A phase transition in commuting Gaussian multi-matrix models

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Abstract

We analyze in detail a second order phase transition that occurs in large N Gaussian multi-matrix models in which the matrices are constrained to be commuting. The phase transition occurs as the relative masses of the matrices are varied, assuming that there are at least four matrices in the lowest mass level. We also discuss the phase structure of weakly coupled large N 3+1 dimensional gauge theories compactified on an S^3 of radius R. We argue that these theories are well described at high temperatures ($T \gg 1/R$) by a Gaussian multi-matrix model, and that they do not exhibit any phase transitions between the deconfinement scale ($T \sim 1/R$) and the scale where perturbation theory breaks down ($T \sim 1/\lambda R$, where λ is the 't Hooft coupling).

1 Introduction

In this paper we analyze a phase transition that occurs in large N Gaussian matrix models for which the matrices commute with each other, as a function of the masses. Our original motivation for considering such matrix models came from the hope [1, 2, 3] that they would be relevant for describing the high-temperature regime of $3+1$ dimensional weakly coupled gauge theories on S^3. As we describe in section 2, it seems that this hope is not realized, since at high temperatures the commutator squared terms in the action are not large enough to constrain the matrices to commute with each other. However, such matrix models have been suggested to be useful for studying BPS states in supersymmetric gauge theories on S^3 at strong coupling [4], and we hope that other applications may be found for them as well.

We begin in section 2 by reviewing the phase structure of weakly coupled large N gauge theories on S^3, and arguing that these theories do not exhibit any phase transition in the range between the deconfinement temperature and the temperature where perturbation theory breaks down. In section 3 we analyze in detail a phase transition that occurs in commuting matrix models as the masses are varied. A special case of this transition was studied in [3]; here we generalize that analysis to general numbers of matrices, and describe analytically how the theory looks just below the transition. This allows us to prove that there is a second order phase transition, and to compute the jump in the second derivative of the action.

2 Thermal gauge theories on S^3

The thermal partition functions of gauge theories on S^3 can be computed by performing the Euclidean path integral of the theory on $S^3 \times S^1$, where the S^1 has circumference $\beta = 1/T$. We will denote the radius of the S^3 by R. This path integral can naturally be performed by expanding all the fields into Kaluza-Klein (KK) modes on $S^3 \times S^1$, and then integrating over these modes (which are just matrices). The derivative terms in the original action now become mass terms for the various matrices, with contributions proportional to T^2 coming from the Euclidean time derivatives, and contributions proportional to $1/R^2$ coming from the spatial derivatives.

Classically, such a gauge theory has a single massless mode – the zero mode α of A_0, both on the S^3 and on the S^1. All other modes are massive (assuming that any scalar fields either have a classical mass, or have a conformal coupling to the curvature that gives them a mass on S^3), with masses at least of order $1/R$ or of order T. It is then natural to integrate out all the other modes, and obtain an effective action for α, which due to the symmetry under large gauge transformations on the S^1 is really an action for the unitary matrix $U = e^{i\beta\alpha}$. This effective action turns out to be non-trivial already in the free theory, since the one-loop path integral with some background value for α is non-trivial. We assume for simplicity

that all fields in the gauge theory are in the adjoint representation; the generalization to other cases is straightforward. The effective action of such free gauge theories, as computed in [5, 6], takes the form, up to an overall additive constant,

$$S = -\sum_{n=1}^{\infty} \frac{1}{n} \left(z_B(n\beta) + (-1)^{n+1} z_F(n\beta) \right) \text{Tr}(U^n) \text{Tr}(U^{-n}), \tag{1}$$

where $z_B(\beta)$ $(z_F(\beta))$ is the generating function for the bosonic (fermionic) modes in this classical theory on S^3, given by the sum of $e^{-\beta E_i}$ over all bosonic (fermionic) states of energy E_i, counting each mode in the adjoint representation once. The phase structure of the matrix model (1) was analyzed in [5, 6]; the action S provides an attraction between the eigenvalues (at least at short distance), but as in any unitary matrix model there is a repulsion between the eigenvalues coming from the measure. We can write the action in terms of the eigenvalues $e^{i\theta_p}$ $(p = 1, \cdots, N)$ of U in the form:

$$S = \sum_{p \neq q} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - z_B(n\beta) - (-1)^{n+1} z_F(n\beta) \right) \cos(n(\theta_p - \theta_q)), \tag{2}$$

