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EMITTANCE GROWTH FROM MULTIPLE COULOMB SCATTERING IN A PLASMA WAKEFIELD ACCELERATOR*

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Abstract

Emittance growth is an important issue for plasma wakefield accelerators (PWFAs). Multiple Coulomb scattering (MCS) is one factor that contributes to this growth. Here, the MCS emittance growth of an electron beam traveling through a PWFA in the blow out regime is calculated. The calculation uses well established formulas for angular scatter in a neutral vapor and then extends the range of Coulomb interaction to include the effects of traveling through an ion column. Emittance growth is negligible for low Z materials; however, becomes important for high Z materials.

INTRODUCTION

The first section of this paper reviews results for multiple scattering in a neutral vapor. These are extended in the second section to include the contribution from the ion column. The next two sections deal with the effect on beam propagation and emittance growth, respectively.

SCATTERING THROUGH NEUTRAL VAPOR

The emittance growth of a beam through a neutral vapor can be found from the angular scatter through the vapor, which is related to the radiation length. Equation 1 shows how to determine the radiation length, L_r [1].

$$L_r = \frac{N_A \cdot 716.4 \cdot \mathrm{cm}^{-2}}{n \cdot Z \cdot (Z+1) \cdot \ln(287/\sqrt{Z})} \tag{1}$$

where Z is the atomic number of traversed material, N_A is Avogadro's number in units of mole^{-1}, and n is the density of the neutral vapor.

The rate of angular scatter of an ultra-relativistic electron through a neutral vapor can be found from the following equation [2]:

$$\frac{d<\theta^2>_{vapor}}{dz} = \frac{1}{2 \cdot \gamma^2 \cdot L_r}\left(\frac{20\mathrm{MeV}}{m_e \cdot c^2}\right)^2 \tag{2}$$

where γ is the Lorentz factor, θ is angle in the x plane, c is the speed of light, m_e is the electron mass, and z is distance along the accelerator.

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SCATTERING THROUGH AN ION COLUMN

In recent PWFA experiments the electric field from the drive beam was strong enough to completely expel electrons from its volume, which created an ion column in the plasma [3]. The radius of the ion column is called the blow out radius, R_b. Scattering through an ion column extends the range of the Coulomb interaction. In a neutral vapor, if the incident particle doesn't come within the atomic dimensions of an atom, then the nucleus of the atom is shielded by its electrons. In order to account for the fact that there are charged ions, another Coulomb scattering term is added with a range from the atomic radius, R_a, to the blow out radius.

The radial kick in momentum that an incident electron receives from a charged ion is found by time integrating over the radial electric field. This kick is then projected onto the x plane and is turned into an angle after dividing by the electron's momentum.

$$\theta = \frac{\Delta P_x}{P} = \frac{e^2 \cdot Q \cdot \cos\phi}{2\pi \cdot P \cdot v \cdot \varepsilon_0 \cdot b} \tag{3}$$

where e is the charge of a proton, P is the electron's momentum, b is the ion's impact parameter, v is the electron velocity, $Q \cdot e$ is the ion charge, ε_0 is the permittivity of free space, and φ is the azimuthal angle of the ion. Next, equation 3 is turned into a squared angle expectation value by integrating over an ion that is randomly placed with $R_a < b < R_b$. Since the incident particle is ultra-relativistic, $P \cdot v = \gamma \cdot m_e \cdot c^2$.

$$<\theta^2> = \left(\frac{Q \cdot e^2}{2 \cdot \pi \cdot \gamma \cdot m_e \cdot c^2 \cdot \varepsilon_0}\right)^2 \cdot \frac{\ln(R_b/R_a)}{(R_b{}^2 - R_a{}^2)} \tag{4}$$

The angular scatters from individual ions add in quadrature. The reason can be seen by looking at the expectation value for the addition of two angles.

$$<\theta_f^2> = <(\theta_1 + \theta_2)^2> = <\theta_1{}^2> + <\theta_2{}^2> \tag{5}$$

Angular scatter from individual ions are just as likely to be positive as negative, so the $<\theta_1\theta_2>$ term is zero. The total rate of change in mean square scatter can then be found by multiplying by the number of ions that the incident particle will intercept per unit length by the mean square scatter from one.

