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Gravitational Instability of a Nonrotating Galaxy *

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Abstract Gravitational instability of the distribution of stars in a galaxy is a well-known phenomenon in astrophysics. This work is a preliminary attempt to analyze this phenomenon using the standard tools developed in accelerator physics. By applying this analysis, it is found that a stable nonrotating galaxy would become unstable if its size exceeds a certain limit that depends on its mass density.

1 Introduction

There are some notable examples in the past when developments in astrophysics later are found to be profoundly connected to important topics in accelerator physics. Two major topics in accelerator physics has been nonlinear dynamics and collective effects. It turns out that each of these topics has its origin traced back to astrophysics.

In nonlinear dynamics, Henri Poincaré (1854-1912) was believed to be the first person who noted the behavior of nonlinear dynamical chaos. In 1887, he entered a contest sponsored by the king of Sweden and Norway, and the problem was to prove that the solar system (as a 3-body system) was dynamically stable. He did not succeed in proving it, but his work won the prize anyway. Poincaré was also the person who introduced the Poincareé section, which accelerator physicists today use everyday. In fact, what a beam position monitor detects in a circular accelerator is a special case of Poincaré section. Dynamic aperture and chaotic motion are also typically observed as Poincaré sections (this time on a computer printout), and have become daily language of nonlinear dynamists in accelerator physics.

In collective effects, one notable preview was the impressive work by James Clerk Maxwell (1831-1879). In 1857, Maxwell also won a prize – the Adams Prize – when he proved analytically that the Saturn rings can not be stable unless they consisted of many small solid satellites instead of a single solid piece. Today, we call this mechanism of Maxwell "negative mass instability" in accelerator physics.

Following these pioneering founders, one might ask if today, after years of evolution, might there be some studies that the accelerator physicists have developed, and that can be applied to astrophysics in return. One such attempt is ventured in this paper. We will try to apply modern accelerator techniques to the well-known problem of a gravitational instability of a nonrotating galaxy.

Consider a distribution of stars in a galaxy described by a distribution density $\rho(\vec{x},\vec{v},t)$ in the phase space $(\vec{x},\vec{v})$ at time t. We wish to analyze the stability of this distribution of stars under the influence of their collective gravitational force. To simplify the problem, we will use a flat Euclidean space-time and will consider Newtonian, nonrelativistic dynamics only. In other words, we ignore both the special theory and the general theory of relativity. The instability thus does not assume a specific cosmological model other than Newtonian gravity. If this approach turns out fruitful, a large arsenal of analysis tools can be transported from accelerator physics to this and other problems in astrophysics.

The instability we are interested in is self-generated, i.e. it occurs spontaneously. In particular, it does not require an initial "seed" fluctuation at the

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birth of the galaxy. The instability growth pattern as well as its rate of growth are intrinsic properties of the system. This gravitational instability is a well-known problem; its first analysis appeared almost a century ago [1]. What we do in the following is to treat the same problem using the standard techniques developed in the study of collective instabilities in circular accelerators[2].

2 Dispersion Relation

Consider a particular star in the galaxy. The equations of motion of this star are

$$\begin{aligned}\dot{\vec{x}} &= \vec{v} \\ \dot{\vec{v}} &= G\int d\vec{v}'\int d\vec{x}'\,\frac{\rho(\vec{x}',\vec{v}',t)(\vec{x}'-\vec{x})}{|\vec{x}'-\vec{x}|^3}\end{aligned} \tag{1}$$

where G is the gravitational constant. Note that these equations do not depend on the mass of the star under consideration.

Evolution of ρ is described by the Vlasov equation [3]

$$\begin{aligned}& \frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial\vec{x}}\cdot\dot{\vec{x}}+\frac{\partial\rho}{\partial\vec{v}}\cdot\dot{\vec{v}} \\ =\ & \frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial\vec{x}}\cdot\vec{v}+\frac{\partial\rho}{\partial\vec{v}}\cdot G\int d\vec{v}'\int d\vec{x}'\frac{\rho(\vec{x}',\vec{v}',t)(\vec{x}'-\vec{x})}{|\vec{x}'-\vec{x}|^3} \\ =\ & 0\end{aligned} \tag{2}$$

To examine the stability of the system, let the galaxy distribution be given by an unperturbed distribution ρ_0 plus some small perturbation. Let the unperturbed distribution ρ_0 depend only on $\vec{v}$,

$$\rho_0=\rho_0(\vec{v}) \tag{3}$$

This unperturbed distribution is uniform in $\vec{x}$, i.e. it is uniform in the infinite 3-D space. The function $\rho_0(\vec{v})$ is so far unrestricted.