where the first term comes from the change of measure from the unitary matrix to its eigenvalues. From now on we will discuss only the large N limit of the gauge theory, in which we can assume that there is some smooth distribution of the eigenvalues. At low temperatures the repulsion wins and the eigenvalue distribution is uniform, corresponding to a confined phase in which the expectation values of the Polyakov-Susskind loops $\text{Tr}(U^n)$ vanish. At a temperature T_d of order $1/R$, given by the solution to $z_B(\beta) + z_F(\beta) = 1$, there is a weakly first order phase transition to a phase where the eigenvalue distribution is non-uniform and gapped, and this deconfined phase governs the high temperature behavior. Adding higher loop corrections turns this weakly first order transition either into a first order transition or into a second order transition followed by a third order transition, depending on a coefficient which requires a three-loop computation [6, 7].

For $T \gg 1/R$ the eigenvalue distribution becomes highly localized. In this limit the functions z_B and z_F go as $2 n_B (TR)^3$ and as $2 n_F (TR)^3$, respectively, where n_B (n_F) is the number of bosonic (fermionic) adjoint degrees of freedom in the theory: two from the vector field plus one for every additional scalar field. We can then expand the action (2) for small values of θ. The action includes a quadratic term, which is a one-loop mass term for the zero mode α; recalling that we are working in a normalization of the fields with a factor of N/λ in front of the classical action (where λ is the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$), this quadratic term corresponds to a physical mass proportional to $\lambda T^2 (n_B + \frac{1}{2} n_F)$. This is precisely the one-loop "electric mass term" that we expect to find for A_0 in the large volume limit (see [8] and references therein); in the large TR limit the diagrams giving this mass term in the theory on S^3 become identical to the same diagrams on $\mathbb{R}^3$ (they are not IR-divergent). The presence of this mass term means that even though classically α is always the lightest mode, in the theory with finite coupling this is no longer true when $\lambda T^2 \sim 1/R^2$, since then

2

the KK modes of other fields could start becoming lighter than α, and it no longer makes sense to integrate out the other fields and keep only α.

In the range $TR \gg 1$ it is easy to see that the higher order interaction terms in (2) are negligible, so the effective action of U is simply a Gaussian matrix model, and the eigenvalue distribution approaches a semi-circle. When $T \gtrsim 1/\sqrt{\lambda}R$, we need to add also additional fields in our effective action, since the lowest KK modes of the other massless bosonic fields have masses of the same order (they have classical masses of order $1/R$, and all of them, except for the spatial components of the gauge field A_i, also have one-loop mass terms of order $\sqrt{\lambda}T$). However, at such high temperatures, all fields are massive enough so that their eigenvalues are small, and all interaction terms can be ignored. Balancing the mass term against a logarithmic eigenvalue repulsion for each field of mass m, one sees that the effects of the classical commutator squared interaction terms scale as $\lambda T/m^4 R^3$ compared to the mass terms, so they can be consistently ignored whenever $\lambda T \ll m^4 R^3$; this is true for all our fields until we reach $T \sim 1/\lambda R$. At this scale perturbation theory breaks down since the interactions of the A_i fields (with $m \sim 1/R$) become large (note that up to this temperature, perturbation theory is valid if we add to it the effect of the "electric mass terms", as discussed in section 5 of [7]). The effects of additional interaction terms arising from quantum corrections are even smaller than the effect of these classical interactions, so they can also be ignored for all $T \gg 1/R$.

Thus, for all $1/\lambda R \gg T \gg 1/R$, gauge theories on S^3 are well-approximated by a Gaussian multi-matrix model for all the KK modes, with the masses given by a combination of the classical KK masses and the one-loop masses. The eigenvalue distribution of these matrices is (to a very good approximation) a semi-circle distribution, with a size proportional to $1/m$. There is no correlation between the different modes, so that different modes do not generically commute with each other[1]. When $T \ll 1/\sqrt{\lambda}R$, α is much lighter than the other modes, so its eigenvalues will be larger and dominate the dynamics. On the other hand, when $T \gg 1/\sqrt{\lambda}R$, the KK modes of A_i will be much lighter than those of all other fields (since they are the only ones which do not obtain an "electric mass term" at one-loop order), so they will dominate the dynamics.

