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A sub-horizon framework for probing the relationship between the cosmological matter distribution and metric perturbations

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ABSTRACT
The relationship between the metric and nonrelativistic matter distribution depends on the theory of gravity and additional fields, hence providing a possible way of distinguishing competing theories. With the assumption that the geometry and kinematics of the homogeneous universe have been measured, we present a procedure for understanding and testing the relationship between the cosmological matter distribution and metric perturbations (along with their respective evolution) using the ratio of the physical size of the perturbation to the size of the horizon as our small expansion parameter. We expand around Newtonian gravity on linear, subhorizon scales with coefficient functions in front of the expansion parameter. Our framework relies on an ansatz which ensures that (i) the Poisson equation is recovered on small scales (ii) the metric variables (and any additional fields) are generated and supported by the nonrelativistic matter overdensity. The scales for which our framework is intended are small enough so that cosmic variance does not significantly limit the accuracy of the measurements and large enough to avoid complications due to nonlinear effects and baryon cooling. From a theoretical perspective, the coefficient functions provide a general framework for contrasting the consequences of ΛCDM and its alternatives. We calculate the coefficient functions for general relativity (GR) with a cosmological constant and dark matter, GR with dark matter and quintessence, scalar-tensor theories (STT), $f(R)$ gravity and braneworld (DGP) models. For observers, constraining the coefficient functions provides a streamlined approach for testing gravity in a scale dependent matter. We briefly discuss the observations best suited for an application of our framework.

Key words: large scale structure of Universe, dark matter, gravitational lensing.

1 INTRODUCTION

A successful model of the universe must include a background geometry, an inventory of its contents, a kinematical description of its expansion and a dynamical explanation of how its constituents interact, drive the expansion and develop structure. Recent observations [for example (Riess et al. 1998; Perlmutter et al. 1999; Freedman et al. 2001; Allen et al. 2002; Tegmark et al. 2004; Eisenstein et al. 2005; Spergel et al. 2007) and references therein] have led to a "Flat ΛCDM cosmology", (henceforth FΛCDM) dominated by dark energy (cosmological constant Λ) and matter (predominately dark and initially cold) and the observed expansion rate and growth of structure agree with the predictions of this model at the ten percent level. Future observations should be capable of testing this model at the one percent level. If they verify its predictions, they will affirm a remarkable, simple description of the universe, implicit in the earliest relativistic investigations of Einstein, Friedmann and Lemaître, analogous to the affirmation of general relativity (GR) that took place twenty years ago (Will 2001). If FΛCDM passes this test, then the challenge will be to account for this outcome in terms of physical processes operating at earlier epochs; if it fails, then we shall either have learned something important about gravitational physics or uncovered a new constituent. Many alternatives, with and without GR, to FΛCDM have been proposed. At this stage, none of them stands out. There is therefore a need to provide a framework for describing future observations and theoretical investigations in general terms which will facilitate a distinction between FΛCDM and its alternatives. The provision of one such framework is the goal of this paper.

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Further observational progress is anticipated over the coming decade. The analysis of Planck observations (Planck-Collaboration 2006) of the microwave background, coupled with local measurements of the contemporary Hubble parameter, H_0, should result in an extremely accurate description of the physical conditions and the statistical properties of the density fluctuation spectrum at the epoch of recombination when the universe had a scale factor $a \equiv (1+z)^{-1} \sim 10^{-3}$ relative to today. Combining the calculated physical sizes of the acoustic peaks in the background radiation spectrum with the Hubble constant and the Copernican Principle leads to a measurement of spatial curvature, which is already known to contribute to the kinematics at a level of less than a few percent (Spergel et al. 2007). We shall adopt a value of zero for illustration purposes. Essentially kinematic measurements, for example, those involving Type Ia supernova explosions, baryonic acoustic oscillations and baryonic gas fractions in clusters should provide a record of the comoving distance, $d(a) = \int cdt/a$, from which the evolution of the Hubble parameter $H(a) = d\ln a/dt$ and the acceleration parameter $q(a) = d\ln Ha/d\ln a$ can be inferred[1]. For the rest of the paper we shall assume that these evolutions are known. Note that we are using a instead of the cosmic time t as the time coordinate as this relates directly to the observable photon frequency shift. For recent constraints on the expansion history, see for example (Rapetti et al. 2007) and references therein.

Given an understanding of the geometry and kinematics, the task is then to see if the dynamical evolution of the universe is consistent with GR or mandates an alternative theory. Now, GR provides a relationship between the space-time geometry on a cosmological scale measured by the Einstein tensor $\mathbf{G}[g_{\mu\nu}]$ and the total Energy-Momentum Tensor (EMT) of its contents $\mathbf{T}$, $\mathbf{G}[g_{\mu\nu}] = 8\pi G\mathbf{T}$. The discovery that $\mathbf{G}[g_{\mu\nu}] \neq 8\pi G\mathbf{T}[\text{“obs”}]$ where $\mathbf{T}[\text{“obs”}]$ includes known forms of matter such as electromagnetic radiation, baryons etc. has led to the addition of dark matter and dark energy contributions to the EMT. Dark matter candidates include Weakly Interacting Massive Particles and axions which would presumably behave gravitationally like baryonic matter. However other possibilities exist which might behave differently. Dark energy is most simply characterized as a temporarily and spatially constant vacuum energy field with zero enthalpy [see (Carroll 2001) for a review]. However, it could also have quite different dynamical properties and might include contributions from additional scalar (Ratra & Peebles 1988), vector (Armendariz-Picon 2004) or tensor fields with possible interactions between each other (Farrar & Peebles 2004) and with known forms of matter. Historically, the first representation of dark energy was Einstein's cosmological constant, which was seen as an augmentation to $\mathbf{G}$, not $\mathbf{T}$ [see for example (Carroll et al. 2004)]. This original proposal has also been generalized in many ways so that $\mathbf{G}[g_{\mu\nu}] + \mathbf{F}[g_{\mu\nu}, \varphi] = 8\pi G\mathbf{T}[\text{“obs”}]$, where $\mathbf{F}[g_{\mu\nu}, \varphi]$ depends on the metric and more generally some additional gravitational fields, φ. For example φ could be the additional gravitational scalar field in Scalar-Tensor Theories (STT) [see for example (Santiago et al. 1998; Perrotta et al. 2000)]. Nature could of course be unkind and we might have

$$\mathbf{G}[g_{\mu\nu}] + \mathbf{F}[g_{\mu\nu}, \varphi] = 8\pi G\mathbf{T}[\text{“obs”}] + 8\pi G\mathbf{T}[\text{“dark”}]. \qquad (1)$$

Considerable effort has been made in constructing models that fall into the above mentioned categories and more recently in finding ways to distinguish between them [for example see (Lue et al. 2004; Ishak et al. 2006; Bludman 2006; Zhang et al. 2007; Huterer & Linder 2007)].

Now, modifying the physics beyond GR with cold dark matter and Λ can have three quite separate manifestations. Firstly it can lead to a change in expansion of the universe, secondly, it can influence the growth of structure and thirdly, it can confront local tests of the theory of gravity. The approach that we follow is to assume that the theory is constrained by the first and third manifestations and that it is the growth of structure that is providing the test. This oversimplifies the data analysis but does lead to a transparent and simple approach. One important consequence of adopting local gravitational tests is that photons and baryons, at least, will follow geodesics and that the photons will be subject to cosmological redshifting of their frequencies, $\nu \propto a^{-1}$. This simplifies the interpretation of observational data.

