \log|x-y|$.

Due to chirality of the underlying theory, the interaction between pairs with the same topological charge vanishes identically: $\langle V_{\text{BPS}}(x) V_{\text{BPS}}(y) \rangle_0 = \langle V_{\text{BPS}}(x) V_{\text{KK}}(y) \rangle_0 = 0$.

Since the topological charge of the magnetic bion is zero, it does not have any fermion zero mode attached to it. Since magnetic bions and antibions has ± 2 magnetic charges, they will lead to Debye phenomena. The appropriate effective potential induced by bions is indeed what we wrote based on symmetry arguments:

$$V(\sigma) = [e^{-S_0} \cos \sigma]^2 \sim e^{-2S_0} (1 + 1 + e^{2i\sigma} + e^{-2i\sigma}) \quad (2.21)$$

The terms in the potential has an interpretation as the contribution of respectively $\text{BPS}\overline{\text{BPS}} + \text{KK}\overline{\text{KK}} + \text{BPS}\overline{\text{KK}} + \text{KK}\overline{\text{BPS}}$.

More precisely, the interaction terms in the lagrangian are due to monopole and bion contributions. The monopole contributions necessarily involve the fermion interactions. Schematically, the nonperturbatively induced interaction terms will always be

$$L_{\text{int}} = \underbrace{\sum_{\text{bions}} V_{\text{bion}}}_{\int F\tilde{F}=0} + \underbrace{\sum_{\text{monopoles}} V_{\text{monopole}}}_{\int F\tilde{F}=\pm\frac{1}{2}} \quad (2.22)$$

Therefore, the dual QCD lagrangian for $SU(2)$ QCD(adj) on small $S^1 \times \mathbb{R}^3$ is given by

$$L^{\text{dQCD}} = \frac{1}{2}(\partial\sigma)^2 - b e^{-2S_0} \cos 2\sigma + i\bar{\psi}^I \gamma_\mu \partial_\mu \psi_I + c e^{-S_0} \cos \sigma (\det_{I,J} \psi^I \psi^J + \text{c.c.}) \quad (2.23)$$

up to higher order (insignificant) terms in e^{-S_0} .

The potential term for the dual photon, when expanded around one of its two minima (located at 0 and π), provides a mass term for the dual photon. In Euclidean viewpoint, the photon mass is the inverse Debye screening length in the plasma of magnetic bions. On a fixed time slice of a timelike Wilson loop, the inverse photon mass is the thickness of the chromoelectric flux tube formed between two external electric test charges. Just like Polyakov model [6] on \mathbb{R}^3 , the QCD(adj) on small $S^1 \times \mathbb{R}^3$ exhibits linear confinement,

$$V_{\text{linear}}(R) \sim e^{-S_0} R \tag{2.24}$$

the potential energy of a pair of the electric source separated by a distance R grows linearly with separation.

Remark: The results and approach of this work should not be confused with 't Hooft's abelian projection scheme [34], which only leaves an $U(1)^{N-1}$ gauge symmetry. Hence, monopoles in that case are gauge artifacts, which is fine in the prescribed gauge. In our case, the gauge symmetry breaking $SU(N) \rightarrow U(1)^{N-1}$ is dynamical, and is a well-controlled effect due to radiatively induced Coleman-Weinberg potential. The QCD(adj) in $LA \ll 1$ regime tells us that, in the presence of fermions, the idea of monopole condensation no longer holds due to fermion zero modes. Despite this fact, the qualitative and beautiful idea of dual superconductivity of 't Hooft and Mandelstam [34, 35] is still realized at a quantitative level, albeit via condensation of the pairs with combined magnetic and topological charges $(\pm 2, 0)$.

As emphasized, the presence of monopoles is not sufficient to induce confinement, or monopole condensation. Better appreciation of the above picture can come with the study of a Yang-Mills Higgs system with adjoint fermions on \mathbb{R}^3 , a system with monopoles and yet, no confinement.

2.4 Noncompact Higgs with adjoint fermions on \mathbb{R}^3 , and the lack of confinement

Affleck, Harvey and Witten studied extensions of the Polyakov's model in the presence of an adjoint Dirac fermion on \mathbb{R}^3 [11]. The generalization of their argument to multiple flavors is obvious. They analyzed (among other things) a Yang-Mills Higgs system with possess the same action as Eq.2.3, except the fact that the compact adjoint Higgs field in Eq.2.3 is substituted by a non-compact one.

$$V_{\text{eff}}^{\text{compact}}(|\Phi|) \rightarrow V_{\text{eff}}^{\text{noncompact}}(|\Phi|) \tag{2.25}$$

Since the chiral anomaly is absent in odd dimensions, the noncompact model has a genuine $U(n_f)$ symmetry whose $U(1)$ part is fermion number. Ref. [11] showed quite explicitly that such a model does not confine. Photons remain infinite range nonperturbatively, and it is indeed the Goldstone boson of the spontaneously broken $U(1)$ fermion number symmetry. Their arguments is essentially based on symmetries, and index theorems by Callias [36], and explicit zero mode construction by Rebbi and Jackiw [37]. Here, we wish to provide a simple dynamical explanation for this phenomena.

Since gauge symmetry breaking occurs via a *noncompact* adjoint Higgs field, there is no longer a KK monopole. Thus, in order to obtain the long distance effective action from our

discussion in previous section, we must delete all KK monopole related terms from our effective action. Hence, the interaction lagrangian is $L_{\text{int}} \sim V_{\text{BPS}} + V_{\overline{\text{BPS}}} + V_{\text{BPS}\overline{\text{BPS}}}$. Consequently,

$$L_{\text{eff}}^{\text{noncompact}} = \frac{1}{2}(\partial\sigma)^2 + i\bar{\psi}^I \gamma^\mu \partial_\mu \psi_I + ae^{-S_0} (e^{i\sigma} \det \psi^I \psi^J + \text{c.c.}) \quad (2.26)$$

where we ignored a trivial cosmological constant which may be induced by $\text{BPS}\overline{\text{BPS}}$ pair. This is indeed the generalization of the Ref. [11] to multiflavor ($n_f > 1$). The effective action is respectful to all the symmetries of the underlying theory, in particular $SU(n_f) \times U(1)$ symmetry, where the former is manifest. The $U(1)$ fermion number symmetry acts as

$$\psi^I \rightarrow e^{i\alpha} \psi^I, \quad \bar{\psi}^I \rightarrow e^{-i\alpha} \bar{\psi}^I, \quad \sigma \rightarrow \sigma - 2n_f \alpha. \quad (2.27)$$

and prohibits any kind of mass term (or potential) for the dual photon. This is the symmetry which breaks down spontaneously, and dual photon is the Goldstone boson.

Clearly, the only topologically neutral object (which may contribute to the bosonic potential) is $\text{BPS}\overline{\text{BPS}}$ pair. But such an object has vanishing magnetic charge. Since there are no topologically null, but magnetically charged carriers in the vacuum of the model studied in [11], the Debye mechanism is not possible. Hence, the photon remains infinite range non-perturbatively. The inability to form magnetically charged bions is the dynamical reason for the absence of confinement in the extension of Polyakov's model in the presence of adjoint fermions.

This discussion also shows that the presence of monopoles in the Yang-Mills Higgs systems with adjoint fermions is a necessary, but not a sufficient condition to have confinement. In particular, it also exhibits that, in such systems, condensations of objects with non-vanishing topological charge (monopole condensation) does not occur.

2.5 Magnetic bions in $\mathcal{N} = 1$ SYM on small $S^1 \times \mathbb{R}^3$

The generalization of the discussion in section 2.3 to $SU(2)$ $\mathcal{N} = 1$ supersymmetric gauge theory is easy, yet important. All one needs to take care is an extra massless scalar which remains massless in perturbation theory. Hence it should be incorporated into long distance physics. With the inclusion of the scalar, the number of Coulomb interaction channels increases and we obtain

Type	Type	σ -int	ϕ -int.	combined	$(\int F, \int F\tilde{F})$
$\text{BPS} - e^{-\phi+i\sigma}$	$\text{BPS} - e^{-\phi+i\sigma}$	rep.	att.	0	$(1, \frac{1}{2}) + (+1, +\frac{1}{2}) = (2, 1)$
BPS	$\overline{\text{BPS}} - e^{-\phi-i\sigma}$	att.	att.	2(att.)	$(1, \frac{1}{2}) + (-1, -\frac{1}{2}) = (0, 0)$
BPS	$\text{KK} - e^{+\phi-i\sigma}$	att.	rep.	0	$(1, \frac{1}{2}) + (-1, +\frac{1}{2}) = (0, 1)$
BPS	$\overline{\text{KK}} - e^{+\phi+i\sigma}$	rep.	rep.	2(rep.)	$(1, \frac{1}{2}) + (+1, -\frac{1}{2}) = (2, 0)$
...