We will allow the small perturbation around ρ_0 to depend on t and in $\vec{x}$ although the unperturbed distribution ρ_0 has been assumed not to depend on t or in $\vec{x}$. We Fourier decompose the perturbation and write

$$\rho(\vec{x},\vec{v},t)=\rho_0(\vec{v})+\Delta\rho(\vec{v})\,e^{-i\omega t+i\vec{k}\cdot\vec{x}} \tag{4}$$

where $\vec{k}$ is the wavenumber vector and ω is the angular oscillation frequency of the perturbation. We anticipate that for a given $\vec{k}$, there will be a specific oscillation frequency ω. In general, $\vec{k}$ is considered to be real, but for a given $\vec{k}$, the corresponding ω can be complex. With the time dependence of the perturbation given by $\sim e^{-i\omega t}$, we see that the imaginery part of ω is the instability growth rate (growth rate if $\mathrm{Im}(\omega)>0$, damping rate if $\mathrm{Im}(\omega)<0$). The quantity $\Delta\rho$ is considered to be infinetisimal compared with ρ_0.

Substituting Eq.(4) into Eq.(2) and keeping only first order in $\Delta\rho$ yield

$$\begin{aligned}&-i(\omega-\vec{v}\cdot\vec{k})\Delta\rho(\vec{v}) \\ &\qquad +G\left(\int d\vec{v}'\,\Delta\rho(\vec{v}')\right)\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})=0\end{aligned} \tag{5}$$

where

$$\vec{q}(\vec{k})\equiv\int d\vec{x}'\,\frac{e^{i\vec{k}\cdot(\vec{x}'-\vec{x})}(\vec{x}'-\vec{x})}{|\vec{x}'-\vec{x}|^3}=\int d\vec{y}\,\frac{e^{i\vec{k}\cdot\vec{y}}\,\vec{y}}{|\vec{y}|^3} \tag{6}$$

is a well-defined quantity depending only on $\vec{k}$; it is the Fourier transform of the Newton kernel $\vec{x}/|\vec{x}|^3$, and might be called the graviton propagator following a terminology in quantum field theory. In fact, aside from the singularity at the origin $\vec{k}=\vec{0}$, it can be shown that

$$\vec{q}(\vec{k})=\frac{4\pi i}{|\vec{k}|^2}\,\vec{k} \tag{7}$$

In accelerator physics, the Newton kernel $\vec{x}/|\vec{x}|^3$ stands for the wake function while its Fourier transform $\vec{q}$ stands for the impedance.

Eq.(5) can be rewritten as

$$\Delta\rho(\vec{v})=-iG\left(\int d\vec{v}'\,\Delta\rho(\vec{v}')\right)\frac{\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})}{\omega-\vec{v}\cdot\vec{k}} \tag{8}$$

Integrating both sides over $\vec{v}$ and canceling out the mutual factor of $\int d\vec{v}'\,\Delta\rho(\vec{v}')$ then gives a dispersion relation that must be satisfied by ω and $\vec{k}$,

$$1=-iG\int d\vec{v}\frac{\frac{\partial\rho_0(\vec{v})}{\partial\vec{v}}\cdot\vec{q}(\vec{k})}{\omega-\vec{v}\cdot\vec{k}} \tag{9}$$

Given $\rho_0(\vec{v})$, we solve this dispersion relation for ω as a function of $\vec{k}$. This solution is then used to find the most unstable pattern of perturbation and its corresponding growth rate will be described next.