Our procedure is to adopt a general form for the metric of a linearly perturbed homogeneous and isotropic universe which introduces two potentials $\Phi(\mathbf{x}, a)$ and $\Psi(\mathbf{x}, a)$ (scalar metric perturbations in the Newtonian gauge), where $\mathbf{x}$ denotes the three spatial coordinates. We also introduce an associated fractional density perturbation $\delta_m(\mathbf{x}, a)$ in nonrelativistic matter and relate it to the potentials. We assume that there is a dominant nonbaryonic contribution to the clustering nonrelativistic matter. In practice, it is easier to work with Fourier modes and this allows us to focus attention on the range of length scales that are most relevant observationally: comfortably smaller than the horizon so that we can observe enough independent volumes within our current horizon allowing for a high precision measurement despite “cosmic variance” and yet large enough that nonlinear effects and baryonic cooling are not a factor. Within this range of length scales, we adopt the following ansatz regarding the relationship between linearized metric and density perturbations, written as an expansion in powers of $(aH/k)^2$, where k is the magnitude of the comoving wavevector $\mathbf{k}$.

$$\begin{aligned}
\Phi(\mathbf{k}, a) &= -\frac{4\pi G\rho_m}{H^2}\left(\frac{aH}{k}\right)^2 \delta_m(\mathbf{k}, a)\left[\beta_0(a) + \beta_1(a)\left(\frac{aH}{k}\right)^2 + \dots\right] \\
\Psi(\mathbf{k}, a) &= -\frac{4\pi G\rho_m}{H^2}\left(\frac{aH}{k}\right)^2 \delta_m(\mathbf{k}, a)\left[\gamma_0(a) + \gamma_1(a)\left(\frac{aH}{k}\right)^2 + \dots\right] \\
\delta_m(\mathbf{k}, a) &= \delta_{mi}(\mathbf{k})\left[\delta_0(a) + \delta_1(a)\left(\frac{aH}{k}\right)^2 + \dots\right],
\end{aligned} \qquad (2)$$

where the background mass density $\rho_m \propto a^{-3}$ and $\delta_{mi}(\mathbf{k})$ is determined from initial conditions which can in principle be taken close to the surface of last scattering, $a_i \sim 10^{-3}$, as long as the modes are sufficiently sub-horizon. Assuming a simple prescription for evolving the perturbations from last scattering to the post-reionization era $a \gtrsim 10^{-1}$, in this paper we take $a_i \sim 10^{-1}$. The coefficient functions $\{\beta_n, \gamma_n, \delta_n\}$ with $n = 0, 1$ are arbitrary functions of the scalefactor. The leading terms in the expansion agree with

[1] Our acceleration parameter differs from the conventional deceleration parameter by a minus sign.

Poisson's equation on small scales, subsequent terms allow for a scale-dependent departure from Newtonian theory on larger scales where the relative velocities approach that of light. This approach introduces a length-scale dependence to the perturbations which is equivalent to an expansion in powers of $(\Phi/c^2) \sim (v/c)^2$, similar to what is done in the Parametrized Post Newtonian development of theories of gravitation. Relative to Newtonian gravitation, in which $c \to \infty$, GR introduces corrections. The theories that we discuss below introduce different corrections (different coefficient functions $\{\beta_n, \gamma_n, \delta_n\}$) and these differences are measurable. From an observer's perspective, constraining the coefficient functions with measurements of Φ, Ψ and δ_m provides a streamlined approach to characterizing gravity on cosmological scales in a scale dependent manner. On the other hand, from a theorist's perspective, substituting the ansatz into the field equations for a given theory allows for a (mostly straigtforward) calculation of the coefficient functions. The coefficient functions provide a means of comparing the consequences of different theories. We shall discuss our assumptions, limitations and our ansatz in detail in the next section.

This is certainly not the first time that an attempt at constructing and applying such a framework has been made. The Parametrized Post Newtonian formalism (see (Will 2001) and references therein) has been a powerful framework for understanding and constraining gravity on solar system (and other isolated system) scales. It is hoped that we can construct a similar framework for cosmological scales. Recently a few attempts have been made in this direction. However most of these are either concerned with the expansion history alone, deal with specific aspects of departures from GR such as effective gravitational constant on small scales (Tsujikawa 2007), growth of perturbations on small scales (density perturbations & effective gravitational constant in modified gravity models of dark energy Lin), the gravitational slip (Caldwell et al. 2007), or deal with superhorizon scales (Bertschinger 2006). The authors in (Amendola et al. 2007) take into account growth of structure, anisotropic stress and the modification to the Poisson equation and parametrize departures from Einstein's gravity with a growth index and two functions of the scale factor which are relevant for weak lensing surveys. However, they do not consider scale dependent departures. Another popular phenomenological approach for characterizing the effects of the unknown physics (additional fields, their interactions, or modified gravitational laws) is to define an effective fluid energy momentum tensor for everything other than the standard model matter, effectively move $\mathbf{F}$ in equation (1) to the right hand side and define $\mathbf{T}_{eff} = -(8\pi G)^{-1}\mathbf{F}[\varphi, g_{\mu\nu}] + \mathbf{T}[\text{“dark”}]$. This effective energy momentum tensor is then parametrized in terms of the equation of state, sound speed, anisotropic stress, etc. (Hu & Eisenstein 1999; Bashinsky 2007). This approach, however, seems to put an unnecessary restriction of a fluid interpretation which might be misleading, especially when the effective dark energy is due to modified gravity or extra dimensions. We are unaware of a systematic approach undertaken where the framework includes a scale *dependent* departure in the relationship between the matter distribution and the metric perturbations along with their respective evolution on cosmological scales up to post-Newtonian order.

The rest of the paper is organized as follows. Section 2 discusses our assumptions and the particular form of the ansatz in detail. In Section 3 we apply our framework to GR, STT, quintessence, $f(R)$ models (Carroll 2001) and DGP gravity (Dvali et al. 2000). In particular, we calculate the coefficient functions in these theories and comment on our ansatz in the context of these theories. Section 4 is devoted to how our framework might be employed by observers. We briefly discuss the observations that could be used to constrain the different coefficient functions. Section 5 presents a short summary and future directions for extending the framework.

2 OUR ANSATZ AND ASSOCIATED ASSUMPTIONS

With an eye towards observations in the next decade, we assume that the geometry (spatial curvature) and kinematics (expansion history) of the universe have been measured to a percent level accuracy. What remains to be understood and measured accurately (at the few percent level) is the relationship between the metric fluctuations and the nonrelativistic matter distribution along with their respective evolution on linear, subhorizon scales. This relationship will depend on the theory of gravity or the presence of yet unknown components, thus providing a test for distinguishing different theories. To explore this relationship in an (almost) model independent way, we provide an ansatz, equation (2), relating the scalar metric perturbations (in Newtonian gauge) and the nonrelativistic matter overdensity in Fourier space. In this section we discuss the particular form of the ansatz and the underlying assumptions in detail. We introduce our notation and conventions followed by some physical arguments regarding our choice of the particular form of the ansatz. We end with a discussion of the range of scales for which our ansatz is expected to be useful.