(2.28)

The change in monopole vertices are

$$\begin{aligned} \text{BPS} &: e^{-\phi+i\sigma} \psi\psi, & \text{KK} &: e^{+\phi-i\sigma} \psi\psi, \\ \overline{\text{BPS}} &: e^{-\phi-i\sigma} \bar{\psi}\bar{\psi}, & \overline{\text{KK}} &: e^{+\phi+i\sigma} \bar{\psi}\bar{\psi}, \end{aligned} \quad (2.29)$$

The bosonic potential is due to the sector of the theory with net zero topological charge, so that there will not be any fermion zero mode insertion in it. Thus

$$\begin{aligned} \text{BPS}\overline{\text{BPS}} + \text{KK}\overline{\text{KK}} + \text{BPS}\overline{\text{KK}} + \text{KK}\overline{\text{BPS}} &= e^{-2S_0}(e^{-2\phi} + e^{+2\phi} - e^{i2\sigma} - e^{-2i\sigma}) \\ &= e^{-2S_0}|e^z - e^{-z}|^2 \end{aligned} \quad (2.30)$$

where we defined $z = -\phi + i\sigma$. This result is nice because the bions already know that there is an underlying superpotential, given by ¹⁰

$$\mathcal{W}(z) = e^{-S_0}(e^z + e^{-z}) \quad (2.31)$$

The long distance effective action for SYM on small $S^1 \times \mathbb{R}^3$ is

$$\begin{aligned} L_{\text{eff}}^{\text{SYM}} &= \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\phi)^2 - c^2 e^{-2S_0}(\cos 2\sigma - \cosh 2\phi) \\ &\quad + i\bar{\psi}\gamma_\mu\partial_\mu\psi + c e^{-S_0} \left[(e^{-\phi+i\sigma} + e^{+\phi-i\sigma})\psi\psi + (e^{-\phi-i\sigma} + e^{+\phi+i\sigma})\bar{\psi}\bar{\psi} \right] \end{aligned} \quad (2.32)$$

The $\mathbb{Z}_{2N} = \mathbb{Z}_4$ discrete chiral symmetry of the original theory is also manifest in the effective theory

$$\psi^I \rightarrow e^{i2\pi/4}\psi^I, \quad \sigma \rightarrow \sigma + \pi \quad (2.33)$$

This symmetry breaks down spontaneously to $\mathbb{Z}_2 = (-1)^F$ where F is fermion number leading to the appearance of two isolated vacua.

The dynamics of the $\mathcal{N} = 1$ SYM on $\mathbb{R}^3 \times S^1$ is previously analyzed by imbedding it into F theory in Ref. [12], and by using the elliptic curves of $\mathcal{N} = 2$ SYM combined with the mass deformation in [13]. The works of Davies et.al [14,15] provided a clear field theory exposition of the nonperturbatively induced effects in such theories. The general strategy of these papers had been to calculate monopole vertex first, then use supersymmetry as a completion device to find the bosonic potential and superpotential. For fermionic terms, our strategy is the same as in these earlier works. For the bosonic potential, our strategy is different. Rather than using supersymmetry as a completion tool, we preferred to delineate on its microscopic (physical) origin. Essentially, we identified topologically null configurations which are topologically indistinguishable from the perturbative vacuum, and hence can contribute to the potential. Summing up their contributions gives us the bosonic potential, which can nicely be derived from the superpotential.

These two approaches in the case of $\mathcal{N} = 1$ SYM are identical. The latter approach has a higher value in our opinion due to the fact that it does not make any reference to supersymmetry, and we applied it first to nonsupersymmetric theories. Our analysis makes it manifest that the mechanism of confinement in $\mathcal{N} = 1$ SYM is not monopole condensation, i.e., condensation of excitations with topological charge $\pm\frac{1}{2}$, rather of objects with topological charge 0. This physical fact was not appreciated before. We conclude this section by pointing that the mechanism of the confinement in supersymmetric $\mathcal{N} = 1$ SYM is same as the one in nonsupersymmetric QCD(adj) theories in the $LA \ll 1$ regime, both of which is magnetic bion condensation.

¹⁰Strictly speaking, this superpotential is the form acquired after the superHiggs mechanism.

In the dimensional reduction of $\mathcal{N} = 1$ SYM down to \mathbb{R}^3 , confinement does not occur as shown in [11]. The distinctions are so important that it is worthwhile rederiving their results following the consideration of this paper, and explain the absence of confinement on dynamical grounds.

2.6 The $\mathcal{N} = 2$ SYM on \mathbb{R}^3 and lack of confinement, again

Delete all terms in the effective action 2.32 which is related to KK monopole. (This is the same statement as ϕ field becomes noncompact on \mathbb{R}^3 limit.) This leaves us with BPS and $\overline{\text{BPS}}$ induced vertex in fermionic terms and $\text{BPS}\overline{\text{BPS}}$ induced term in the bosonic potential in the lagrangian 2.32:

$$L_{\text{eff}}^{\text{n.c.}} = \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\phi)^2 - c^2 e^{-2S_0} e^{-2\phi} + i\bar{\psi}\gamma_\mu\partial_\mu\psi + c e^{-S_0} \left[e^{-\phi+i\sigma}\psi\psi + e^{-\phi-i\sigma}\bar{\psi}\bar{\psi} \right] \quad (2.34)$$

which is the same as the lagrangian in [11]. The \mathbb{Z}_{2N} discrete chiral symmetry of SYM on locally four dimensional settings elevates to the full $U(1)$ fermion number on \mathbb{R}^3 due to absence of chiral anomaly in odd dimensions. The continuous $U(1)$ symmetry acts as

$$\psi^I \rightarrow e^{i\alpha}\psi^I, \quad \bar{\psi}^I \rightarrow e^{-i\alpha}\bar{\psi}^I, \quad \sigma \rightarrow \sigma - 2\alpha. \quad (2.35)$$

and prohibits any kind of mass term (or potential) for the dual photon. This is the symmetry which breaks down spontaneously, and dual photon is the Goldstone boson. The runaway potential $e^{-2\phi}$ does not have a vacuum at finite ϕ .

On dynamical grounds, the absence of confinement is due to the inability to form long range magnetic bions in SYM vacuum on \mathbb{R}^3 . The $\text{BPS}\overline{\text{BPS}}$ pairs are neutral, and photon remains infinite range in a medium of neutral molecules. In other words, it remains massless nonperturbatively as demanded from a Goldstone particle, and this implies the absence of confinement.

3. $SU(N)$ QCD(adj), bions, and secret integrability?

The $SU(N)$ QCD(adj) theory undergoes gauge symmetry breaking on sufficiently small spatial S^1 due to a perturbative Coleman-Weinberg potential. The gauge symmetry breaking is $SU(N) \rightarrow U(1)^{N-1}$. For simplicity, we will add a decoupled "center of mass" degree of freedom to the original theory and consider gauge symmetry breaking of the form $U(N) \rightarrow U(1)^N$. This is a technical trick, and in the spontaneously broken gauge theory, the center of mass mode decouples from the dynamics. Hence, our goal is to determine the dynamics of the $N - 1$ modes $\frac{U(1)^N}{U(1)_{\text{c.m.}}}$.

The monopoles may be described by their magnetic charges, topological charge and their action. The magnetic charges of the N types of (BPS and KK) monopoles under unbroken gauge symmetry $U(1)^N$ is proportional to the simple roots and affine root of the Lie algebra, respectively. The simple roots are given by

$$\begin{aligned} \alpha_1 &= (1, -1, 0, \dots, 0) = e_1 - e_2 \\ \alpha_2 &= (0, 1, -1, \dots, 0) = e_2 - e_3 \end{aligned}$$

$$\begin{aligned}
\alpha_i &= (0, \dots, 1, -1, \dots, 0) = e_i - e_{i+1} \\
&\dots \\
\alpha_{N-1} &= (0, \dots, 0, 1, -1) = e_{N-1} - e_N
\end{aligned} \tag{3.1}$$

and the affine root is

$$\alpha_N \equiv - \sum_{j=1}^{N-1} \alpha_j = (-1, 0, 0, \dots, 1) = e_N - e_1 \tag{3.2}$$

It is convenient to define the simple Δ^0 and affine (extended) Δ_{aff}^0 root systems of the the associated Lie algebra:

$$\Delta^0 \equiv \{\alpha_1, \alpha_2, \dots, \alpha_{N-1}\}, \quad \Delta_{\text{aff}}^0 \equiv \{\alpha_1, \alpha_2, \dots, \alpha_{N-1}, \alpha_N\}, \tag{3.3}$$

The latter is the one relevant for QCD(adj) on $\mathbb{R}^3 \times S^1$. More generally, in the Yang-Mills Higgs systems with adjoint fermions, if the Higgs field is noncompact, the monopole and antimonopole charges are valued in Δ^0 and, $-\Delta^0$, respectively. If the Higgs field is compact, then there is an extra monopole, and the charges take values in $\pm\Delta_{\text{aff}}^0$.