3 Uniform Isotropic Galaxy

We next consider an unperturbed distribution that depends only on the magnitude of $\vec{v}$, i.e., let

$$\rho_0=\rho_0(|\vec{v}|^2) \tag{10}$$

which gives

$$\frac{\partial\rho_0}{\partial\vec{v}}=2\vec{v}\,\rho_0'(|\vec{v}|^2) \tag{11}$$

This is the case of a uniform isotropic (uniform in $\vec{x}$, isotropic in $\vec{v}$) galaxy. Normalization condition is

$$\begin{aligned}\int_0^\infty 4\pi v^2 dv\,\rho_0(v^2) &= \rho_m \\ &= \text{mass density of} \\ &\quad\ \text{stars per unit volume}\end{aligned} \tag{12}$$

Substituting Eqs.(7) and (11) into Eq.(9) then gives

$$1 = \frac{8\pi G}{|\vec{k}|^2}\int d\vec{v}\,\rho_0'(|\vec{v}|^2)\frac{\vec{v}\cdot\vec{k}}{\omega-\vec{v}\cdot\vec{k}} \tag{13}$$

Let $\vec{k} = (0,0,k)$, and choose coordinates so that $\vec{v} = v(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, Eq.(13) becomes, with a change of variable $u=\cos\theta$,

$$1 = \frac{16\pi^2 G}{k}\int_0^\infty v^3 dv\,\rho_0'(v^2)\int_{-1}^{1} du\,\frac{u}{\omega-kvu} \tag{14}$$

One must refrain from performing the integration over u at this time because that integral involves a singularity. Proper treatment of the singularity follows the standard technique used in accelerator physics (and plasma physics) on Landau damping [4]. Omitting the details, the treatment amounts to adding an infinitesimal positive imaginary part to ω, i.e. $\omega \to \omega + i\epsilon$,

$$\begin{aligned}\mathcal{I}(\omega,kv) &\equiv \int_{-1}^{1} du\,\frac{u}{\omega-kvu} \to \int_{-1}^{1} du\,\frac{u}{\omega+i\epsilon-kvu} \\ &= \text{P.V.}\int_{-1}^{1} du\,\frac{u}{\omega-kvu} - \frac{i\pi\omega}{k^2v^2}H\left(1-\left|\frac{\omega}{kv}\right|\right) \\ &= -\frac{2}{kv} - \frac{\omega}{k^2v^2}\ln\left|\frac{\omega-kv}{\omega+kv}\right| - \frac{i\pi\omega}{k^2v^2}H\left(1-\left|\frac{\omega}{kv}\right|\right)\end{aligned} \tag{15}$$

where P.V. means taking the principal value of the integral, and $H(x)=1$ for $x>0$ and 0 for $x<0$ is the step function. By taking P.V., the singularity is avoided in a well-defined manner.

To be specific, we next take a uniform distribution of ρ_0 (uniform in $\vec{x}$, isotropic in $\vec{v}$, and ρ is constant up to v_0),

$$\rho_0(v^2) = \begin{cases}\frac{3\rho_m}{4\pi v_0^3} & \text{if } v^2 < v_0^2 \\ 0 & otherwise\end{cases} \tag{16}$$

This distribution is called the "waterbag model" in accelerator physics. The quantity v_0^2 is related to the "temperature" of the stars. Substituting Eq.(16) into Eq.(14) gives the dispersion relation

$$\lambda = \frac{1}{2 + x\ln\left|\frac{x-1}{x+1}\right| + i\pi x\,H(1-|x|)} \tag{17}$$

where

$$\lambda = \frac{6\pi G\rho_m}{k^2 v_0^2} \quad \text{and} \quad x = \frac{\omega}{kv_0} \tag{18}$$

In accelerator physics, λ is replaced by the impedance. One simplification for the gravitational instability is that λ is a real quantity, while the impedance is complex in general.

4 Stability Condition

We next need to compute the instability growth rate, which is given by the imaginary part of ω, as a function of k. The star distribution $\rho_0(\vec{v})$ would be unstable if, for any $\vec{k}$, its corresponding ω is complex with a positive imaginary part. We need to compute x as a function of λ using Eq.(17) in order to obtain ω as a function of k. Unfortunately Eq.(17) gives λ as a function of x, and its inversion to give x as a function of λ is difficult. Here we apply another technique of accelerator physics as follows.