$$\frac{d<\theta^2>_{ion}}{dz} = <\theta^2>\cdot n\cdot\pi\cdot(R_b{}^2 - R_a{}^2) \quad (6)$$

$$\frac{d<\theta^2>_{ion}}{dz} = \frac{n\cdot Q^2\cdot e^4}{4\pi\cdot\gamma^2\cdot m_e^2\cdot c^4\cdot\varepsilon_0^2}\cdot\ln({}^{R_b}\!/_{R_a}) \quad (7)$$

The total rate of angular scatter is then the addition of the vapor and ion scattering terms. Equations 9 and 10 are the result of some algebra.

$$\frac{d<\theta_T^2>}{dz} = \frac{d<\theta^2>_{ion}}{dz} + \frac{d<\theta^2>_{vapor}}{dz} \quad (8)$$

$$\frac{d<\theta_T^2>}{dz} = \frac{k_p^2\cdot r_c}{\gamma^2}\cdot S \quad (9)$$

$$S = Q\cdot\left(\ln\left(\frac{R_b}{R_a}\right) + \frac{1.78\cdot Z\cdot(Z+1)}{Q^2}\cdot\ln\left(\frac{287}{\sqrt{Z}}\right)\right) \quad (10)$$

where θ_T is the total angle, $k_p = n_p\cdot e^2/(m_e\cdot\varepsilon_0\cdot c^2)$, n_p is the plasma density ($n_p = Q\cdot n$), and r_c is the classical electron radius. For Z=1, the terms from the ion column and neutral vapor are roughly the same size; however, for higher Z the term from the neutral vapor dominates.

BEAM PROPAGATION

Beam size in the plasma relates the rate of angular scatter to the emittance growth. The beam size can be found by understanding the focusing forces. The following differential equation is appropriate for describing an ultra relativistic electron oscillating through the beam axis of the bare ion column [4].

$$\frac{dP_x}{dt} = \frac{-n_p\cdot e^2\cdot x}{2\cdot m_e\cdot\varepsilon_0} \quad (11)$$

where x is the electron position coordinate in the x plane. For an ultra relativistic electron with constant energy, equation 11 can be turned into 12.

$$\ddot{x} = \frac{-k_p^2}{2\cdot\gamma}\cdot x \quad (12)$$

where the dots represent derivatives in z. There are conditions in which the beam size doesn't change along the accelerator. When these conditions are satisfied the beam is matched to the plasma. The conditions for matching can be found by taking derivatives of the beam size.

$$\frac{d<x^2>}{dz} = 2<x\dot{x}> \quad (13)$$

$$\frac{d^2<x^2>}{dz^2} = 2\left(<\dot{x}^2> + <x\ddot{x}>\right) \quad (14)$$

The angle brackets refer to expectation values over the entire beam. The second derivative of x can be replaced from equation 12 into 14. By setting the first two derivates equal to zero, it insures that all higher order derivatives are also zero. This makes $<x^2>$ a constant, with the following criterion for a matched beam:

$$<x\cdot\dot{x}> = 0,\quad <\dot{x}^2> = \frac{k_p^2}{2\cdot\gamma}<x^2> \quad (15)$$

The following is the definition for the geometric emittance:

$$\varepsilon^2 = <x^2><\dot{x}^2> - <x\dot{x}>^2 \quad (16)$$

Plugging in from equation 15 into 16 gives the relationship between the emittance and beam size.

$$\varepsilon = \frac{k_p}{\sqrt{2\cdot\gamma}}<x^2> \quad (17)$$

It can be shown that an accelerating beam that starts matched retains the conditions expressed in equation 15 [5].

EMITTANCE GROWTH

The rate of emittance growth is found by taking a z derivative of equation 16.

$$\varepsilon\cdot\dot{\varepsilon} = <x^2><\dot{x}\ddot{x}> - <x\dot{x}><x\ddot{x}> \quad (18)$$

The next step is to substitute for $\ddot{x}$. In order to do this it is now important to include not only the energy change term but also rate of angular growth from scattering. By converting time derivatives to z derivatives and by adding the angular growth from the scatterers shown in equation 9, equation 11 can be turned into the following differential equation:

$$\ddot{x} = \frac{-\dot{\gamma}}{\gamma}\cdot\dot{x} + \frac{-k_p^2}{2\cdot\gamma}\cdot x + \dot{\theta}_T \quad (19)$$

Equation 19 is next substituted into equation 18.

$$\varepsilon\cdot\dot{\varepsilon} = \frac{-\dot{\gamma}}{\gamma}\varepsilon^2 + <x^2><\dot{x}\dot{\theta}_T> + <x\dot{x}><x\dot{\theta}_T> \quad (20)$$