The topological charges $\int F\tilde{F}$ is correlated with the sign of the two sets $\pm\Delta_{\text{aff}}^0$. Thus, the quantized magnetic and topological charges are

$$\int_{S^2} F^i = \pm \frac{2\pi}{g} \alpha^i, \quad \int F\tilde{F} \equiv \frac{g^2}{32\pi^2} \int \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} = \pm \frac{1}{N}, \tag{3.4}$$

The action of a monopole with charge α_i and topological charge $\int F\tilde{F} = \pm\frac{1}{N}$ is given by $S_{0,i} = \frac{8\pi^2}{g^2} \int F\tilde{F} = \frac{8\pi^2}{g^2 N}$. Due to the presence of the effective potential for the Wilson line, the monopoles of QCD(adj) theory (except for $n_f = 1$ which is supersymmetric) do not saturate the BPS bound. But the correction are perturbative in g^2 and we will neglect them.

The long range Coulomb interaction of monopoles (in the absence of fermions) is given by ¹¹

$$V(\alpha_i, \pm\alpha_j, r) = \frac{\alpha_i \cdot (\pm\alpha_j)}{4\pi r} = \pm \frac{2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}}{4\pi r}, \quad i, j = 1, \dots, N \tag{3.5}$$

which translate to self and nearest neighbor interaction between monopoles in the Dynkin space. The inner product of the roots of the associated Lie algebra is a basis independent statement, though the above choice of the basis 3.1 is due its visual simplicity.

We are now ready to generalize the derivation of effective potential for $SU(2)$ QCD(adj) to $SU(N)$ with $1 < n_f \leq 4$. Our discussion will be brief.

Were the adjoint fermions absent, a monopole with charge α_j would be associated with vertex $e^{i\alpha_j\sigma}$. Due to index theorem 2.9, any object with nonvanishing topological charge $(1/N)$ must have $\Delta Q_5 = 2n_f$ fermions attached to it. As discussed in footnote 9, the underlying QCD(adj) theory has an $[SU(n_f) \times \mathbb{Z}_{2Nn_f}]/\mathbb{Z}_{n_f}$ continuous and discrete chiral symmetries. The manifestly $SU(n_f)$ invariant fermion vertex with $2n_f$ fermion insertion is

¹¹We set $\frac{2\pi}{g}$ to unity as in our discussion of $SU(2)$ to lessen the clutter in expressions. All physical quantities are measured in units of L , which is also set to unity. We will restore both quantities if necessary.

given by $\det_{IJ} \alpha_i \psi^I \alpha_i \psi^J$ where the determinant is over the flavor index. Here, we use a vector notation

$$\sigma = (\sigma_1, \dots, \sigma_N), \quad \psi^I = (\psi_1^I, \dots, \psi_N^I), \quad \alpha_i \sigma = \sigma_i - \sigma_{i+1} \quad (3.6)$$

As stated earlier, the center of mass mode is extraneous and decouples from the dynamics completely. Hence, the appropriate monopole and antimonopole vertices are

$$V_{\alpha_i} = e^{i\alpha_i \sigma} \det_{IJ} \alpha_i \psi^I \alpha_i \psi^J, \quad V_{-\alpha_i} = e^{-i\alpha_i \sigma} \det_{IJ} \alpha_i \bar{\psi}^I \alpha_i \bar{\psi}^J \quad (3.7)$$

This means, the interaction Lagrangian at $O(e^{-S_0})$ is given by

$$e^{-S_0} \sum_{\alpha_i \in \Delta_{\text{aff}}^0} \left(e^{i\alpha_i \sigma} \det_{IJ} \alpha_i \psi^I \alpha_i \psi^J + e^{-i\alpha_i \sigma} \det_{IJ} \alpha_i \bar{\psi}^I \alpha_i \bar{\psi}^J \right) \quad (3.8)$$

This vertex is invariant under $(SU(n_f) \times \mathbb{Z}_{2Nn_f}) / \mathbb{Z}_{n_f}$ as desired. The discrete chiral symmetry acts as

$$\psi^I \rightarrow e^{i2\pi/(2Nn_f)} \psi^I, \quad \bar{\psi}^I \rightarrow e^{-i2\pi/(2Nn_f)} \bar{\psi}^I, \quad \sigma \rightarrow \sigma - \frac{2\pi}{N} \sum_{j=1}^{N-1} \mu_k \quad (3.9)$$

where μ_k are the $N - 1$ fundamental weights (not the weight of fundamental representation) of the associated Lie algebra. They are defined by the reciprocity relation,

$$\frac{2\alpha_i \mu_j}{\alpha_i^2} = \alpha_i \mu_j = \delta_{ij} \quad (3.10)$$

The shift in the photon field is called the Weyl vector, and we will often abbreviate it as

$$\rho \equiv \sum_{j=1}^{N-1} \mu_k, \quad \text{such that} \quad e^{i\frac{2\pi}{N} \rho \alpha_i} = e^{i\frac{2\pi}{N}}, \quad i = 1, \dots, N \quad (3.11)$$

The action of the discrete chiral symmetry on $SU(n_f)$ singlets is a \mathbb{Z}_N symmetry transformation,

$$\det_{I,J} \alpha_i \psi^I \alpha_i \psi^J \rightarrow e^{i2\pi/N} \det_{I,J} \alpha_i \psi^I \alpha_i \psi^J, \quad e^{i\alpha_i \sigma} \rightarrow e^{-i2\pi/N} e^{i\alpha_i \sigma} . \quad (3.12)$$

Consequently, the monopole induced interaction terms (which are of order e^{-S_0}) are respectful the discrete (and continuous) symmetries of the underlying theory.

Exactly as in the $SU(2)$ discussion, this is the net effect of the topologically nontrivial sector of the theory which saturates the lagrangian at order e^{-S_0} . In particular, a would be (confining) potential term for σ field

$$e^{-S_0} \sum_{\alpha_i \in \Delta_{\text{aff}}^0} (e^{i\alpha_i \sigma} + e^{-i\alpha_i \sigma}) \quad (3.13)$$

is forbidden by the \mathbb{Z}_N shift symmetry $\sigma - \frac{2\pi}{N} \sum_{j=1}^{N-1} \mu_k$ of the dual photon. This is a consequence of having adjoint fermions in the system. In the absence of fermions, such as pure Yang-Mills compact Higgs system, this *is* the leading term which renders all the photons massive, with masses of order $e^{-S_0/2}$. We will see that in QCD(adj), the masses of photons are of order e^{-S_0} , and there is a \mathbb{Z}_N shift symmetry respecting potential at order e^{-2S_0} .

3.1 Attractive channels, bions, and a prepotential

We must examine the combinations of the monopole antimonopole pairs with magnetic charges from the two sets Δ_{aff}^0 and $-\Delta_{\text{aff}}^0$ with respective topological charges $\frac{1}{N}$ and $-\frac{1}{N}$. Due to the presence of many pairs one can construct, this may a priori seem like a mess. However, the theory does something truly remarkable. At order e^{-2S_0} , it only pairs the monopoles with charge α_j with their nearest neighbor antimonopoles, with charges $-\alpha_{j\pm 1}$ in the Dynkin space. These combinations are the bion states. (There are also neutral monopoles and antimonopole pairing of same kind, but the charge of such an object is zero and not so interesting in nonsupersymmetric QCD(adj). It has an effect in SYM.)

Let us first find the attractive channels. We can extract the interaction of a monopole with charge α_i and antimonopole with charge $-\alpha_j$ by inspecting its connected correlator in the functional integral of the free theory with the action $S_{d,0}(\sigma, \psi, \bar{\psi})$.

$$\begin{aligned} \langle V_{\alpha_i}(x)V_{-\alpha_j}(y)\rangle_0 &= \langle e^{i\alpha_i\sigma(x)} \det_{IJ} \alpha_i \psi^I \alpha_i \psi^J(x) e^{-i\alpha_j\sigma(y)} \det_{I'J'} \alpha_j \bar{\psi}^{I'} \alpha_j \bar{\psi}^{J'}(y)\rangle_0 \\ &= \langle e^{i\alpha_i\sigma(x)} e^{-i\alpha_j\sigma(y)} \rangle_0 \langle \det_{IJ} \alpha_i \psi^I \alpha_i \psi^J(x) \det_{I'J'} \alpha_j \bar{\psi}^{I'} \alpha_j \bar{\psi}^{J'}(y)\rangle_0 \\ &\sim e^{+\alpha_i \cdot \alpha_j G(x-y)} (\alpha_i \alpha_j)^{2n_f} [S_F(x-y)]^{2n_f} \end{aligned} \quad (3.14)$$

The connected correlator is only nonzero if $\alpha_i \alpha_j$ is nonzero, and induces a logarithmic binding potential of the form

$$V_{\text{eff}}(x-y) = \begin{cases} +\frac{1}{4\pi|x-y|} + 4n_f \log|x-y| & \text{for } i = j \pm 1 \\ -\frac{2}{4\pi|x-y|} + 4n_f \log|x-y| & \text{for } i = j \\ 0 & \text{otherwise .} \end{cases} \quad (3.15)$$

If $i = j$, then both Coulomb and fermion exchange forces are attractive. If $i = j \pm 1$, then Coulomb interaction is repulsive, but the attractive fermion exchange term easily dominates.