In general x is complex, but at the edge of instability, x is real. The edge of stability can therefore be seen by plotting the RHS of Eq.(17) as x is scanned along the real axis from $-\infty$ to ∞. Fig.1 shows the real and imaginary parts of the RHS of Eq.(17) in such a scan. The horizontal and vertical axes are the real and imaginary parts of the RHS of Eq.(17) respectively. As x is scanned from $-\infty$ to ∞, the RHS of Eq.(17) traces out a cherry-shaped diagram, including the "stem" of the cherry running from $-\infty$ to 0 along the real axis. If λ lies inside this cherry diagram (including the stem), the galaxy distribution is stable. Since λ is necessarily real and positive, the stability condition therefore reads

$$\lambda < \frac{1}{2} \tag{19}$$

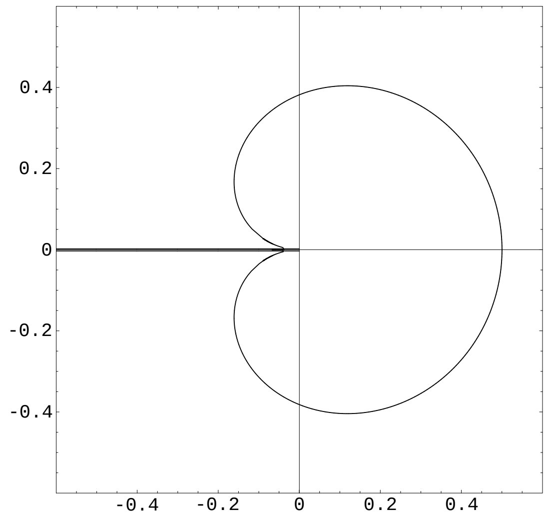

Figure 1 Stability diagram for the galaxy distribution.

Eq.(19) indicates that a hot universe (high temperature, i.e. large v_0) is more stable than a cold universe. This is expected due to the Landau damping mechanism. It also indicates that the star distribution is unstable for long-wavelength perturbations (small k). The threshold wavelength is given by

$$x_{\rm th} = \frac{2\pi}{k_{\rm th}} \tag{20}$$

where

$$k_{\rm th} = \frac{\sqrt{12\pi G\rho_m}}{v_0} \tag{21}$$

Perturbations with wavelength longer than $x_{\rm th}$ are unstable. One might expect that the galaxy will have a dimension of the order of $x_{\rm th}$ because if the galaxy had a larger dimension, it would have broken up due to the instability until it is reduced to the stable size.

5 Spontaneous Gravitational Instability

When $\lambda > 1/2$, ω will be complex. The instability growth rate is determined by the imaginary part of ω,

$$\tau^{-1} = {\rm Im}(\omega) \tag{22}$$

We need to modify Eq.(17) slightly for complex ω. In the unstable region, let

$$\frac{\omega}{kv_0} = x + iy, \quad (y > 0) \tag{23}$$

Eq.(17) reads

$$\frac{1}{\lambda} = 2 + \frac{x+iy}{2} \ln \frac{(x-1)^2+y^2}{(x+1)^2+y^2} + (ix-y) \tan^{-1} \frac{x+1}{y} - \tan^{-1} \frac{x-1}{y} \tag{24}$$

When $y \to 0^+$, we obtain Eq.(17) as it should.

We will need to solve Eq.(24) for x and y for given $\lambda > \frac{1}{2}$. It turns out that in this range there is always one solution with purely imaginary ω, i.e. $x = 0$, and therefore

$$\lambda = \frac{1}{2 - 2y\tan^{-1}\frac{1}{y}} \tag{25}$$

or, written out explicitly,

$$\frac{6\pi G\rho_m}{k^2 v_0^2} = \frac{1}{2 - \frac{2\tau^{-1}}{kv_0}\tan^{-1}\frac{kv_0}{\tau^{-1}}} \tag{26}$$

We need to find τ^{-1} as a function of k. To do so, we first scale the variables by

$$u = \frac{kv_0}{\sqrt{6\pi G\rho_m}}, \quad v = \frac{\tau^{-1}}{\sqrt{6\pi G\rho_m}} \tag{27}$$

and then

$$\frac{1}{u^2} = \frac{1}{2 - 2\frac{v}{u}\tan^{-1}\frac{u}{v}} \tag{28}$$

Fig.2 shows the result.