We focus on a perturbed FRW universe (spatially flat) with scalar metric fluctuations in the Newtonian gauge (Bardeen 1980). In this gauge the metric takes the following form ($c = 1$)

$$ds^2 = -[1 + 2\Phi(\mathbf{x}, t)]dt^2 + a^2(t)[1 - 2\Psi(\mathbf{x}, t)]d\mathbf{x} \cdot d\mathbf{x}$$

Here the metric perturbations $|\Phi(\mathbf{x}, t)|, |\Psi(\mathbf{x}, t)| \ll 1$. We choose to work in the Newtonian gauge because $\Phi(\mathbf{x}, t)$ is the generalization of the Newtonian gravitational potential and the potentials $\Phi(\mathbf{x}, t)$ and $\Psi(\mathbf{x}, t)$ are gauge invariant Bardeen variables when we specialize to the Newtonian gauge. The energy density perturbation $\delta_m(\mathbf{x}, t)$ is also gauge invariant, corresponding to the energy density perturbation on the zero shear spatial hypersurface which is closest to Newtonian time slicing [see equation (3.14) in (Bardeen 1980)]. In what follows, we use the scale factor a as the independent variable instead of cosmic time t with $a(\text{today}) = 1$. With this change of variables, the metric takes the form

$$ds^2 = -[1 + 2\Phi(\mathbf{x}, a)](aH)^{-2}da^2 + a^2[1 - 2\Psi(\mathbf{x}, a)]d\mathbf{x} \cdot d\mathbf{x}$$

We shall work primarily in Fourier space and use the convention $f(\mathbf{x}, a) = \int d^3\mathbf{k} f(\mathbf{k}, a)e^{i\mathbf{k}\cdot\mathbf{x}}$. To avoid unnecessary clutter we write the Fourier transform of the metric perturbations $\Phi(\mathbf{k}, a)e^{i\mathbf{k}\cdot\mathbf{x}}$ as Φ. The same is true for Ψ and

δ_m. The background quantities depend on a. We shall often suppress this dependence; for example by H we mean $H(a)$.

Our ansatz provides a relationship between Φ, Ψ and δ_m on linear (in Φ, Ψ and δ_m), subhorizon scales. We now turn to the discussion of some important features of this ansatz. On scales that are much smaller than the size of the horizon, $aH/k \ll 1$, the leading term has the form of a linearised Newtonian gravitational field equation. For the purpose of this paper the Newtonian form of the field equation refers to the the following relation between the time-time metric perturbation $\Phi(\mathbf{x}, a)$ and the nonrelativistic matter density contrast $\delta_m(\mathbf{x}, a)$, $\nabla^2\Phi(\mathbf{x}, a) \propto \delta_m(\mathbf{x}, a)$, which in Fourier space becomes $\Phi \propto (aH/k)^2\delta_m$. Now, in the Newtonian gauge $\Phi(\mathbf{x}, a)$ plays the role of the Newtonian potential once the background has been subtracted out. The proportionality allows for a possible temporal variation in the effective Newton's constant which could depend on the cosmological background evolution.

From GR we know that this Newtonian relation starts breaking down as the size of the perturbation becomes comparable to the size of the horizon. In general, different theories of gravity will introduce different scale dependent departures from this equation, changing the metric-matter relationship. Our claim is that for a large class of theories, our ansatz, equation (2), captures the scale dependence of the relationship between the nonrelativistic matter distribution and cosmological metric perturbations. In particular, our ansatz faithfully reproduces the scale dependence of the metric-matter relationship in the fiducial case of GR with cold dark matter and a cosmological constant. In the presence of additional fields one might expect this relationship to break down; however, this is usually not the case. Suppose that an additional field enters the equations, for example as a source (quintessence), as a time varying gravitational constant (Brans Dicke) or indirectly encapsulating the effect of higher dimensions, etc. Perturbations $\delta\varphi$ in such a scalar field φ (consider quintessence or scalar-tensor theories) will be involved in the relationship between δ_m and Φ. However, from the field equation for $\delta\varphi$, equations (18) and (19), we can see that $\delta\varphi \propto \Phi(aH/k)^2$ for quintessance and $\delta\varphi \propto \Phi$ for scalar-tensor theories when $aH/k \ll 1$. Thus, even if additional scalar fields are present, our ansatz should be a good approximation for the relationship between the matter distribution and the metric at the scales of interest. Note that we have assumed $\mathcal{O}[\Phi] = \mathcal{O}[\Psi]$ for this argument.

Another feature of our ansatz is that Φ and Ψ are directly proportional to δ_m. This might seem unusual, since it implies that in the absence of nonrelativistic matter perturbations, there would be no metric perturbations. This is certainly not true in principle if an additional scalar field is present. However observationally, we know that nonrelativistic perturbations are present and they dominate over perturbations in other fields. The following argument provides a more detailed justification. Since on the smallest scales, to lowest order in $(aH/k)^2$, the potential $\Phi \propto \delta_m(aH/k)^2$, we also have $\delta\varphi \propto \delta_m(aH/k)^4$ or $\delta\varphi \propto \delta_m(aH/k)^2$. This means that the potentials and pertubations in other scalar fields are supported by the nonrelativistic matter perturbations.

We do not expect to see the effects of the initial power spectrum of these additional fields up to the order of the terms considered in our ansatz, with the initial power spectrum of the additional field possibly playing a role in higher order terms. This is one of the reasons for not extending the power series in aH/k beyond the order considered in the ansatz.

We note that the above arguments are made under the assumption that the additional gravitational or nongravitational contribution to the field equations is due to a scalar field (quintessence or scalar tensor theories). It is certainly possible to construct theories where this ansatz will fail to capture some aspect of the scale dependence. One such situation could arise in extra dimensional theories where boundary conditions on our 4 dimensional brane might give rise to a scale dependence involving odd powers of k as well.

Regarding Ψ, we assume that the relationship between Ψ and δ_m has the same (aH/k) dependence as Φ and δ_m since from GR we expect $\Phi = \Psi$ when no anisotropic stress is present. The form of $\delta_m(\mathbf{k}, a)$ in the ansatz can be motivated from the conservation equation for nonrelativistic matter at first order in Φ, Ψ and δ_m:

$$a^2\partial_a^2\delta_m + (2+q)a\partial_a\delta_m = -\left(\frac{k}{aH}\right)^2\Phi + 3\left[a^2\partial_a^2\Psi + (2+q)a\partial_a\Psi\right]. \quad (3)$$

As discussed above at lowest order in $(aH/k)^2$, the metric perturbations $\Phi, \Psi \propto \delta_m(aH/k)^2$, thus the largest term on the RHS of equation (3) is proportional to $\delta_m(\mathbf{k}, a)$. At this order we get a homogeneous equation for δ_m which has a solution of the form $\delta_m(\mathbf{k}, a) = \delta_{mi}(\mathbf{k})\delta_0(a)$. This is the usual approximation used when investigating the growth function on small scales. Perturbatively including the next order term on the RHS, we can see that our ansatz captures the general form of the solution to that order. Again, we use this argument as motivation for the form of the ansatz, being aware of the fact that nonrelativistic dark matter is not covariantly conserved in some models. In δ_m, we include both baryonic and nonbaryonic dark matter, with an understanding that baryonic matter contibutes a small fraction to the total. We assume that baryons are covariantly conserved and follow timelike geodesics, serving as test particles whose motion can be used to probe the metric.