Now, we are ready to define the magnetic bions in the spontaneously broken $SU(N)$ gauge theory. A bion is a bound state of the monopole associated with magnetic charge α_i and anti-monopole associated with charge $-\alpha_{i+1}$ with null topological charge:

$$Q_i = \alpha_i - \alpha_{i-1} = 2e_i - e_{i+1} - e_{i-1}, \quad \int F \tilde{F} = 0 \quad i = 1, \dots, N \quad (3.16)$$

Restoring the prefactors and writing more explicitly, the magnetic bion (antibion) charges are given under the $U(1)^N$ gauge group as

$$Q_i = \pm \frac{2\pi}{g} \left(0, \dots, \underbrace{-1}_{i-1}, \underbrace{2}_i, \underbrace{-1}_{i+1}, \dots, 0 \right) \quad (3.17)$$

This means, bions interact via a next-to-nearest neighbor interaction in the Dynkin space: For high rank gauge groups ($N \geq 5$),

$$Q_i Q_j = 6\delta_{ij} - 4\delta_{i,j+1} - 4\delta_{i,j-1} + \delta_{i,j+2} + \delta_{i,j-2}, \quad N \geq 5 \quad (3.18)$$

In order to find the bion-bion interactions in low rank gauge groups $N \leq 4$, we need to identify nodes $j \equiv j + N$ in the (affine) Dynkin diagram, since there are less than five nodes. Consequently,

$$\begin{aligned} Q_i Q_j &= 6\delta_{ij} - 4\delta_{i,j+1} - 4\delta_{i,j-1} + 2\delta_{i,j+2}, & N &= 4 \\ Q_i Q_j &= 6\delta_{ij} - 3\delta_{i,j+1} - 3\delta_{i,j-1}, & N &= 3 \\ Q_i Q_j &= 8\delta_{ij} - 8\delta_{i,j+1}, & N &= 2 \end{aligned} \quad (3.19)$$

The long range interactions of magnetic bions are given by Coulomb's potential and is equal to

$$V(Q_i, \pm Q_j, r) = \frac{Q_i \cdot (\pm Q_j)}{4\pi r} = \pm \frac{6\delta_{ij} - 4\delta_{i,j+1} - 4\delta_{i,j-1} + \delta_{i,j+2} + \delta_{i,j-2}}{4\pi r} \quad (3.20)$$

The meaning of this formula is clear. Bion pairs with the charges (Q_i, Q_i) repel, $(Q_i, Q_{i\pm 1})$ attract, $(Q_i, Q_{i\pm 2})$ repel, and no interactions for pairs (Q_i, Q_{i+k}) with $k > 2$. The overall sign of the interactions is reversed for the bion-antibion pairs.

Now, we can convert the Coulomb gas of magnetic bions into a field theory following Polyakov's treatment [6]. We only quote the result, since the manipulations are standard. The vertex appropriate for a bion molecule located at $x \in \mathbb{R}^3$ is

$$e^{iQ_i\sigma(x)} = e^{i\alpha_i\sigma(x)} e^{-i\alpha_{i-1}\sigma(x)} \quad (3.21)$$

Clearly, this is manifestly invariant under the \mathbb{Z}_N shift symmetry of the photon which acts as $e^{i\alpha_i\sigma(x)} \rightarrow e^{-i2\pi/N} e^{i\alpha_i\sigma(x)}$. The bosonic effective potential is a sum over all bion and antibion contributions given by

$$V(\sigma) = -e^{-2S_0} \sum_{i=1}^N (e^{iQ_i\sigma} + e^{-iQ_i\sigma}) = -2e^{-2S_0} \sum_{i=1}^N \cos Q_i\sigma \quad (3.22)$$

There is something remarkable about this potential, in fact surprising. It can be derived from a *prepotential*, just like a bosonic potential in the supersymmetric system may be derived from a superpotential. In order to see this, rewrite the potential $V(\sigma)$ as

$$\begin{aligned} V(\sigma) &= -e^{-2S_0} \sum_{i=1}^N (e^{i\alpha_i\sigma} e^{-i\alpha_{i-1}\sigma} + e^{-i\alpha_i\sigma} e^{i\alpha_{i-1}\sigma}) \\ &= e^{-2S_0} \sum_{i=1}^N |e^{i\alpha_i\sigma} - e^{i\alpha_{i-1}\sigma}|^2 + \text{constant} \end{aligned} \quad (3.23)$$

where constant is unimportant. Define the prepotential as

$$\mathcal{W}(\sigma) = e^{-S_0} \sum_{\alpha_i \in \Delta_{\text{aff}}^0} e^{i\alpha_i\sigma} . \quad (3.24)$$

Hence, the potential may be written as

$$V(\sigma) = \sum_{i=1}^N \left| \frac{\partial \mathcal{W}}{\partial \sigma_i} \right|^2 = e^{-2S_0} \sum_{i=1}^N |e^{i\alpha_i\sigma} - e^{i\alpha_{i-1}\sigma}|^2, \quad \text{QCD(adj)} \ n_f > 1 \quad (3.25)$$

The reader familiar with the supersymmetric affine Toda theories will recognize the form of our (nonsupersymmetric) prepotential as the superpotential. In order to describe the infrared of $\mathcal{N} = 1$ SYM on small S^1 , one must incorporate the extra massless scalars into the potential: All one needs to do is a holomorphic completion of our formula. Not surprisingly,

$$V(z, \bar{z}) = \sum_{i=1}^N \left| \frac{\partial \mathcal{W}}{\partial z_i} \right|^2 = \sum_{i=1}^N |e^{i\alpha_i z} - e^{i\alpha_{i-1} z}|^2, \quad \text{SYM} \quad (3.26)$$

The fact that the potential can be derived from a prepotential as above implies that the classical equations of motions for the σ field can be reduced to a first order one. Anticipating ourselves a little bit, it also implies that the area law coefficients for large Wilson loops are analytically calculable. The k -string tensions and thicknesses are calculable. This is not less than magical, because underlying theory is not supersymmetric (except, of course $n_f = 1$ QCD(adj), which is $\mathcal{N} = 1$ SYM.)

Let us finalize this section by writing the final form of the dual of the QCD(adj) lagrangian on small $S^1 \times \mathbb{R}^3$ with $1 < n_f \leq 4$ flavors:

$$L^{\text{dQCD}} = \frac{1}{2}(\partial\sigma)^2 - b e^{-2S_0} \sum_{\alpha_i \in \Delta_{\text{aff}}^0} |e^{i\alpha_i \sigma} - e^{+i\alpha_{i-1} \sigma}|^2 + i\bar{\psi}^I \gamma_\mu \partial_\mu \psi_I + c e^{-S_0} \sum_{\alpha_i \in \Delta_{\text{aff}}^0} \left(e^{i\alpha_i \sigma} \det_{IJ} \alpha_i \psi^I \alpha_i \psi^J + e^{-i\alpha_i \sigma} \det_{IJ} \alpha_i \bar{\psi}^I \alpha_i \bar{\psi}^J \right) \quad (3.27)$$

The dualQCD lagrangian and the physics it encapsulates, which will be discussed next, is the essential result of this paper.

3.2 The vacuum structure of QCD(adj)

The bosonic potential of nonsupersymmetric QCD(adj) has N gauge inequivalent isolated vacua, aligned along the Weyl vector ρ

$$\sigma = \left\{ 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \dots, \frac{(N-1)2\pi}{N} \right\} \rho \quad (3.28)$$

in the field space. This is same as $\mathcal{N} = 1$ SYM studied in [15]. Since each component of σ is a periodic variable with periodicity 2π , there exist a physical congruence between σ and σ' which are separated by an element of the root lattice Λ_r .

$$\sigma \equiv \sigma + 2\pi\alpha \quad \text{for some } \alpha \in \Lambda_r \quad (3.29)$$

Since the sum of all fundamental weights is a root, $\rho = \sum_{j=1}^{N-1} \mu_j \in \Lambda_r$, this implies there only exist N gauge inequivalent vacua when the (global) gauge symmetry redundancies are removed. Let us the abbreviate and label the vacuum states in Hilbert space as

$$\begin{aligned} |\Omega_{\frac{2\pi k}{N}\rho + \Lambda_r}\rangle &\equiv |\Omega_k\rangle \equiv |\Omega_{k+N}\rangle, \quad k = 0, \dots, N-1 \\ \text{Ground states} &= \left\{ |\Omega_0\rangle, |\Omega_1\rangle, \dots, |\Omega_{N-1}\rangle \right\} \end{aligned} \quad (3.30)$$

which forms a one dimensional representation of \mathbb{Z}_N shift symmetry, (which is intertwined with \mathbb{Z}_N discrete chiral symmetry, see footnote 9.) This means, the (large) physical Hilbert space spits into N superselection sectors, each of which may be build upon the associated vacuum. The choice of the vacuum breaks the \mathbb{Z}_N discrete chiral symmetry (which is same as \mathbb{Z}_N shift symmetry of the dual photon) spontaneously. Note that QCD(adj) also possess a $\mathcal{G}_s = \mathbb{Z}_N$ spatial center symmetry which does not break regardless of the size of the S^1 .