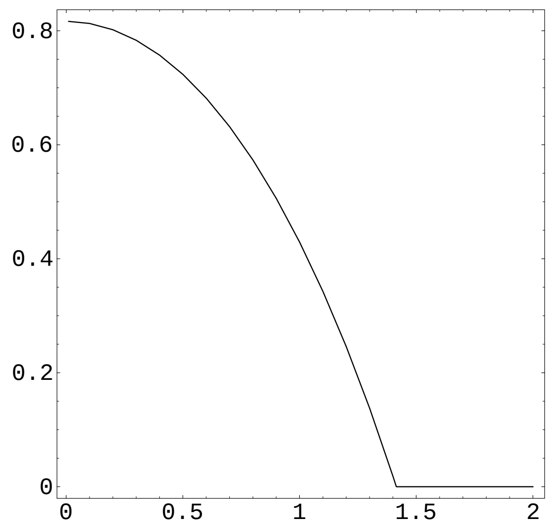

Figure 2 v vs u according to Eq.(28).

As seen from Fig.2, the growth rate vanishes ($v = 0$) when $u = \sqrt{2}$, corresponding to $\lambda = 1/2$, i.e. at the stability boundary. This is of course expected. Fig.2 also shows that instability occurs fastest for small u, i.e. small k and large wavelength of the perturbation. The growth rate is maximum at $u = 0$ with $v = \sqrt{2/3}$. This means the maximum growth rate occurs for perturbation of infinite wavelength, and is given by

$$(\tau^{-1})_{\rm max} = \sqrt{4\pi G\rho_m} \tag{29}$$

Note that the growth rate is independent of v_0, even though that for instability, there is still the condition $\lambda > 1/2$, which does depend on v_0 and can be cast into the form (see Eq.(21))

$$k < \frac{\sqrt{3}}{v_0}(\tau^{-1})_{\rm max} \tag{30}$$

The fastest instability corresponds to $k = 0$, or an instability of infinite wavelength.*

According to Eq.(30), all stable galaxies must have a dimension smaller than a critical value, i.e.

$$\text{galaxy dimension} < \frac{2\pi v_0}{\sqrt{12\pi G\rho_m}} \tag{31}$$

The stability is provided through Landau damping. When the temperature $v_0 \to 0$, no galaxies can be stable. Eqs.(29) and (31) are our main results.

6 Numerical Estimates

For a numerical application, we take estimates from the Milky Way,

$$\begin{aligned} \rho_m &= 2 \times 10^{-23} \text{ g/cm}^3 \\ v_0 &= 200 \text{ km/s} \end{aligned}$$

We obtain a maximum growth time of $\tau_{\max} = 7 \times 10^6$ years for perturbations with very large wavelengths. For stability, the galaxy dimension must be smaller than 14000 light-years, which seems to be consistent with the size of the Milky Way.

7 Discussions

- The case studied so far is that of a galaxy with uniform distribution of stars. One direction of generalization is to consider galaxies with a finite extent. One attempt was made in [5]. It is found that a spherically symmetric distribution of the Haissinski type [6] (potential-well distortion in accelerator physics) does not exist.
- An attempt to extend to a planar galaxy, still nonrotating, had also been made in [5]. The unperturbed Haissinski distribution does exist. However, this planar distribution is found to be always stable against perturbations that do not involve transverse structures. Any instability of the planar galaxy will therefore have to have a sufficiently complex pattern.
- It is conceivable that the same analysis can be applied to the dynamics of galaxies in a galaxy cluster, instead of stars in a galaxy. In that case, $\rho(\vec{x}, \vec{v}, t)$ describes the distribution of galaxies in the galaxy cluster. We might then take the corresponding numerical values

$$\begin{aligned} \rho_m &= 10^{-31} \text{ g/cm}^3 \\ v_0 &= 1000 \text{ km/s} \end{aligned}$$

  We obtain a growth time of $\tau_{\max} = 1 \times 10^{11}$ years. The galaxy cluster dimension should be smaller than 1×10^9 light-years. These values do not seem to be unreasonable.
- For more detailed applications, we will have to include the rotation of the galaxy into the analysis. The unperturbed distribution will then involve also the angular momentum. The analysis will be more involved.
- Still further extensions might include the special relativity and general relativity to replace Newtonian gravity and to avoid the action-at-a-distance problem.

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*This result depends on our assumption of Newtonian dynamics of action-at-a-distance. Under this assumption, perturbation at one location instantly affects locations infinitely far away. If this assumption is appropriately removed, it is expected that the instability for perturbations with very large wavelengths will be weakened.