We now turn to a discussion of the range of scales where we expect our procedure to be applicable. Our ansatz uses the ratio of the physical size of the perturbation $d_p(a)$ to the size of the Hubble horizon $d_H(a) \equiv 1/H(a)$ as our small (post-Newtonian) expansion parameter. In Fourier space $d_p(a) \sim a/k$ and we need $d_p(a)/d_H(a) \sim aH/k \ll 1$ for the expansion in aH/k to be meaningful. We first give a rough upper bound on H_0/k. From an observational standpoint, the largest scales of interest are the ones where cosmic variance does not significantly limit the precision of our measurements. A perturbation with a given k corresponds roughly to a multipole $l \sim kd(a)$ where $d(a)$ is the comoving distance. Taking $l \sim 30$ as the largest angular scale where cosmic variance does not significantly limit measurement precision, the corresponding comoving wavevector of the perturbation at $a \sim 0.5$ is $k \sim 10^{-2}\,h\,\mathrm{Mpc}^{-1}$ or equivalently $H_0/k \sim 3 \times 10^{-2}$. As seen in Figure 1, this yields $(aH/k)^2 \ll 1$ in the range $10^{-1} \lesssim a \lesssim 1$, safely consistent with the ansatz. This upper bound $H_0/k \lesssim 3 \times 10^{-2}$ can be relaxed depending on the range of redshift in which the observations are made.

Now, for the lower bound on H_0/k we get $H_0/k \gtrsim$

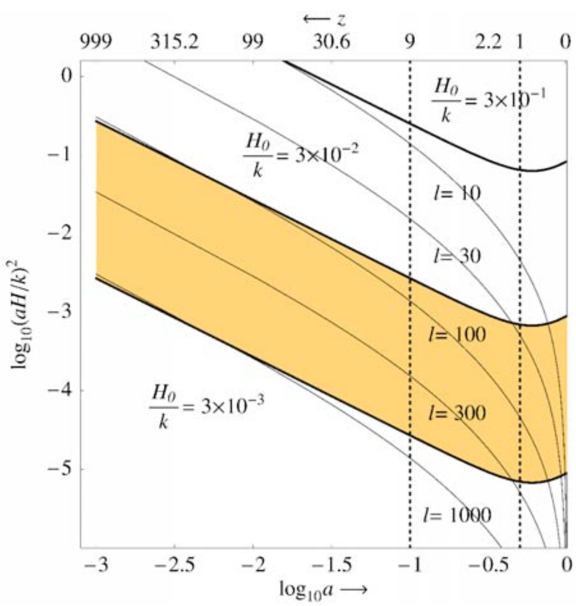

Figure 1. The ratio of the physical size of the perturbation to the size of the horizon is used as an expansion parameter in our anzatz. We plot the square of this ratio $(aH/k)^2$ as a function of a from last scattering to the present for the concordance model (yellow region). The upper and lower bounds of the yellow region are determined by considering scales that are small enough so that cosmic variance does not dominate the errors and at the same time large enough so that nonlinear evolution and baryon cooling are not a significant factor. Most of the observations in the next decade will yield information in the range $10^{-1} \lesssim a \lesssim 1$. If we are interested in observations that only care about a smaller range of the scale factor, then the allowed range of H_0/k increases. We also plot lines of constant multipole $l \sim kd(a)$, which provides a rough estimate of the relationship between k and angular scales at different redshifts.

3×10^{-3}. This corresponds to $k_{nl} \sim 10^{-1}\, h\,\mathrm{Mpc}^{-1}$ which is at the boundary between linear and nonlinear evolution of δ_m today. At these scales the linear and nonlinear matter power spectrum differ by a few percent today (and less in the past). Since the scalar metric fluctuations $\mathcal{O}[\Phi(\mathbf{x},a),\Psi(\mathbf{x},a)] \sim 10^{-5}$ on these scales, as indicated by measurements of the cosmic microwave background (CMB), we can linearize the field equations in Φ, Ψ and δ_m at these scales. Another reason for this lower bound is that on scales larger than these we do not expect a significant bias between the baryonic and nonbaryonic matter. We can relax the lower bound if the observations are restricted to smaller scale factors since the scale factor dependence of the boundary between linear and nonlinear evolution is given by $k_{nl}(a) \sim 10^{-1} a^{-3/2}\, h\,\mathrm{Mpc}^{-1}$.

Figure 1 shows the typical order of magnitude of $(aH/k)^2$ for the range $3 \times 10^{-3} \lesssim H_0/k \lesssim 3 \times 10^{-2}$ (filled yellow region). Finally, the range of scalefactors we have in mind for our framework is $10^{-1} \lesssim a \lesssim 1$. Gravitational dynamics at late times (large a) is particularly interesting due to cosmic acceleration. The next generation of observations including lensing, baryon acoustic oscillations, cluster counts, galaxy power spectra etc. will be made within this range.

Before we end this section we provide a concrete example of what the coefficient functions look like in a simple case, the Einstein-de Sitter universe:

$$\begin{aligned} \beta_0 &= \gamma_0 = 1, \\ \beta_1 &= \gamma_1 = -3, \\ \delta_0 &= a/a_i, \\ \delta_1 &= 3(a/a_i)(1 - a/a_i). \end{aligned} \tag{4}$$

We turn to the calculation of the coefficient functions in the next section.

3 APPLICATION OF THE FRAMEWORK WITH EXAMPLES

In this section we calculate the coefficient functions for GR with a cosmological constant and nonrelativistic matter, GR with quintessence, scalar-tensor theories, $f(R)$ theories and DGP gravity. In general, the nonrelativistic matter consists of baryons, massive neutrinos and nonbaryonic dark matter with (possibly) nongravitational interactions between them and other fields. For simplicity we will ignore massive neutrinos and baryons in this section. Local tests of gravity provide strong constraints on baryons and photons and their interactions. They do not yet provide similar constraints on the interactions of nonbaryonic matter. Hence, nonbaryonic matter need not be covariantly conserved. However in the examples considered, we treat dark matter as a perfect fluid that is covariantly conserved for simplicity. This allows us to use the conservation equation (3), which is sometimes easier to use than a gravitational field equation that would otherwise take its place.

The basic strategy is to substitute our ansatz into the field equations and conservation equations and solve for the coefficient functions. We begin by substituting our ansatz (2) into the conservation equation for nonrelativistic perfect fluid dark matter (3), collecting terms with like powers of $(aH/k)^2$ and setting their coefficient terms equal to zero to obtain

$$\begin{aligned} &\left[a^2\partial_a^2 + (2+q)a\partial_a\right]\delta_0 - \frac{4\pi G\rho_m}{H^2}\delta_0\beta_0 = 0, \\ &\left[a^2\partial_a^2 + (2+q)a\partial_a\right][(aH)^2\delta_1] - \frac{4\pi G\rho_m}{H^2}[(aH)^2\delta_1] \\ &= -\frac{12\pi G\rho_m}{H^2}(aH)^2\left[a^2\partial_a^2 + qa\partial_a - q - \frac{\beta_1}{3\gamma_0}\right][\gamma_0\delta_0], \end{aligned} \tag{5}$$

where q and H are assumed to be known from the background evolution. The above equations are second order differential equations for δ_0 and δ_1. The equation for δ_0 can be solved once β_0 is known. $G\beta_0$ is the effective gravitational *constant*. If $\beta_0 = 1$, the equation for δ_0 is the usual equation for the fractional matter overdensity on linear and small scales in GR with nonrelativistic matter as the only clustering component.