As long as the relative angular growth in one betatron oscillation is small, then a beam that starts match will remain closely matched. This means we can drop the

$< x\dot{x} >$ term. The scatter term was calculated in the earlier part of this paper.

$$< \dot{x}\dot{\theta}_T >= \frac{1}{2} \cdot \frac{d < \theta_T^2 >}{dz} \qquad (21)$$

The relationship to the normalized emittance growth can be found by substituting equations 9 and 21 into 20, multiplying by γ, and dividing by ε.

$$\gamma \cdot \dot{\varepsilon} + \dot{\gamma} \cdot \varepsilon = \frac{k_p^2 \cdot r_c \cdot S}{2 \cdot \varepsilon \cdot \gamma} < x^2 >= \dot{\varepsilon}_N \qquad (22)$$

Now substituting back in for the relationship between the emittance and the size:

$$\dot{\varepsilon}_N = \frac{k_p \cdot r_c \cdot S}{\sqrt{2 \cdot \gamma}} \qquad (23)$$

where ε_N is the normalized emittance. The derivative in z can be turned into a derivative in γ [6].

$$\frac{d\varepsilon_N}{d\gamma} = \frac{k_p \cdot r_c \cdot S}{\dot{\gamma} \cdot \sqrt{2 \cdot \gamma}} \qquad (24)$$

This can then be integrated from the initial Lorentz factor, γ_i, to the final Lorentz factor, γ_f, which gives the following formula for the change in normalized emittance:

$$\Delta\varepsilon_N = \frac{\sqrt{2} \cdot k_p \cdot r_c \cdot S}{\dot{\gamma}} \cdot \left(\sqrt{\gamma_f} - \sqrt{\gamma_i}\right) \qquad (25)$$

Equation 25 can be simplified a step further by taking into account the scale of the acceleration. By assuming the beam is accelerated by an electric field of mc^2k_p/e then $d\gamma/dz = k_p$.

$$\Delta\varepsilon_N = \sqrt{2} \cdot r_c \cdot S \cdot \left(\sqrt{\gamma_f} - \sqrt{\gamma_i}\right) \qquad (26)$$

One current scheme for a PWFA is to use it at the end of a conventional linear collider to double the energy of a witness electron bunch [7]. As an example equation 26 was used to calculate the emittance growth from doubling the energy of an electron beam initially at 500 GeV through various materials that have been singly ionized (see Fig. 1). The blow out radius was set to $2.5 \cdot 10^{-5}$ m, and the atomic radius was set to 10^{-10} m. ILC projected emittances are $\varepsilon_{N,y} = 4 \cdot 10^{-8}$m, and $\varepsilon_{N,x} = 9.6 \cdot 10^{-6}$m [8]. At Z ~ 60 the y normalized emittance will double after energy doubling. The red dotted lines were put in for the normalized emittance growth of the following elements: Li = $2.0 \cdot 10^{-10}$ m, Na = $1.7 \cdot 10^{-9}$ m, K = $4.6 \cdot 10^{-9}$ m, Rb = $1.6 \cdot 10^{-8}$ m, and Cs = $3.3 \cdot 10^{-8}$ m.

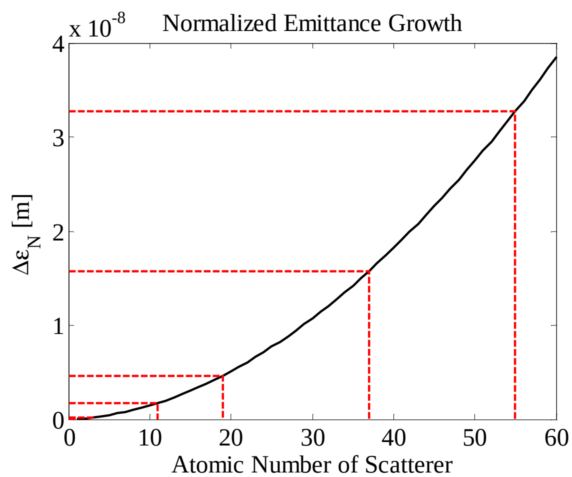

Figure 1: Normalized emittance growth from doubling the energy of an electron beam initially at 500 GeV through singly ionized materials with various atomic numbers.

CONCLUSION

A calculation is shown for the normalized emittance growth of a beam traversing a PWFA operated in the blow out regime. Emittance growth is negligible for low Z materials; however, becomes important for high Z materials.

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