3.2.1 Mass gap in the gauge sector

The small fluctuations around one of the N minima of the $-\cos Q_i \sigma$ potential shows that the $N - 1$ dual photon acquires masses proportional to e^{-S_0} . In order to see this, let us expand the nonperturbative bion induced potential to quadratic order in dual photon σ

$$\begin{aligned} V(\sigma_i) &= -e^{-2S_0} \sum_{i=1}^N \cos Q_i \sigma = -e^{-2S_0} \sum_{i=1}^N \cos(2\sigma_i - \sigma_{i+1} - \sigma_{i-1}) \\ &= \frac{1}{2} e^{-2S_0} \sum_i (6\sigma_i^2 - 4\sigma_i \sigma_{i+1} - 4\sigma_i \sigma_{i-1} + \sigma_i \sigma_{i+2} + \sigma_i \sigma_{i-2}) \quad \text{bion induced (3.31)} \end{aligned}$$

If the fermions were absent, and the gauge symmetry was still broken by a compact adjoint Higgs field, the quadratic fluctuations would be described by the nearest neighbor coupled harmonic oscillator

$$V(\sigma_i) = \frac{1}{2} e^{-S_0} \sum_i (2\sigma_i^2 - \sigma_i \sigma_{i+1} - \sigma_i \sigma_{i-1}) \quad \text{monopole induced (3.32)}$$

which is not the case in QCD(adj). The bion induced ‘hopping’ terms are next to nearest neighbor and of order e^{-2S_0} as opposed to the monopole induced hopping terms which is just nearest neighbor, and of order e^{-S_0} . This interesting structure also has a remarkable consequence for the calculability of the string tensions in the former, due to the fact that it arises from a prepotential. Unfortunately, this is not the case in the latter, the monopole case. (Except for $SU(2)$ of course, in which case both reduce to Sine-Gordon model).

The quadratic fluctuations can be diagonalized by using the discrete Fourier transform $\sigma_p = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \omega^{jp} \sigma_j$ in Dynkin space:

$$\begin{aligned} V(\sigma_p) &= \frac{1}{2} e^{-2S_0} \sum_p (6 - 4\omega^{-p} - 4\omega^p + \omega^{-2p} + \omega^{2p}) \sigma_p \sigma_{-p} \\ &= \frac{1}{2} e^{-2S_0} \sum_p (\omega^{p/2} - \omega^{-p/2})^4 \sigma_p \sigma_{-p} = \frac{1}{2} e^{-2S_0} \sum_p (2 \sin \frac{p\pi}{N})^4 \sigma_p \sigma_{-p} \quad (3.33) \end{aligned}$$

Restoring the dimensions, we obtain the mass spectrum of the $N - 1$ dual photons as

$$m_p \sim \left(\Lambda (\Lambda L)^{b_0-1} = \Lambda (\Lambda L)^{(8-2n_f)/3} \right) \times (2 \sin \frac{p\pi}{N})^2, \quad p = 1, \dots, N - 1 \quad (3.34)$$

The nonperturbative spectrum 3.34 is a remarkable result. It exhibits that the gauge sector of the QCD(adj) theory is quantum mechanically gapped due to non-perturbative effects, and permanently confines external electric charges at small $S^1 \times \mathbb{R}^3$ limit.

The masses are graded according to the \mathbb{Z}_N center group of $SU(N)$ in one to one correspondence with the representations \mathcal{R}_p of $SU(N)$ under the center group. There are two equivalent physical interpretation for the mass gap: one as the inverse Debye screening length in a magnetic conductor (in a Euclidean setting), and the other is the inverse thickness of the chromoelectric flux tubes in a magnetic superconductor (at a fixed time in a Minkowski setting). (See [38] for a parallel discussion in the context of the Polyakov model.)

Imagine a large, planar Wilson loop in a representation with charge p under the center \mathbb{Z}_N . In the small S^1 regime (where gauge symmetry is broken to the abelian subgroup), we may regard the Wilson loop as carrying an electric current along the contour of the loop. Hence, by Maxwell's equation, the current generates a magnetic field along the axis perpendicular to the plane of the loop, within the boundary C of the loop surface Σ . The external magnetic field can not penetrate into the magnetic conductor above a penetration depth, due to Debye screening. The mobile magnetic charge carriers (bions) form a dipole layer in the vicinity of the surface Σ to prevent the penetration of the external magnetic field into the magnetic conductor, which is the vacuum of QCD(adj) from Euclidean viewpoint. The thickness of the dipole layer for the Wilson loop with \mathbb{Z}_N charge p is the inverse of the photon mass m_p^{-1} .

We may visualize a Wilson loop at a fixed time slice. This is a system with $\pm p \mathbb{Z}_N$ chromoelectric sources located at two boundaries of the fixed time slice of Wilson loop. There exist a stable chromoelectric flux tube in between the two. Since the dual superconductor expels the electric field, and the electric flux lines are trapped within tubes with quantized electric flux. The $N - 1$ classes of the photon masses are indeed the inverse characteristic sizes of the $N - 1$ types of the chromoelectric flux tubes, both of which is a class function of the \mathbb{Z}_N center group. In weakly coupled regime, making L larger reduces the thicknesses of the stable flux tubes

$$l_p \sim \Lambda^{-1} (\Lambda L)^{-(8-2n_f)/3} \times (2 \sin \frac{p\pi}{N})^{-2}, \quad p = 1, \dots, N - 1 \quad (3.35)$$

We expect it to saturate to an L independent value above the scale of gauge symmetry restoration. Also, intermediate N -ality tubes seems to be much more slimmer than the small and large N -ality ones.

Due to compactification, in the weakly coupled regime, the characteristic size of the flux tubes and their tensions are no longer parametrically related. In the next section, we explicitly calculate the string tensions.

3.2.2 Area law of confinement and monodromy

We wish to exhibit the area law of confinement for all but adjoint representations \mathcal{R}_p of the $SU(N)$ gauge group. The representations of the Wilson loops under the center group \mathbb{Z}_N are in one to one correspondence with the monodromies, $\int_C d\sigma$ in the dual theory [29]. Both forms a representation of \mathbb{Z}_N . This is also the quantized electric flux of external charges sourcing the associated Wilson loop. More precisely, the exponent of the flux passing through some surface surrounding the color electric source is essentially $e^{i\frac{2\pi}{N}k}$ for source with charge k .

The evaluation of a Wilson loop in a representation with charge k under the \mathbb{Z}_N center group in the original theory translates into finding the field configurations for the dual

scalar theory with monodromies equal to $\frac{2\pi k}{N}\rho$. Thus, we need to find the action of the the soliton configurations which connects vacua separated by k units. Due to the the existence of prepotential, the lower bound of the action in each topological sector can be calculated analytically.

The expectation values of the Wilson loop falls into N categories. Let H denote the Hamiltonian of the dual theory: The Hilbert space interpretation of Polyakov's result is

$$\lim_{A(\Sigma)\rightarrow\infty} \langle W_{\mathcal{R}_k}(C) \rangle|_{C=\partial\Sigma} = \langle \Omega_k | e^{-zH} | \Omega_0 \rangle \equiv \int_{\sigma(z=-\infty)=0}^{\sigma(z=+\infty)=\frac{2\pi k}{N}\rho} D\sigma e^{-S(\sigma)} \quad (3.36)$$

where z is Euclidean time, and interpolates between $[-\infty, \infty]$. The expectation value of arbitrarily large Wilson loops (where Σ is \mathbb{R}^2 filling) are equal to *tunneling amplitudes* in the dual theory, where we used a functional integral representation. Formally, the tunneling amplitude on $\mathbb{R}^2 \times \mathbb{R}$ is $e^{-\text{Area}(\mathbb{R}^2)S_k^*}$ where S_k^* is the least action associated with $x, y \in \mathbb{R}^2$ independent soliton (kink) solution with topological charge k .

$$t = \int_{-\infty}^{\infty} dz \frac{d\sigma}{dz} = \frac{2\pi k}{N}\rho \quad (3.37)$$

The kink with topological charge k is localized within the m_k^{-1} proximity of the surface Σ .