We digress a bit to note that for $\bar{\delta}_1 \equiv (aH)^2\delta_1$, the differential operator acting on $\bar{\delta}_1$ and δ_0 is $[a^2\partial_a^2 + (2+q)a\partial_a - 4\pi G\rho_m\beta_0/H^2]$. This feature continues if we were to go to higher order terms as well, hence it might be useful to find a Green's function for this operator. In general, to solve for δ_1 we need to know $\beta_0, \gamma_0, \delta_0$ and β_1 along with two initial conditions. To progress further we turn to specific theories of gravitation. Our aim is to show how to apply the formalism rather than discuss in detail the various models considered.

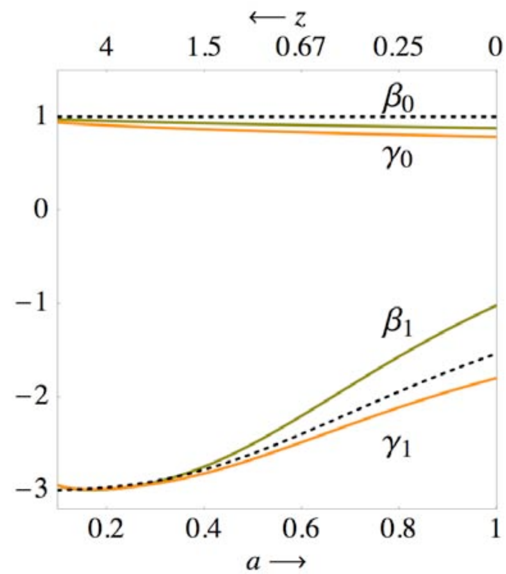

Figure 2. The dimensionless coefficient functions characterizing the relationship between the metric perturbations and matter distribution are show above for ΛCDM(dashed lines) and the scalar-tensor theory (STT) (solid lines). The STT model is chosen so that its expansion history is consistent with observations. In the case of ΛCDM $\beta_0 = \gamma_0 = 1$ and $\beta_1 = \gamma_1$. At early time (matter domination) $\beta_1 = \gamma_1 = -3$ with the cosmological constant causing a departure from this value at late times. The variation of β_0 with the scale factor in the STT can be interpreted as a variation of Newton's constant "$G\beta_0$" as far as growth of perturbations is concerned. Also note that for STT, $\beta_0 \neq \gamma_0$ and $\beta_1 \neq \gamma_1$. For STT, the difference in the coefficient functions is due to $\Phi - \Psi = -\alpha(\varphi)\delta\varphi \neq 0$. We remind the reader that in the ansatz (2) the coefficients β_1 and γ_1 are multiplied by $(aH/k)^2$, whose magnitude is shown in Figure 1, making them accessible at large scales only.

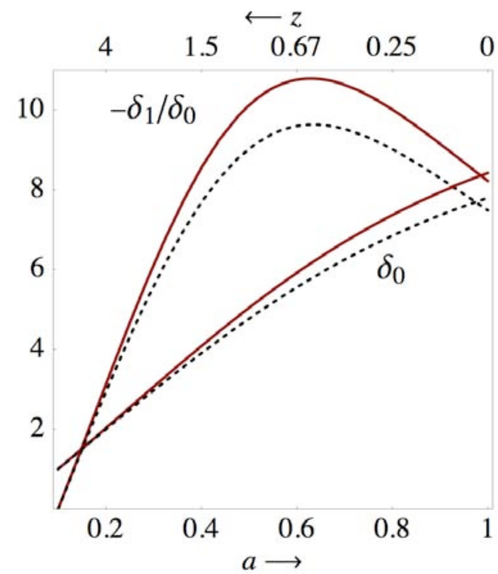

Figure 3. The dimensionless coefficient functions characterizing growth of structure are show above for ΛCDM(dashed lines) and the scalar-tensor theory (STT) (solid lines). The STT model is chosen so that its expansion history is consistent with observations. δ_0 is the usual growth function on small scales, whereas δ_1 characterizes the departures as we move to larger scales. We note that δ_1 is the coefficient of $(aH/k)^2$, which is small withing the scales of interest (see Figure 1). The initial conditions for δ_0 and δ_1 are chosen at $a_i \sim 10^{-1}$ and are consistent with growth of structure in a matter dominated era.

We leave out the detailed steps, which are straightforward but tedious.

3.1 General relativity with cold dark matter and the cosmological constant

We start with the usual Einstein Hilbert action:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R - 2\Lambda] + \int d^4x\sqrt{-g}\mathcal{L}_m, \tag{6}$$

with $\mathcal{L}_m$, the lagrangian density for perfect fluid cold dark matter The corresponding field equations are

$$G^\mu_\nu + \Lambda\delta^\mu_\nu = 8\pi G T^\mu_\nu, \tag{7}$$

where $G^\mu_\nu = R^\mu_\nu - \delta^\mu_\nu R/2$ and T^μ_ν is the energy-momentum tensor for a pressureless perfect fluid. As usual, we separate the field equations into the background and perturbed parts (first order in Φ, Ψ and δ_m). Upon substitution of our ansatz into the perturbed field equations we get the following expressions/equations for the coefficient functions.

$$\begin{aligned}
&\left[a^2\partial_a^2 + (2+q)a\partial_a\right]\delta_0 - \frac{4\pi G\rho_m}{H^2}\delta_0 = 0,\\
&\left[a^2\partial_a^2 + (2+q)a\partial_a\right][(aH)^2\delta_1] - \frac{4\pi G\rho_m}{H^2}[(aH)^2\delta_1]\\
=& -\frac{12\pi G\rho_m}{H^2}(aH)^2\left[a^2\partial_a^2 + (q+1)a\partial_a - q\right]\delta_0,\\
&\beta_0 = \gamma_0 = 1,\\
&\beta_1 = \gamma_1 = -3\frac{a\partial_a\delta_0}{\delta_0},
\end{aligned} \tag{8}$$

where we used the 00 and $i \neq j$ Einstein equations along with the coefficient form of the conservation equations (5). We need to to provide 4 constants of integration for the two second order differential equations. We take these to be

$$\begin{aligned}
\delta_0(a_i) &= 1, \qquad a_i\partial_a\delta_0(a_i) = 1,\\
\delta_1(a_i) &= 0, \qquad a_i\partial_a\delta_1(a_i) = -3.
\end{aligned} \tag{9}$$

This ensures that $\delta_m(\mathbf{k}, a_i) = \delta_{mi}(\mathbf{k})$, thus defining $\delta_{mi}(\mathbf{k})$ in our ansatz (2). The derivatives are chosen to agree with the case of pure matter domination at early times, where the explicit solution takes the form $\delta_0 = a/a_i$ and $\delta_1 = 3(a/a_i)(1 - a/a_i)$ after rejecting the decaying modes. We shall use these initial conditions for all the models we consider in this section.