This implies that such weakly coupled gauge theories do not have phase transitions in the range of temperatures between the deconfinement scale $T \sim 1/R$ and the scale $T \sim 1/\lambda R$ where perturbation theory breaks down. For any $T \ll 1/\sqrt{\lambda}R$ the theory is well-approximated by the single-matrix model (1), which has one or two phase transitions around $T \sim 1/R$ but no additional transitions at higher temperatures. For all $1/\lambda R \gg T \gg 1/R$ the theory is well-approximated by a Gaussian multi-matrix model, and such models do not have any phase transitions. This contradicts previous claims about phase transitions in this range of temperatures that were made in [1, 3]; these claims were based on the assumption that the matrices commute with each other. This assumption seems to be wrong (at least

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[1] We are very grateful to Shiraz Minwalla for pointing this fact out to us, and correcting the first version of this paper in which we wrongly assumed that the different modes commute.

for $T \gg 1/R$), and the dominant configuration, in which the matrices do not necessarily commute, does not exhibit any phase transitions in this range.

3 Commuting Gaussian multi-matrix models

In this section we analyze the behavior of Gaussian multi-matrix models of the form

$$S = N \sum_i m_i^2 \text{Tr}((\vec{\Phi}^i)^2), \tag{3}$$

where $\vec{\Phi}^i$ is a vector of n_i Hermitean matrices, and the matrices are coupled by the requirement that they commute with each other in the dominant configuration. Similar models have been discussed recently in [3, 4, 9]. We will not discuss here the justification for the assumption that the matrices commute; in general one might expect that if there are strong enough commutator interactions that would force the matrices to commute, they would also lead to strong interactions between the diagonal elements, but we will assume that imposing the constraint that the matrices commute does not lead to any other interactions in our model. Perhaps this can be justified in a model describing only BPS states [4, 9].

With the assumption that all the matrices commute, we can diagonalize all of them simultaneously with eigenvalues $\vec{\phi}_p^i$ ($p = 1, \cdots, N$). The action for the eigenvalues, including the term coming from the change in the measure from the matrices to their eigenvalues, then takes the form

$$S = N \sum_i m_i^2 \sum_p |\vec{\phi}_p^i|^2 - \tfrac{1}{2} \sum_{p \neq q} \ln \left(\sum_i |\vec{\phi}_p^i - \vec{\phi}_q^i|^2 \right). \tag{4}$$

We assume that the masses are ordered so that $m_1^2 < m_2^2 < \cdots$; if some masses are equal we can join the corresponding modes together into a single vector. We will discuss the phase structure of the theory as the ratios between the masses are varied. The equations of motion following from the action (4) are

$$m_i^2 \vec{\phi}_p^i = \frac{1}{N} \sum_{q \neq p} \frac{\vec{\phi}_p^i - \vec{\phi}_q^i}{\sum_j |\vec{\phi}_p^j - \vec{\phi}_q^j|^2}. \tag{5}$$

3.1 The spherical phase and its stability

In this model, unlike the general (unconstrained) Gaussian matrix model that was mentioned in the previous section, we do not have a separate repulsive interaction between the eigenvalues of each matrix, but just a single repulsive potential (4). Thus, the simplest assumption to make is that only the lightest mode $\vec{\phi}^1$ condenses, and all others do not. In the large N limit where we have a smooth distribution $\rho(\vec{\phi})$ for the eigenvalues of $\vec{\phi}^1$, we can write the equation of motion for such a solution as

$$m_1^2 \vec{\phi} = \int d^{n_1} \vec{\phi}' \rho(\vec{\phi}') \frac{\vec{\phi} - \vec{\phi}'}{|\vec{\phi} - \vec{\phi}'|^2}. \tag{6}$$

4

A simple argument shows that (for $n_1 > 2$) there are no solutions in which ρ is a smooth function on $\mathbb{R}^{n_1}$ [4, 3]. It is easy to see that (6) has a solution where all eigenvalues are distributed on an S^{n_1-1} sphere [10, 9, 3], of the form

$$\rho(\vec{\phi}) = \frac{\delta(|\vec{\phi}| - r_1)}{|\vec{\phi}|^{n_1-1}\text{Vol}(S^{n_1-1})},\tag{7}$$

where the sphere has radius

$$r_1 = \frac{1}{\sqrt{2}m_1}.\tag{8}$$

The fact that the radius does not depend on the dimension was noted in [10]. This solution is valid for all $n_1 > 1$; for $n_1 = 1$ the solution turns out to be a semi-circle, and for $n_1 = 2$ the lowest action saddle is in fact a disc rather than the circle described above, but in this note we will limit ourselves to theories with $n_1 \geq 4$. The action for this configuration takes the form

$$S = N^2 \left(\frac{1}{2} + \ln(m_1) - \frac{1}{2}f_{n_1} \right),\tag{9}$$

where f_n is a complicated expression involving hypergeometric functions, which is monotonically increasing with n (approaching zero for large n). Its values for small n are $f_8 = 37/60 - \ln(2) \simeq -0.0765$, $f_7 = -47/60 + \ln(2) \simeq -0.0902$, $f_6 = 7/12 - \ln(2) \simeq -0.1098$, $f_5 = -5/6 + \ln(2) \simeq -0.1402$, $f_4 = 1/2 - \ln(2) \simeq -0.1931$, $f_3 = -1 + \ln(2) \simeq -0.3069$, $f_2 = -\ln(2) \simeq -0.6931$.