As stated earlier, the fact that the potential may be derived from a prepotential leads to the reduction of the equations of motions of the solitons to the first order, and allows us to find the global minimum of the action in each topologically distinct sector of the effective theory. We have

$$\langle \Omega_k | e^{-zH} | \Omega_0 \rangle \equiv \int_{\sigma(z=-\infty)=0}^{\sigma(z=+\infty)=\frac{2\pi k}{N}\rho} D\sigma e^{-S(\sigma)} = e^{-\text{Area}(\Sigma)S_k^*}, \quad \Sigma \sim \mathbb{R}^2 \quad (3.38)$$

Thus, the k -string tension is equal to the global minimum of the action (divided by area of the surface Σ), i.e., $T_k \equiv S_k^*$, given by

$$T_k = |\mathcal{W}(\sigma(\infty)) - \mathcal{W}(\sigma(-\infty))| = |\mathcal{W}(\frac{2\pi k}{N}\rho) - \mathcal{W}(0)| \quad (3.39)$$

in terms of prepotential. Hence,

$$T_k \sim e^{-S_0} N |e^{i\frac{2\pi k}{N}} - 1| = e^{-S_0} 2N \sin \frac{\pi k}{N} \quad k = 1, \dots, N-1. \quad (3.40)$$

Restoring the dimensions and using the one loop renormalization group result for the strong scale, we obtain

$$T_k \sim \left(\Lambda^2 (\Lambda L)^{b_0-2} = \Lambda^2 (\Lambda L)^{(5-2n_f)/3} \right) \times 2N \sin \frac{\pi k}{N}. \quad (3.41)$$

This exhibits the area law of permanent confinement in QCD(adj) in the $\Lambda L \ll 1$ regime, and the existence of the linearly confining potential between two external electric sources with charges $\pm k \in \mathbb{Z}_N$

$$V_k(R) = T_k R, \quad \text{linear confinement} \quad (3.42)$$

We expect the tension to saturate to a size independent value, a c-number times Λ^2 for $L\Lambda > 1$.

On the other hand, $0 \equiv 2\pi\rho$ monodromy can be induced by no-soliton, and even soliton sector of the dual theory. Hence, $\langle W_{\text{adj}}(C) \rangle = 1 + O(e^{-NT\text{Area}(\Sigma)})$, and no area law as expected. In the strongly coupled regime, this must become perimeter law.

To summarize, in QCD(adj), both the string tensions and thicknesses of flux tubes (which is the inverse masses of the dual photons) are class functions of the center group \mathbb{Z}_N . The class functions depend on the N -ality of the source, but are blind to the particular representative of a class. Also, exchanging (color) source and sink is just the mirror image, and tells us that class functions must obey $X_k = X_{N-k}$, where X is any class function. Interesting physical quantities (which are all measurable in lattice) are the ratios of the string tensions, (inverse) string thicknesses and their energy densities given by

$$\frac{T_p}{T_1} = \frac{\sin \frac{p\pi}{N}}{\sin \frac{\pi}{N}}, \quad \frac{m_p}{m_1} = \left(\frac{\sin \frac{p\pi}{N}}{\sin \frac{\pi}{N}} \right)^2, \quad \frac{\mathcal{E}_p}{\mathcal{E}_1} = \left(\frac{\sin \frac{p\pi}{N}}{\sin \frac{\pi}{N}} \right)^5. \quad (3.43)$$

These observable obey

$$X_p \equiv X_{N+p}, \quad X_p = X_{N-p}, \quad p = 1, \dots, N-1 \quad (3.44)$$

Therefore, there are $\lfloor \frac{N}{2} \rfloor$ types of flux tubes, where bracket labels the integer part of the $N/2$. The ratio of the string tensions yields the ‘‘sine-law’’ for the tensions.

In the $n_f = 1$ case, the sine law for tension has previously been derived by Douglas and Shenker [5] on \mathbb{R}^4 by deforming the $\mathcal{N} = 2$ theory by a perturbative mass term for the chiral multiplet, and by Hanany et. al. [39] by realizing the same deformation in M-theory fivebrane version, referred as mQCD.¹² Both [5, 39] achieves a weakly coupled $\mathcal{N} = 1$ SYM theory on \mathbb{R}^4 by adding extra matter into the theory.¹³ In our derivation, no extra matter is needed. But in order to achieve a weakly coupled formulation, we compactify the theory on $\mathbb{R}^3 \times S^1$ and benefit from asymptotic freedom. In both cases, the physics is rather similar, it is spontaneously broken $U(1)^{N-1}$ gauge theory, and abelian duality in $d = 3$ and $d = 4$ plays a fundamental role. The formula receives $O(e^{-S_0})$ corrections, which is insignificant in the regime we derived it, but will be essential in the large radius regime. Consequently, our result does not imply that the tension will obey a sine law in large S^1 or in \mathbb{R}^4 , even in the $n_f = 1$ case which is $\mathcal{N} = 1$ pure SYM.

Remark on other QCD-like theories: Either the mass gap in the gauge sector or the area law for large Wilson loops are equally valid indicators of confinement for theories in

¹²Our result for nonsupersymmetric theories is new, and directly testable on the lattice in the appropriate regime. Our derivation for the SYM is also different from earlier work [5, 39] and does not make any reference to supersymmetry, or the underlying theory being realizable in string theory. Due to the generality of our approach, it is applicable to nonsupersymmetric QCD-like theories which are more interesting.

¹³An important issue here is to realize that this theory is not pure $\mathcal{N} = 1$ SYM in \mathbb{R}^4 . As the authors of [5] discusses, this mechanism holds so long as $m/\Lambda \ll 1$, a perturbation. In order to obtain pure $\mathcal{N} = 1$ SYM in the IR, we must take $m \gg \Lambda$, which is not a perturbation, and calculational control of the softly broken $\mathcal{N} = 2$ do get lost. Currently, there is no analytical derivation of mass gap or confinement in pure $\mathcal{N} = 1$ SYM on \mathbb{R}^4 .

which the only dynamical degrees of freedom are adjoint fermions. For theories such as QCD with two adjoint and one fundamental fermions (which also breaks its gauge symmetry at small S^1), the mass gap should still emerge, but area law must become a perimeter law. The theory should still be confining, but the ability to form stable flux tubes must be lost due to the fact that charged fermions can be pair created out of the vacuum, and break the flux tube to reduce its energy. If we can construct the dual of such QCD-like theories, they must have unique vacuum (unlike QCD(adj) which has multiple vacua), and a manifest mass gap for photon. This should be so, because the tunneling interpretation of the Polyakov's result tells us that if dual theory has multiple vacua, the area law will naturally follow. A thorough discussion of such theories is left for upcoming work.

3.2.3 Chiral symmetry realizations

The choice of the vacuum state $|\Omega_k\rangle$ spontaneously breaks the \mathbb{Z}_N shift symmetry, which is intertwined with the \mathbb{Z}_N discrete chiral symmetry. The chiral order parameter which is a singlet under continuous flavor symmetry, and which only probes the discrete chiral symmetry is the determinantal condensate $\det \text{tr} \lambda^I \lambda^J$ in the original theory. In the infrared of the theory on small S^1 , the off-diagonal modes of the λ^I are heavy due to gauge symmetry breaking and can not contribute to the determinantal chiral condensate. We may decompose $\lambda^I = \lambda^{I,a} t^a$ into massless components along the Cartan subalgebra and heavy off diagonal modes, $\text{tr} \lambda^I \lambda^J \sim L^{-3} \sum_j (\alpha_j \psi^J) (\alpha_j \psi^J) + \text{heavy}$, where L^{-3} is due to dimensional reasons. The vacuum expectation value of the flavor singlet chiral condensate in $SU(N)$ QCD(adj) with $1 \leq n_f \leq 4$ flavor can be found by integrating over the zero mode wave functions (which are essentially proportional to monopole profiles) in the background of a monopole in the small S^1 regime, where the gauge symmetry is broken. On large S^1 , we do not know a reliable analytical technique in the $1 < n_f \leq 4$ case to evaluate the condensate. However, we expect the modulus of the chiral condensate to saturate to a c-number times Λ^{3n_f} . Consequently,

$$\langle \Omega_k | \det \text{tr} \lambda^I \lambda^J | \Omega_k \rangle \sim \begin{cases} \Lambda^{3n_f} (\Lambda L)^{\frac{11}{3}(1-n_f)} e^{\frac{i2\pi k}{N}} & L \ll L_c \\ \Lambda^{3n_f} e^{\frac{i2\pi k}{N}}, & L > L_c \end{cases} \quad (3.45)$$

where the phase is \mathbb{Z}_N valued. In the $n_f = 1$ case, this produces the correct L independence of chiral condensate (which is due to supersymmetry) [15], and N isolated vacua. We believe that the scale at which the determinantal condensate becomes L independent is the scale of the gauge symmetry restoration.

In the far infrared of the QCD(adj), since σ is massive, the long distance theory further reduce to a purely fermionic theory, which schematically looks like an NJL-type Lagrangian:

$$L_{\text{NJL}} = \sum_{j=1}^N \left[i \bar{\psi}_j^I \gamma_\mu \partial_\mu \psi_j^I + c e^{-S_0} (\det_{I,J} \alpha_j \psi^I \alpha_j \psi^J + \text{c.c.}) \right] \quad (3.46)$$

The Lagrangian is invariant under $SU(n_f) \times \mathbb{Z}_{2n_f}$ chiral symmetry. The \mathbb{Z}_{2n_f} is the unbroken subgroup of the \mathbb{Z}_{2Nn_f} discrete symmetry. We wish to know whether the continuous chiral symmetry is broken spontaneously.