The dashed lines in Figures 2 and 3 show these dimentionless coefficient functions for FΛCDM with $\Omega_m = 8\pi G\rho_{m0}/3H_0^2 = 0.3$. Since $\beta_0 = \gamma_0 = 1$, there are no corrections to the Newtonian gravitational constant as far as growth of perturbations is concerned on small scales. The fact that $\beta_1 = \gamma_1 \neq 0$ reflects corrections because of GR to the relationship between matter and metric perturbations, whereas $\beta_1 = \gamma_1 \neq -3$ reflects the effect of the cosmological constant. δ_0 characterizes the growth of structure on small scales. It deviates from $\delta_0 = a/a_i$ because of Λ. δ_1 reflects the corrections to the growth function as we move to larger scales. Note that β_1 and γ_1 and δ_1 are multiplied by $(aH/k)^2$, whose magnitude is shown in Figure 1. The terms $\beta_1(aH/k)^2$, $\gamma_1(aH/k)^2$ and $\delta_1(aH/k)^2$ are much smaller than β_0, γ_0 and δ_0, making it difficult to observe their effects unless we investigate large scales.

3.2 Scalar-tensor theory with cold dark matter (matter representation)

Scalar-tensor theories are popular alternatives to GR. In the matter representation (also called the Jordan frame), the action contains two free functions $f(\varphi)$ and $V(\varphi)$

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[f(\varphi)R + \mathcal{L}_\varphi] + \int d^4x\sqrt{-g}\mathcal{L}_m. \tag{10}$$

Note that we have decided to make φ dimensionless since we wish to treat the perturbation in this field $\delta\varphi$ on the same footing as the metric perturbations Φ and Ψ. Also $\mathcal{L}_\varphi = -(\partial\varphi)^2/2 - V(\varphi)$ and $\mathcal{L}_m$ does not contain φ. The field equations for this theory are

$$\begin{aligned} &G^\mu_\nu + \frac{1}{f}\left[\delta^\mu_\nu\Box - \nabla^\mu\nabla_\nu\right] f \\ =&\frac{8\pi G}{f}T^\mu_\nu + \frac{1}{2f}\left[\partial^\mu\varphi\partial_\nu\varphi - \delta^\mu_\nu\left(\frac{1}{2}\partial^\sigma\varphi\partial_\sigma\varphi + V\right)\right]. \end{aligned} \tag{11}$$

The field equation for φ is

$$\Box\varphi - V_\varphi + f_\varphi R = 0, \tag{12}$$

where $f_\varphi = \partial_\varphi f$ and $V_\varphi = \partial_\varphi V$. These field equations at the background level can be found in the literature [for example see (Boisseau et al. 2000)]. Using our ansatz in the perturbed gravitational field equations and the field equations for φ at first order in Φ, Ψ, δ_m and $\delta\varphi$, collecting terms with like powers of $(aH/k)^2$, and setting the expression in front of each power of $(aH/k)^2$ equal to zero, we get the following expressions/equations for the coefficient functions:

$$\begin{aligned} \beta_0 &= \frac{1}{f}\left(\frac{1+4f\alpha^2}{1+3f\alpha^2}\right) \approx \frac{1}{f} + \mathcal{O}[\alpha^2], \\ \gamma_0 &= \frac{1}{f}\left(\frac{1+2f\alpha^2}{1+3f\alpha^2}\right) \approx \frac{1}{f} + \mathcal{O}[\alpha^2], \\ \beta_1 &= -\frac{3}{f}\frac{a\partial_a\delta_0}{\delta_0} + \frac{1}{4f^2}(a\partial_a\varphi)^2 \\ &\quad + \left[-3(a\partial_a\varphi)\frac{a\partial_a\delta_0}{\delta_0} + \frac{1}{2}(a\partial_a\varphi)^2\frac{\alpha_\varphi}{\alpha} + 3(a\partial_a\varphi) + \frac{3V_\varphi}{2H^2}\right]\frac{\alpha}{f} \\ &\quad + \mathcal{O}[\alpha^2], \\ \gamma_1 &= -\frac{3}{f}\frac{a\partial_a\delta_0}{\delta_0} + \frac{1}{4f^2}(a\partial_a\varphi)^2 \\ &\quad + \left[(a\partial_a\varphi)\frac{a\partial_a\delta_0}{\delta_0} + \frac{1}{2}(a\partial_a\varphi)^2\frac{\alpha_\varphi}{\alpha} - (a\partial_a\varphi) - \frac{V_\varphi}{2H^2})\right]\frac{\alpha}{f} \\ &\quad + \mathcal{O}[\alpha^2], \end{aligned} \tag{13}$$

where $\alpha = f_\varphi/f$ is the coupling function and all the functions depend on the scale factor a . We have calculated the full expressions for β_1 and γ_1, which are rather long. The first two terms are listed as a power series in the coupling function $\alpha \ll 1$ with $\alpha \sim \alpha_\varphi, \alpha_{\varphi\varphi}...$. We used the $i \neq j$ equation, $\alpha\delta\varphi = \Psi - \Phi$, to eliminate $\delta\varphi$ from the field equations. The 00 equation and the field equation for $\delta\varphi$ yield β_n and γ_n with $(n = 0, 1)$. The equations for δ_0 and δ_1 are given by equations (5) with $\beta_0, \beta_1, \gamma_0$ and γ_1 given above. Again using the initial conditions (9), we can solve for all the coefficient functions once $f(\varphi)$ and $V(\varphi)$ have been provided. Note that the difference $\Phi - \Psi$ depends on $\beta_n - \gamma_n$ $(n = 0, 1)$. This is usually small for $\alpha \ll 1$ since $\beta_0 - \gamma_0 \sim \alpha^2$ and $\beta_1 - \gamma_1 \sim \alpha$.

We plot the coefficient functions in Figures 2 and 3. We have chosen $f(\varphi) = 1 + c_1\varphi^2$ and $V(\varphi) = 2\Lambda(1 + c_2\varphi^2)$ with $c_1 = c_2 = 0.1$. The initial conditions were chosen to ensure that the expansion history remains consistent with observations (consistent with ΛCDM to within a few percent). The difference between β_n and γ_n $(n = 0, 1)$ is due to nonminimal coupling $(\alpha \neq 0)$. We stress that we have not included baryons in this illustrative calculation. Including baryons would lead to very strong constraints on the function $f(\varphi)$ today from solar system tests (Schimd et al. 2005). For an example of a STT that includes dark matter and baryons with different couplings to gravity see (Bean 2001).

3.3 General relativity with cold dark matter and quintessence

GR with quintessence is a special case of the scalar-tensor theories discussed above, with $f(\varphi) = 1$. The action and corresponding field equations are

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R + \mathcal{L}_\varphi] + \int d^4x\sqrt{-g}\mathcal{L}_m \tag{14}$$

$$G^\mu_\nu = 8\pi G T^\mu_\nu + \frac{1}{2}\left[\partial^\mu\varphi\partial_\nu\varphi - \delta^\mu_\nu\left(\frac{1}{2}\partial^\sigma\varphi\partial_\sigma\varphi + V\right)\right] \tag{15}$$

$$\Box\varphi - V_\varphi = 0 \tag{16}$$

The coefficient functions are given by

$$\begin{aligned} \beta_0 &= \gamma_0 = 1 \\ \beta_1 &= \gamma_1 = -3\frac{a\partial_a\delta_0}{\delta_0} + \frac{1}{4}(a\partial_a\varphi)^2 \end{aligned} \tag{17}$$

where $(a\partial_a\varphi)^2/4 = 1 - q - 4\pi G\rho_m/H^2$. The $i \neq j$ Einstein equation yields $\beta_n = \gamma_n$ $(n = 0, 1)$. We used the $0i$ equation to eliminate $\delta\varphi$ from the field equations. As before δ_0 and δ_1 are provided by equation (5).