We can also write down many other saddle points of (6), where only some of the components of $\vec{\phi}^1$ condense on a lower dimensional sphere of radius (8). These clearly have higher action than the saddle point above due to the monotonicity of f_n. It is also possible for different sets of components to condense on different spheres, so that the distribution is a product of expressions of the form (7). All possible examples for $n_1 = 6$ were considered in [3], and their action was always found to be larger than that of the S^{n_1-1} distribution. We believe that this is true for all values of $n_1 > 2$.

The stability of this saddle point with respect to fluctuations of the $\vec{\phi}^1$ eigenvalues was checked in [3], and it seems to be stable. There are of course zero modes corresponding to rotations of the S^{n_1-1}, but all other fluctuations seem to raise the action. Next, we can check for the stability of this saddle point with respect to turning on the eigenvalues of the second lightest mode, generalizing a similar computation in [3]. A fluctuation in $\vec{\phi}^2$ leads to the quadratic action

$$\delta S = N \sum_p m_2^2 |\vec{\phi}_p^2|^2 - \frac{1}{2} \sum_{p \neq q} \frac{|\vec{\phi}_q^2 - \vec{\phi}_p^2|^2}{|\vec{\phi}_q^1 - \vec{\phi}_p^1|^2}.\tag{10}$$

In the large N limit, most of these fluctuations start becoming tachyonic when

$$m_2^2 < \frac{n_1 - 2}{n_1 - 3}m_1^2.\tag{11}$$

This analysis requires $n_1 > 3$ for the convergence of the relevant integral.

Thus, assuming that $n_1 > 3$, we have the following picture. If all other masses are larger than $(n_1 - 2)m_1^2/(n_1 - 3)$, then the dominant saddle point is the one with only $\vec{\phi}^1$ condensed on a S^{n_1-1}. Then, as we change the masses (perhaps due to changes in external parameters such as the temperature), this saddle point becomes unstable whenever the mass of some mode goes below this value. The crucial dynamics here is that once one set of eigenvalues has condensed, it prevents the next lightest mode from condensing, over a range of masses.

3.2 The behavior just below the transition

In order to analyze the phase transition in detail we need to know the dominant eigenvalue distribution immediately below the transition. This follows from the action for the joint eigenvalue distribution of ϕ^1 and ϕ^2, which in the large N limit we denote as a continuous function $\rho(\phi^1, \phi^2)$. We will perform this analysis here only for $n_1 \geq 6$. The action is

$$\frac{1}{N^2}S = \int d^{n_1}\phi^1 d^{n_2}\phi^2 \rho(\phi^1, \phi^2) \left(m_1^2 |\vec{\phi}^1|^2 + m_2^2 |\vec{\phi}^2|^2 \right)$$
$$- \frac{1}{2} \int d^{n_1}\phi^1 d^{n_2}\phi^2 d^{n_1}\phi^{1\prime} d^{n_2}\phi^{2\prime} \rho(\phi^1, \phi^2) \rho(\phi^{1\prime}, \phi^{2\prime}) \ln \left(|\vec{\phi}^1 - \vec{\phi}^{1\prime}|^2 + |\vec{\phi}^2 - \vec{\phi}^{2\prime}|^2 \right). \quad (12)$$

We do not know how to minimize this action in general. However, immediately below the transition, we expect the eigenvalue distribution to be localized so that $|\vec{\phi}^2| \ll |\vec{\phi}^1|$. Analytic expressions for the eigenvalue distribution may then be obtained by expanding the action (12) in powers of ϕ^2, using

$$\ln \left(|\vec{\phi}^1 - \vec{\phi}^{1\prime}|^2 + |\vec{\phi}^2 - \vec{\phi}^{2\prime}|^2 \right) = \ln \left(|\vec{\phi}^1 - \vec{\phi}^{1\prime}|^2 \right) + \frac{|\vec{\phi}^2 - \vec{\phi}^{2\prime}|^2}{|\vec{\phi}^1 - \vec{\phi}^{1\prime}|^2} + \cdots . \quad (13)$$

There will be regions of the eigenvalue distribution where $|\vec{\phi}^1 - \vec{\phi}^{1\prime}|$ will be small and this expansion is not strictly valid there. However, our expansion is performed inside an integral weighted by $\rho(\phi^1, \phi^2)$, so that just below the phase transition these regions contribute negligibly to the integral (for $n_1 > 5$).