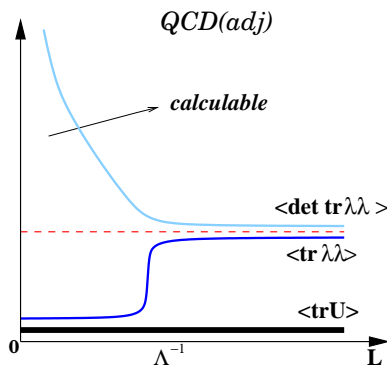


Figure 3: The cartoon of the behavior of the center, discrete and continuous chiral symmetry realization in QCD(adj), for $SU(N)$ where $N = \text{few}$, $n_f = 2$ and $n_f = 1$ ($\mathcal{N} = 1$ SYM). The spatial center symmetry is unbroken at any L in both cases $\langle \text{tr} U \rangle = 0$. In $n_f = 2$, the continuous chiral symmetry is unbroken at small S^1 and broken at large S^1 , and discrete chiral symmetry is always broken. The red (dotted) line is the chiral condensate in $\mathcal{N} = 1$ SYM, and the discrete chiral symmetry is always broken. In the small S^1 regime, the string tensions and thicknesses (the inverse mass gap in gauge sector) are calculable, and $n_f = 2$ theory exhibits confinement without continuous chiral symmetry breaking. The lines slightly on top of the horizontal axis are all zero and are spit to guide the eye.

At small S^1 , we believe the continuous chiral symmetry is unbroken, based on studies on related $d = 3$ dimensional NJL type models. Such models has generically a weakly coupled chirally symmetric phase and a chirally asymmetric strong coupling phase. (See, the review Ref. [40]). Our dimensionless coupling constant is $g \sim e^{-S_0}$, is far too small to induce a chiral transition. Hence, the chiral symmetry must be unbroken, and there must be massless fermions (protected by chiral symmetry) in the spectrum within the region of validity of our long distance effective theory, ($L\Lambda \ll 1$). We believe the naive extrapolation of the NJL Lagrangian Eq.3.46 will exhibit the continuous chiral transition in an expected regime of the underlying QCD theory. (See fig.3.2.3.) However, this will happen outside the region of validity of our effective theory. Consequently, this does not tell us that monopole induced vertex is the dynamical origin of continuous chiral symmetry breaking, even though it is the origin of the discrete chiral symmetry breaking in the small S^1 regime.

The absence of the continuous chiral symmetry breaking in weak coupling regime can also be seen by an independent argument. In the small S^1 regime where theory is weakly coupled, we have control over all nonperturbative objects. A BPS or KK monopole, which may in principle contribute to the condensate, has a minimum of $2n_f$ fermionic zero modes. However, our order parameter $\text{tr} \lambda^I \lambda^J$ can only soak up two zero modes. This implies it cannot acquire a non-trivial vacuum expectation value. The minimal operator which may acquire a condensate must have $2n_f$ fermion insertion, and this is indeed the determinantal condensate $\langle \det \text{tr} \lambda^I \lambda^J \rangle$. The correctness of this argument relies to weak coupling, and an analogous argument cannot be carried to strong coupling.

At large S^1 (and \mathbb{R}^4), the common lore is that the chiral symmetry is spontaneously

broken down to $SO(n_f) \times \mathbb{Z}_2$ by the formation of the chiral condensate

$$\langle \Omega_k | \text{tr } \lambda^I \lambda^J | \Omega_k \rangle \sim \begin{cases} 0 & L < L_c \\ \Lambda^3 e^{\frac{i2\pi\kappa}{Nn_f}}, & L > L_c \end{cases} \quad (3.47)$$

Consequently, there must exist N isolated coset spaces each of which is $SU(n_f)/SO(n_f)$. In this expression, κ ranges in $[0, Nn_f)$. Denote $\kappa = \iota N + k$ where $k = 0, \dots, N-1$ and $\iota = 0, \dots, n_f - 1$. For a given k , there are n_f many ι for which the determinant of the condensate is invariant. Thus, they reside in the same coset space, and there are consequently N isolated coset spaces.

The continuous chiral transition in QCD(adj) is very different from its thermal counterparts. In particular, it occurs in the absence of any change in its spatial center symmetry realization. This is a quantum phase transition at absolute zero temperature, induced solely due to quantum fluctuations. We do not know the order of the phase transition.

Finally, we wish to conjecture that the scale of the chiral phase transition L_c in QCD(adj) is associated with the restoration of the spontaneously broken gauge symmetry. Consequently, we believe that the chiral symmetry breaking is a strong coupling phenomena. Confinement is not necessarily so.

3.3 Noncompact versus compact adjoint Higgs, final pass

Let us reconsider the $SU(N)$ gauge theory with *noncompact* adjoint Higgs field and with one Dirac fermions in adjoint representation on \mathbb{R}^3 . (Multiflavor generalization is obvious.) The theory possess a $U(1)$ fermion number symmetry. The generalization of the argument of ref. [11] shows that the $U(1)$ symmetry is spontaneously broken, and consequently, there only exists one gapless excitation by Goldstone's theorem. The other $N-2$ photons of the spontaneously broken gauge symmetry must acquire masses. We wish to know how this is realized in the microscopic description.

When the $SU(N)$ gauge symmetry breaks down to $U(1)^{N-1}$ via an noncompact adjoint Higgs field rather than a compact one (which was the case in QCD(adj)), monopoles only comes in $N-1$ varieties. The KK monopole is now absent. We may still define the magnetic bions in the spontaneously broken $SU(N)$ gauge theory for $N \geq 3$, but there are only $N-2$ of them. As before, a bion is a bound state of the monopole associated with magnetic charge α_i and anti-monopole associated with charge $-\alpha_{i+1}$ with null topological charge. The magnetic charge of a bion is

$$Q_i = \alpha_i - \alpha_{i-1}, \quad i = 2, \dots, N-1 \quad (3.48)$$

Hence, there are only $N-2$ types of magnetic bions. In other words, the absence of $\alpha_N \equiv \alpha_0$ KK monopole removes two would be bions of the compact theory. Thus, the potential for the σ field is a sum over $N-2$ bions and their conjugates given by

$$V(\sigma) = -e^{-2S_0} \sum_{i=2}^{N-1} (e^{iQ_i\sigma} + c.c) \quad (3.49)$$

The potential generates mass terms only for $N - 2$ dual photons. The massless photon is the Goldstone boson. Equivalently, we may say the sum in the prepotential is restricted to the root system Δ_0 , $\mathcal{W}(\sigma) = e^{-S_0} \sum_{\alpha_i \in \Delta_0} e^{i\alpha_i \sigma}$, and from the study of the analogous supersymmetric theory, we know that the reduction from affine Toda to nonaffine Toda renders the gapped theory gapless [12, 15].

4. Outlook: Confinement, and more on its persona and anima

A microscopic derivation of the mechanism which provides confinement in QCD(adj) quantized on small $S^1 \times \mathbb{R}^3$ is given. This is a QCD-like theory with no elementary scalars in its Lagrangian, and no special properties such as supersymmetry (except the $n_f = 1$ case). We believe the solution provides a significant contribution to our current understanding of QCD-like gauge theories, and teaches us many valuable lessons. We also found the underlying dynamical reasons behind the lack of confinement in Yang-Mills noncompact Higgs systems with adjoint fermions formulated on \mathbb{R}^3 . Let us quote our main result for the $SU(2)$ gauge group:

- QCD(adj) exhibits permanent confinement even at arbitrarily weak coupling (small S^1). In other words, in asymptotically free confining gauge theories, confinement is not *necessarily* a strong coupling phenomena.
- In the presence of massless dynamical fermions, the objects with nonvanishing topological charge must have compulsory fermion zero mode attached to them. Hence, they induce fermion-fermion and fermion-dual photon interactions, neither of which can appear in the bosonic potential of the dual photon. Our arguments rule out monopoles and monopole condensation as the microscopic mechanism of the confinement in QCD-like theories with *dynamical fermions* in general.
- The beautiful and qualitative idea of dual superconductivity is quantitatively realized in the vacuum of QCD(adj). Pairs with magnetic and topological charge $(\pm 2, 0)$ which we referred as bions condenses, and magnetic bion condensation is the mechanism of confinement.
- A new pairing mechanism is at play. The repulsive Coulomb repulsion between the bion constituents [with charges $(1, +\frac{1}{2})$ and $(1, -\frac{1}{2})$] is overwhelmed by a attractive logarithmic force. The pairing mechanism responsible for the bound state is induced by $2n_f$ -fermion exchange in n_f flavor theory.
- Bions generate the bosonic potential which provides the mass gap for the dual photon.
- This rationale also explains why the Yang-Mills with *noncompact* adjoint Higgs field and adjoint fermions do not confine on \mathbb{R}^3 despite the presence of monopoles. The same rationale is also true for $\mathcal{N} = 2$ SYM on \mathbb{R}^3 . These are examples as important as QCD(adj) itself, because we believe it is equally important to understand the lack of confinement in order to understand confinement.

- In general $SU(N)$ case, we demonstrated the area law of confinement for Wilson loops in arbitrary representations. The dual theory hints an integrable (generalized Toda) system behind QCD(adj), in the e^{-S_0} expansion of the action at order e^{-2S_0} . We do not know whether this extends to higher order if we were to find higher order terms in e^{-S_0} expansion. We also do not know whether there may be integrability behind QCD(adj) on \mathbb{R}^4 .