We pause to comment on a difference between minimally and nonminimally coupled scalar-tensor theories. Consider the field equation (12) for $\delta\varphi$:

$$\begin{aligned} &\left[a^2\partial_a^2 + (3+q)a\partial_a\right]\delta\varphi + \left[\left(\frac{k}{aH}\right)^2 + \frac{V_{\varphi\varphi}}{H^2} - 6(1+q)f_{\varphi\varphi}\right]\delta\varphi \\ &= (a\partial_a\varphi - 6f\alpha)a\partial_a\Phi + 3(a\partial_a\varphi - 2(4+q)f\alpha)a\partial_a\Psi \\ &\quad - 2\left(6f\alpha(1+q) + \frac{V_\varphi}{H^2}\right)\Phi + 2f\alpha\left(\frac{k}{aH}\right)^2(\Phi - 2\Psi) \end{aligned} \tag{18}$$

In the minimally coupled case we set $f(\varphi)=1, \alpha(\varphi)=0$ to get

$$\begin{aligned}&\left[a^2\partial_a^2+(3+q)a\partial_a\right]\delta\varphi+\left[\left(\frac{k}{aH}\right)^2+\frac{V_{\varphi\varphi}}{H^2}\right]\delta\varphi\\ =&(a\partial_a\varphi)a\partial_a\Phi+3(a\partial_a\varphi)a\partial_a\Psi-2\frac{V_\varphi}{H^2}\Phi.\end{aligned} \tag{19}$$

From the above equations we can see that in the nonminimally coupled case, for $k/aH \gg 1$ we have $\delta\varphi \propto \alpha(\varphi)\Phi$ whereas in the minimally coupled case $\delta\varphi \propto \Phi(aH/k)^2$. Along with $\Phi \propto \delta_m(aH/k)^2$, at large k the additional field $\delta\varphi$ follows the same aH/k expansion as the potentials with $\delta_{mi}(\mathbf{k})$ multiplying the expansion. This is one of the arguments we had used in Section 2 to justify the form of our ansatz. We have assumed $\mathcal{O}[\Phi]=\mathcal{O}[\Psi]$ in this argument.

3.4 $f(R)$ gravity with cold dark matter

In recent years modifications of the Einstein-Hilbert action in the form of a function of the Ricci scalar has become a popular alternative to quintessence (see for example (Carroll et al. 2004; Nojiri & Odintsov 2007)). The action and field equations are

$$\begin{aligned}&S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R+f(R)\right]+\int d^4x\sqrt{-g}\mathcal{L}_m\\ &(1+f_R)G^\mu_\nu-\delta^\mu_\nu\frac{f}{2}+[\delta^\mu_\nu\Box-\nabla^\mu\nabla_\nu]f_R=8\pi GT^\mu_\nu\,.\end{aligned} \tag{20}$$

In the above expressions $f_R=\partial_R f(R)$. The coefficient functions are

$$\begin{aligned}\beta_0=&\frac{4}{3(1+f_R)},\\ \gamma_0=&\frac{2}{3(1+f_R)},\\ \beta_1=&\frac{1}{(1+f_R)}\left[\frac{2}{3}\frac{a^2\partial_a^2\delta_0}{\delta_0}\right.\\ &-\frac{2}{3}\left\{(24B(j+q-2)+2-q\right\}\frac{a\partial_a\delta_0}{\delta_0}\\ &+4B\left\{10-4q+q^2+2j(q-4)-s\right\}\\ &\left.+72B^2(j+q-2)^2-4(j+q-2)a\partial_aB+2q-\frac{1}{9B}\right],\\ \gamma_1=&\frac{1}{(1+f_R)}\left[-\frac{2}{3}\frac{a^2\partial_a^2\delta_0}{\delta_0}\right.\\ &+\frac{2}{3}\left\{(6B(j+q-2)-7-q\right\}\frac{a\partial_a\delta_0}{\delta_0}\\ &-4B\left\{4-q+q^2+j(2q-5)-s\right\}\\ &\left.+4(j+q-2)a\partial_aB-2q+\frac{1}{9B}\right],\end{aligned} \tag{21}$$

where $j=dq/d\ln a-(1-2q)q$ and $s=dj/d\ln a-(2-3q)j$ are the scalefactor dependent functions, jerk and snap respectively, and $B=H^2f_{RR}/(1+f_R)$[1]. To obtain δ_0 and δ_1 we use equation (5). Note that we have assumed $B\neq 0$ in deriving the above expressions, hence it is not appropriate to take the limit $B\to 0$ after deriving the coefficient functions. Moreover, to lowest order in (aH/k^2), we get $\Phi=2\Psi$ if $B\neq 0$. However, when B is very small, then $\beta_1(aH/k)^2$ and $\gamma_1(aH/k)^2$ might become comparable to β_0 and γ_0, undermining the applicability of our ansatz. More details on the dynamics of $f(R)$ theories in the context of structure formation, solar system tests, etc. can be found in (Faulkner et al. 2006; Song et al. 2007; Bean et al. 2007; Chiba et al. 2007).

3.5 Brane world models: DGP Gravity

As a final example, we provide the expressions and equations governing the coefficient functions for DGP gravity. The action and field equations are [see for example (Koyama & Maartens 2006)]

$$\begin{aligned}S=&\frac{1}{32\pi Gr_c}\int d^5x\sqrt{-g_{(5)}}R_{(5)}\\ &+\frac{1}{16\pi G}\int d^4x\sqrt{-g}R+\int d^4x\sqrt{-g}\mathcal{L}_m\\ G^\mu_\nu=&(16\pi Gr_c)^2\Pi^\mu_\nu-E^\mu_\nu\,,\end{aligned} \tag{22}$$

where

$$\begin{aligned}\tilde{T}^\mu_\nu=&T^\mu_\nu-(8\pi G)^{-1}G^\mu_\nu\,,\\ \Pi^\mu_\nu=&-\frac{1}{4}\tilde{T}^\mu_\alpha\tilde{T}^\alpha_\nu+\frac{1}{12}\tilde{T}^\alpha_\alpha\tilde{T}^\mu_\nu+\frac{1}{24}\left[3\tilde{T}_{\alpha\beta}\tilde{T}^{\alpha\beta}-(\tilde{T}^\alpha_\alpha)^2\right]\delta^\mu_\nu\,,\end{aligned} \tag{23}$$

and E^μ_ν is the trace-free projection of the 5D Weyl tensor. Substituting our ansatz into the field equations and the $4D$ matter conservation equation we get

$$\begin{aligned}\beta_0=&\left[\frac{4-2Hr_c(2+q)}{3-2Hr_c(2+q)}\right]\\ \gamma_0=&\left[\frac{2-2Hr_c(2+q)}{3-2Hr_c(2+q)}\right],\end{aligned} \tag{24}$$

where we have used the quasi-stationary sub horizon approximation (Koyama & Maartens 2006). The calculation of β_1, γ_1 and δ_1 requires solving the perturbation equations on the brane closer to the size of the horizon. Evolution of perturbations on these large scales under a scaling ansatz has been investigated in (Sawicki et al. 2007). We suspect that an odd power of aH/k might arise due to the boundary conditions on our four dimensional brane. We leave the investigation of this issue for a later work.