By imposing that the $SO(n_1) \times SO(n_2)$ symmetry of the action is not broken by the solution, the eigenvalue distribution may be written as

$$\rho(\phi^1, \phi^2) = \int dr_2 \rho(r_2) \frac{\delta(|\vec{\phi}^1| - r_1(r_2))\delta(|\vec{\phi}^2| - r_2)}{\sqrt{1 + (\partial r_1/\partial r_2)^2} |\vec{\phi}^1|^{n_1-1} |\vec{\phi}^2|^{n_2-1} \text{Vol}(S^{n_1-1}) \text{Vol}(S^{n_2-1})} . \quad (14)$$

There are two undetermined functions here, the eigenvalue density $\rho(r_2)$ (whose integral is normalized to one) and the radius of the ϕ^1 sphere $r_1(r_2)$, where r_2 is the radius of the ϕ^2 sphere. When we perform a small ϕ^2 expansion as in (13), we should also expand the undetermined function

$$r_1^2(r_2) = r_1^2(0) + r_1^{(1)}(0)r_2^2 + \cdots . \quad (15)$$

Our strategy is to perform this expansion in the action, and then extremize to solve for $r_1(0)$ and $r_1^{(1)}(0)$. However, $r_1(r_2)$ is not the only undetermined function in the ansatz (14).

We must also solve for the effective eigenvalue density $\rho(r_2)$. For the leading order behavior below the transition it is enough to expand the action to order r_2^4, and to this order (for $n_1 > 5$) the action only depends on the eigenvalue density through the moments

$$\langle r_2^2 \rangle \equiv \int dr_2 \rho(r_2) r_2^2 \,, \quad \langle r_2^4 \rangle \equiv \int dr_2 \rho(r_2) r_2^4 \,. \tag{16}$$

Naively, one may treat $\langle r_2^2 \rangle$ and $\langle r_2^4 \rangle$ as independent variables in the action. However, this leads to inconsistent equations of motion. The reason for this is that the Cauchy-Schwartz inequality requires that $\langle r_2^4 \rangle \geq \langle r_2^2 \rangle^2$, with equality only for the delta function distribution, so these variables are not independent of each other. The procedure we will use will be to define a new variable x by

$$\langle r_2^4 \rangle = x \langle r_2^2 \rangle^2 \,, \tag{17}$$

and to extremize the action with respect to $\langle r_2^2 \rangle$ and x subject to the constraint $x \geq 1$.

Simultaneously to the small r_2 expansion, we must expand about the transition point (11). We do this by introducing a small parameter, ε, defined by

$$m_2^2 = \frac{n_1 - 2}{n_1 - 3} m_1^2 - \varepsilon \,. \tag{18}$$

We then solve for $r_1(0), r_1^{(1)}(0), \langle r_2^2 \rangle$ and x in an expansion in small ε. Our objective is to compute the action of the solution just below the transition to the first non-trivial order, which is order ε^2.

Using the ansatz (14) for the eigenvalue distribution, one can perform the integrals over the spherical directions in the action. Then, we can solve the equations of motion of $r_1(0)$, $r_1^{(1)}(0)$ and $\langle r_2^2 \rangle$ in a power series in ε. We find, up to the order we need:

$$\langle r_2^2 \rangle = \frac{n_2 (n_1 - 3)^2 (n_1 - 4)(n_1 - 5)\, \varepsilon}{2 m_1^4 (n_1 - 2)((n_1 - 4)(n_1 - 3)^2 + n_2 (n_1^2 - 6 n_1 + 7 + (n_1 - 3)x))} + \mathcal{O}(\varepsilon^2) \,,$$

$$\frac{1}{r_1^2(0)} = 2 m_1^2 - \frac{4(n_1 - 2) m_1^4}{(n_1 - 4)(n_1 - 3)} \langle r_2^2 \rangle + \mathcal{O}(\varepsilon^2) \,,$$

$$r_1^{(1)}(0) = -\frac{n_1 - 2}{n_1 - 4} + \mathcal{O}(\varepsilon) \,. \tag{19}$$

It is interesting that the eccentricity of the ellipsoid defined by (15) does not tend to 1 at the phase transition (it does not become a pancake). As we approach the transition from below, the eigenvalues slide along the ellipsoid, which remains of constant shape, and accumulate at $r_2 = 0$. Note also that the relation between r_1 and r_2 does not depend (at leading order) on the number of modes which condense at the transition, n_2. This number only enters in the solution for $\langle r_2^2 \rangle$. The form of the ellipsoid (15) suggests that below the transition the eigenvalue distribution has topology $S^{n_1 + n_2 - 1}$. This is supported by the numerical results in [3].