We wish to express that we are optimistic of future progress which will reveal the inner goings-on of general QCD-like theories:

Incorporating fundamental representation fermions: For example, in a theory with two adjoint and one fundamental fermion (**mixed action**), the back-reaction of fermion is insufficient to induce center symmetry breaking in the small S^1 regime. This theory has both magnetic monopoles, and massless electric charges within the weak coupling regime examined in this paper. This system should teach us something which may be relevant to the QCD of Nature. Unfortunately, our techniques are not directly applicable to pure Yang-Mills or QCD with fundamental fermions due to breaking of (temporal or spatial) center symmetry at small S^1 .

Confinement on QCD-like theories on \mathbb{R}^4 : The techniques of this paper are strictly valid in the gauge symmetry broken phase of the QCD(adj). However, we believe that certain assertions are generalizable to \mathbb{R}^4 , and direct progress will occur in QCD(adj) on \mathbb{R}^4 , where strong coupling necessarily occurs.

Lattice gauge theory: Many assertions made in this paper are directly testable in lattice simulations with available technologies. In particular, the string tensions and characteristic sizes of flux tubes 3.34, 3.43 can be extracted from the lattice simulations of QCD(adj) as in [41]. QCD(adj) also undergoes a zero temperature quantum chiral transition in the absence of any change in center symmetry realization. This should be directly testable on the lattice by modifying the existing simulations (such as [42]) appropriately. It would also be useful to construct the duality between QCD(adj) on $\mathbb{R}^3 \times S^1$ with Lagrangian 2.1 and dual QCD defined in 3.27 directly in lattice formulations.

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References

- [1] J. Greensite, *The confinement problem in lattice gauge theory*, *Prog. Part. Nucl. Phys.* **51** (2003) 1, [[hep-lat/0301023](#)].
- [2] M. Shifman and A. Yung, *Supersymmetric solitons and how they help us understand non-abelian gauge theories*, [hep-th/0703267](#).

- [3] F. Bruckmann, *Topological objects in qcd*, [arXiv:0706.2269](#) [hep-th].
- [4] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in $n=2$ supersymmetric yang-mills theory*, *Nucl. Phys.* **B426** (1994) 19–52, [[hep-th/9407087](#)].
- [5] M. R. Douglas and S. H. Shenker, *Dynamics of $su(n)$ supersymmetric gauge theory*, *Nucl. Phys.* **B447** (1995) 271–296, [[hep-th/9503163](#)].
- [6] A. M. Polyakov, *Quark confinement and topology of gauge groups*, *Nucl. Phys.* **B120** (1977) 429–458.
- [7] N. A. Nekrasov, *Seiberg-witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831–864, [[hep-th/0206161](#)].
- [8] M. Unsal, *Abelian duality, confinement, and chiral symmetry breaking in $qcd(adj)$* , [arXiv:0708.1772](#) [hep-th].
- [9] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Theory of superconductivity*, *Phys. Rev.* **108** (1957) 1175–1204.
- [10] G. W. Gibbons, *Born-infeld particles and dirichlet p -branes*, *Nucl. Phys.* **B514** (1998) 603–639, [[hep-th/9709027](#)].
- [11] I. Affleck, J. A. Harvey, and E. Witten, *Instantons and (super)symmetry breaking in $(2+1)$ -dimensions*, *Nucl. Phys.* **B206** (1982) 413.
- [12] S. H. Katz and C. Vafa, *Geometric engineering of $n = 1$ quantum field theories*, *Nucl. Phys.* **B497** (1997) 196–204, [[hep-th/9611090](#)].
- [13] N. Seiberg and E. Witten, *Gauge dynamics and compactification to three dimensions*, [hep-th/9607163](#).
- [14] N. M. Davies, T. J. Hollowood, V. V. Khoze, and M. P. Mattis, *Gluino condensate and magnetic monopoles in supersymmetric gluodynamics*, *Nucl. Phys.* **B559** (1999) 123–142, [[hep-th/9905015](#)].
- [15] N. M. Davies, T. J. Hollowood, and V. V. Khoze, *Monopoles, affine algebras and the gluino condensate*, *J. Math. Phys.* **44** (2003) 3640–3656, [[hep-th/0006011](#)].
- [16] E. Corrigan, *Recent developments in affine toda quantum field theory*, [hep-th/9412213](#).
- [17] T. J. Hollowood, *Solitons in affine toda field theories*, *Nucl. Phys.* **B384** (1992) 523–540.
- [18] M. A. C. Kneipp, *Bps z_n string tensions, sine and casimir laws and integrable field theories*, [arXiv:0707.3791](#) [hep-th].
- [19] E. D’Hoker, I. M. Krichever, and D. H. Phong, *Seiberg-witten theory, symplectic forms, and hamiltonian theory of solitons*, [hep-th/0212313](#).
- [20] D. Diakonov and V. Petrov, *Ensemble of dyons*, [arXiv:0704.3181](#) [hep-th].
- [21] E. T. Tomboulis, *Confinement for all values of the coupling in four- dimensional $su(2)$ gauge theory*, [arXiv:0707.2179](#) [hep-th].
- [22] F. Lenz, J. W. Negele, and M. Thies, *Confining effective theories based on instantons and merons*, [arXiv:0708.1687](#) [hep-ph].
- [23] A. Di Giacomo, B. Lucini, L. Montesi, and G. Paffuti, *Colour confinement and dual superconductivity of the vacuum. i*, *Phys. Rev.* **D61** (2000) 034503, [[hep-lat/9906024](#)].

- [24] D. Tong, *Tasi lectures on solitons*, [hep-th/0509216](#).
- [25] E. J. Weinberg and P. Yi, *Magnetic monopole dynamics, supersymmetry, and duality*, *Phys. Rept.* **438** (2007) 65–236, [[hep-th/0609055](#)].
- [26] T. Schafer and E. V. Shuryak, *Instantons in qcd*, *Rev. Mod. Phys.* **70** (1998) 323–426, [[hep-ph/9610451](#)].
- [27] S. R. Coleman and E. Weinberg, *Radiative corrections as the origin of spontaneous symmetry breaking*, *Phys. Rev.* **D7** (1973) 1888–1910.
- [28] M. Unsal and L. G. Yaffe, *(in)validity of large n orientifold equivalence*, *Phys. Rev.* **D74** (2006) 105019, [[hep-th/0608180](#)].
- [29] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison, E. Witten, and eds, *Quantum fields and strings: A course for mathematicians. vol. 1, 2*, . Providence, USA: AMS (1999) 1-1501.
- [30] T. C. Kraan and P. van Baal, *Monopole constituents inside $su(n)$ calorons*, *Phys. Lett.* **B435** (1998) 389–395, [[hep-th/9806034](#)].
- [31] K.-M. Lee and C.-h. Lu, *$Su(2)$ calorons and magnetic monopoles*, *Phys. Rev.* **D58** (1998) 025011, [[hep-th/9802108](#)].
- [32] G. 't Hooft, *Computation of the quantum effects due to a four- dimensional pseudoparticle*, *Phys. Rev.* **D14** (1976) 3432–3450.
- [33] G. 't Hooft, *How instantons solve the $u(1)$ problem*, *Phys. Rept.* **142** (1986) 357–387.
- [34] G. 't Hooft, *Topology of the gauge condition and new confinement phases in nonabelian gauge theories*, *Nucl. Phys.* **B190** (1981) 455.
- [35] S. Mandelstam, *Vortices and quark confinement in nonabelian gauge theories*, *Phys. Rept.* **23** (1976) 245–249.
- [36] C. Callias, *Index theorems on open spaces*, *Commun. Math. Phys.* **62** (1978) 213–234.
- [37] R. Jackiw and C. Rebbi, *Solitons with fermion number $1/2$* , *Phys. Rev.* **D13** (1976) 3398–3409.
- [38] J. Callan, Curtis G., R. F. Dashen, and D. J. Gross, *Toward a theory of the strong interactions*, *Phys. Rev.* **D17** (1978) 2717.
- [39] A. Hanany, M. J. Strassler, and A. Zaffaroni, *Confinement and strings in $mQCD$* , *Nucl. Phys.* **B513** (1998) 87–118, [[hep-th/9707244](#)].
- [40] B. Rosenstein, B. Warr, and S. H. Park, *Dynamical symmetry breaking in four fermi interaction models*, *Phys. Rept.* **205** (1991) 59–108.
- [41] B. Lucini, M. Teper, and U. Wenger, *Glueballs and k -strings in $su(n)$ gauge theories: Calculations with improved operators*, *JHEP* **06** (2004) 012, [[hep-lat/0404008](#)].
- [42] F. Karsch and M. Lutgemeier, *Deconfinement and chiral symmetry restoration in an $su(3)$ gauge theory with adjoint fermions*, *Nucl. Phys.* **B550** (1999) 449–464, [[hep-lat/9812023](#)].