In this section we have calculated the coefficient functions for a few examples. Our aim was to give a flavor of the calculations rather than be exhaustive in the investigation of the models considered. It would be interesting to investigate these models in more detail in the context of these coefficient functions to see if there is come generic behavior across a large class of models. Based on the examples considered it might be tempting to conclude that $\beta_n-\gamma_n\neq 0$ indicates physics beyond general relativity. However this is not so. For example a hypothetical dark energy component could also yield significant anisotropic stress. In the early universe, a more mundane source of anisotropic stress was provided by neutrinos. Nevertheless this difference could serve as an indicator of new physics in the matter or gravity sector. We have left out many possibilities including Bekenstein's TeVeS (Bekenstein 2004), models with non-canonical kinetic

[1] Our $B=H^2f_{RR}/(1+f_R)$ differs from the definition of B in (Song et al. 2007) by a factor of $(q-1)/6(j+q-2)$.

terms (Chiba et al. 2000), models of imperfect fluid dark energy with anisotropic stress (Koivisto & Mota 2006), and many others [see (Copeland et al. 2006) for a review]. We now turn our attention to observables and their relationship to the coefficient functions.

4 OBSERVATIONAL IMPLICATIONS

We have outlined a procedure that allows many alternative, dynamical theories to GR with FΛCDM cosmology to be explored within a common framework. Our approach has been devised with future observations in mind as its usefulness is limited to the observations that we expect will be the most prescriptive. To reiterate, the large k expansion connects the inhomogeneous nonrelativstic matter distribution to the perturbed metric in a universe of known (unperturbed) kinematical behavior, *i.e.* with a given relation $H(a)$ [or, equivalently, $a(t)$]. This presumes that these theories define a connection from last scattering to the post-reionization era, $a \gtrsim 10^{-1}$, and an understanding of how the distribution of observable entities such as galaxies relate to that of total mass. This allows us to focus on the manner in which structure can be observed to grow in the linear regime well within the horizon, which avoids the limitations imposed by cosmic variance considerations and the complications associated with gas dynamics. We further suppose that gravitational motion of baryonic matter and photons follows timelike and null geodesics respectively in this spacetime.

From an observational standpoint, our focus is on comoving length scales from ~ 40 Mpc to ~ 400 Mpc or equivalently $300 \gtrsim l \gtrsim 30$ at $z \sim 1$, where we expect the effects to stand out the best. There are three types of observations that are likely to be relevant. Firstly, there are direct measurements of the two point correlation function and its evolution. Counting galaxies (or clusters) in three dimensions will lead to measurements of the evolution of the density function δ_m (or equivalently $\delta_0(a)$ and $\delta_1(a)$) using future survey instruments such as LSST (Tyson 2002; Zhan et al. 2006) limited solely by cosmic variance as the photometric redshift accuracy and biasing errrors will be ignorable on these scales.

The second type of observation that will be carried out involves departures from the Hubble flow. These are dominated by the potential function Φ. Under our assumptions, galaxies will follow timelike geodesics and satisfy the linear conservation equations relating their peculiar velocities to Φ.

Finally there are weak lensing observations which depend upon the sum, $\Phi + \Psi$, presuming photons follow null geodesics. These then allow us to track the evolution of Ψ. A combination of these measurements would not only allow us to understand the scale dependent evolution of Φ, Ψ and δ_m but also allow us to probe the relationship between them. For example, using our ansatz, one can obtain constraints on the coefficient functions by comparing the correlation functions for the potentials, $P_{\Phi+\Psi}(k, a)$ (provided by lensing tomography) and the nonrelativistic matter overdensity $P_{\delta_m}(k, a)$ (provided by growth of structure measurements) using

$$k^4 P_{\Phi+\Psi} \propto P_{\delta_m} (\beta_0 + \gamma_0)^2 \left[1 + 2 \left(\frac{\beta_1 + \gamma_1}{\beta_0 + \gamma_0} \right) \left(\frac{aH}{k} \right)^2 + ... \right]$$

Not surprisingly, the coefficient of the second term is harder to constrain on small scales.

We have limited ourselves to the linear regime. On small scales, the nonlinear matter power spectrum and its evolution can play a role in the observations discussed above. The linear to nonlinear mapping discussed in (Smith et al. 2003) can be used for this purpose. However, without understanding the theories under consideration in the nonlinear regime, this is not fully robust.

Recall that $\{\beta_n, \gamma_n, \delta_n\}$ with $(n = 0, 1)$ are functions of the scale factor, a. If the observations are to be done in a limited range of redshifts then Taylor expanding the coefficient functions around the central value of the redshift might be a simple and model independent way of characterizing these coefficient functions in terms of a few parameters. From a theoretical perspective, the coefficient functions will depend on relevant parameters in the theory or model under consideration. A detailed investigation of the parameterization of the coefficient functions and the possible constraints that can be obtained from current and future observations is beyond the scope of this paper.

5 DISCUSSION

We have outlined a procedure that can be used to test the application of general relativity (more specifically FΛCDM) on cosmological scales in the context where it is most likely to fail and in the regime where observations should be most sensitive to measuring a departure from the general relativistic prediction. The scales are large enough to avoid the complications from nonlinearities and gas physics, yet small enough to avoid strong limitations to the interpretation of observations posed by cosmic variance.

Our procedure assumes that (i) The geometry and kinematics of the universe is understood (ii) baryons and photons behave as ideal test particles following geodesics of the cosmological metric. Given these assumptions, at late times, it is the relationship between the cosmological metric and the nonrelativistic matter distribution (along with their respective evolution) that provides a test for alternatives GR with a cosmological constant and cold dark matter. To probe the dynamics of gravity (or any additional fields) we provided an ansatz, equation(2), which gave a relationship between the cosmological metric and nonrelativistic matter perturbations in the linear, subhorizon regime. This form of the ansatz is consistent with a large class of theories with the differences between different theories evident in the coefficient functions $\{\beta_n(a), \gamma_n(a), \delta_n(a)\}$ with $n = 0, 1$. It is hoped that three scalar functions, the nonrelativistic matter overdensity δ_m and the metric potentials Φ and Ψ can be measured over the next decade, providing constraints on the coefficient functions. Constraining these coefficient functions provides observers with concrete targets for testing gravity in a scale dependent manner.

Our goal was to provide a perturbative framework, similar in spirit to the PPN formalism for testing gravity on solar system scales. However unlike the PPN case, we were left with coefficient functions that depend on the scale factor rather than constant coefficients. Although we have not done so in this paper, if the observations are limited to a small range of scale factors, it is possible to characterize these coefficient functions using a few parameters by expanding around a given scale factor at which the observations are centered.

With our choice of scales, we have restricted ourselves to linear, subhorizon evolution. We leave the connection between superhorizon and subhorizon evolution as well as consideration of nonlinearities for the future[1]. Although, we have restricted ourselves to scalar perturbations, the framework could be extended to include vector and tensor perturbations.

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[1] During the final stages of preparation of this paper, we became aware of recent work (Hu & Sawicki 2007), which deals with superhorizon evolution and a weakly nonlinear regime.