If we now try to solve the equation of motion of x we see that it has no solution. Plugging the solution (19) into the action, we see that within the allowed range of $x \geq 1$ the action

is minimized at the boundary $x = 1$, which is why this equation cannot be satisfied. Thus, the minimal action solution is (19) with $x = 1$ (at leading order in ε). This delta function behavior of the eigenvalue density to leading order in ε implies that just below the transition the eigenvalues will be clustered around $r_2 \sim \langle r_2^2 \rangle^{1/2}$.

We can check the correctness of the results we have just stated by comparing with numerical solutions of the equations of motion. Figure 1 compares our results with a numerically computed eigenvalue distribution with $N = 350$ points, $n_1 = 6$, $n_2 = 1$ and $\varepsilon = 0.04$. There is a good agreement, especially in the region where the bulk of the eigenvalues are clustered, at $r_2 \sim \langle r_2^2 \rangle^{1/2}$. Quantitatively, the numerical analysis gives $\langle r_2^2 \rangle \approx 0.0073$ whereas our formula (19) gives $\langle r_2^2 \rangle \approx 0.0053$. The discrepancy here is of order $\varepsilon^2 = 0.0016$, as we should expect. The numerics furthermore give $\langle r_2^4 \rangle / \langle r_2^2 \rangle^2 \approx 1.21$, which is roughly of order ε away from $x = 1$, again consistent with our approximations.
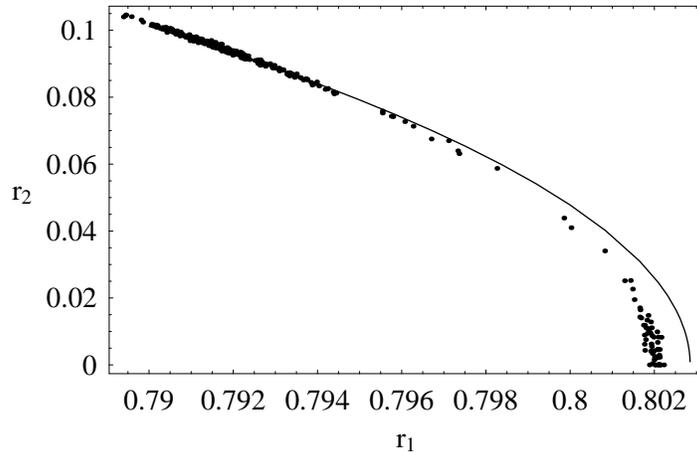

Figure 1: Numerical analysis and leading order analytic results for $n_1 = 6, n_2 = 1$ at masses $m_1^2 = 0.78$ and $m_2^2 = 1$, corresponding to $\varepsilon = 0.04$. The numerical analysis uses $N = 350$ eigenvalues. At leading order in ε the distribution is supposed to be a delta function at $r_2 = \sqrt{\langle r_2^2 \rangle} = 0.0727$. Consistently with this, the fraction of the eigenvalues in the lower grouping is around 15 percent, which is of order ε.

It is now straightforward to take our solution (19) and evaluate the action (12) to order ε^2. Relative to the sphere solution (7), the action is

$$\frac{1}{N^2} \Delta S = -\frac{\varepsilon}{2} \langle r_2^2 \rangle = -\frac{n_2(n_1 - 5)(n_1 - 3)^2}{4(n_1 - 2)((n_1 - 3)^2 + n_2(n_1 - 1))} \frac{\varepsilon^2}{m_1^4}. \tag{20}$$

Thus we see that there is a second order phase transition as $\varepsilon \to 0$.

It would be interesting to see if there are additional transitions in these models, for instance as the mass of a third matrix is decreased. It would also be interesting to generalize our analysis to matrix models that have lower values of n_1. For $n_1 < 4$ it seems that more than one vector of matrices condenses for any ratio of masses, and it would be interesting to see if similar phase transitions to the one we described still occur or not.

Acknowledgements

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