STRETCHED WIRE MECHANICS*

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ABSTRACT

Stretched wires are beginning to play an important role in the alignment of accelerators and synchrotron light sources. Stretched wires are proposed for the alignment of the 130 meter long LCLS undulator. Wire position technology has reached sub-micron resolution yet analyses of perturbations to wire straightness are hard to find. This paper considers possible deviations of stretched wire from the simple 2-dimensional catenary form.

1. INTRODUCTION

Stretched wires have been used for alignment since the building of the pyramids. One of the early applications to accelerator alignment was Panofsky's use of stretched wires on spectrometer magnets in End Station A at SLAC in the 1960's [1], [2]. Both optical readout [3] and electrical readout [1] have been developed to micron resolution. With this level of position resolution for wires 100 meters long, it is reasonable to ask what intrinsic straightness is expected of the wire itself ? At the micron level over 100 meters, it is hard to proof check wire straightness by any conventional survey technique. Absolute calibration is possible only using alignment techniques based on the accelerator beam itself. Gravitational distortions have been considered by F.Becker et.al [4]. This note considers a number of mechanical perturbations to wire straightness. The calculations here are not exact or all-inclusive. They are simply estimates used to set the order of magnitudes for some of the various physical effects.

2. CATENARIES

It is common knowledge that uniform density cables tensioned in a uniform gravity field hang in the form of a catenary with vertical deflection

$$y(z) = -\frac{T}{w}\left[\cosh\left(\frac{wl}{2T}\right) - \cosh\frac{w}{T}\left(\frac{l}{2} - z\right)\right]. \tag{1}$$

Here T is the horizontal component of tension which is uniform from one end of the wire of length l to the other. The weight of the wire per unit length w is also assumed uniform along the wire. For highly tensioned wires, substitution of the first 2 terms of the power series

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expansion for $\cosh(u) = 1 + (u^2/2!) + (u^4/4!) + \ldots$ gives a simple parabolic approximation for highly tensioned wires.

$$y(z) \cong \frac{w}{2T}\left(z^2 - zl\right), \quad y\left(\frac{l}{2}\right) \cong -\frac{wl^2}{8T}. \tag{2}$$

For highly tensioned wires, the error in this approximation is small. Using the power series for $\cosh(u)$, $y\left(\frac{l}{2}\right) = -\frac{T}{w}\left[\frac{1}{2!}\left(\frac{wl}{2T}\right)^2 + \frac{1}{4!}\left(\frac{wl}{2T}\right)^4 + \ldots\right]$. When evaluated for 0.5 mm diameter steel wire 100 meters long with $w = 1.6$ gram/m and tension $T = 14$ kg gives the deflection at mid-span $y(50\text{ m}) = (.1429\text{ m} + 3.89\times10^{-7}\text{ m} + \ldots)$. The 2nd term is completely negligible for wires of this weight and tension. Other deviations of real wires from the catenary form are much larger than the error in this parabolic approximation.

2.1. Nonuniform wires

Variation of weight along the wire will lead to deviations from the catenary form. The general problem of non-uniform cables is considered by Fallis [5]. For highly tensioned cables where the parabolic approximation is accurate, the problem of non-uniform weight is greatly simplified. The force equilibrium on a cable leads to the differential equation:

$$T\frac{d^2y}{dz^2} = w(s)\frac{ds}{dz} \; . \tag{3}$$

Here the vertical y and the horizontal z components of the trajectory are functions of arc length s. For highly tensioned wires, ds/dz is nearly 1 and $w(s)$ can be approximated by $w(z)$ simplifying the equation to

$$T\frac{d^2y}{dz^2} = w(z). \tag{4}$$

This is easily integrated to give the wire's trajectory in terms of its weight integral $W(z)$:

$$y(z) = \frac{1}{T}\left[\int_0^z W(z)dz - \frac{M_1}{l}z\right], \quad W(z) \equiv \int_0^z w(z')dz' , \quad M_1 \equiv \int_0^l W(z)dz . \tag{5}$$

As a simple numerical example of the effect of variable wire weight, consider a 100 meter wire which starts at 90% of w_0 and ends at 110% of w_0. A wire with this linear taper hangs asymmetrically lower on the heavier end compared to the symmetric parabolic form of a uniform wire. The deviations reach nearly 2 mm for a 100 meter wire as shown at right in Figure 1 for $T = 14$ kg and $w = 1.6$ gram/m (.5 mm diameter steel wire).

For equation (5) to be useful, a wire's weight integral $W(s)$ must be measured. One way to do this would be to transfer the wire from one spool to another. If wire spools are placed at opposite ends of a beam balance, transferring the wire from one spool to the other weighs the integral $W(s) = \int_0^s w(s')ds'$. A schematic for such a balance is illustrated on the left of Figure 1 below. Actuator magnet and coil at one end of the beam would hold it level using feedback from an electronic level mounted on the beam. As wire is transferred, actuator coil current would record weight transfer. The 100 meter wire used in this example weighs only 160 grams. For a 20% linear weight variation, each meter weighs only .003 grams more than the last. The balance would have to resolve 1 mg to be useful.

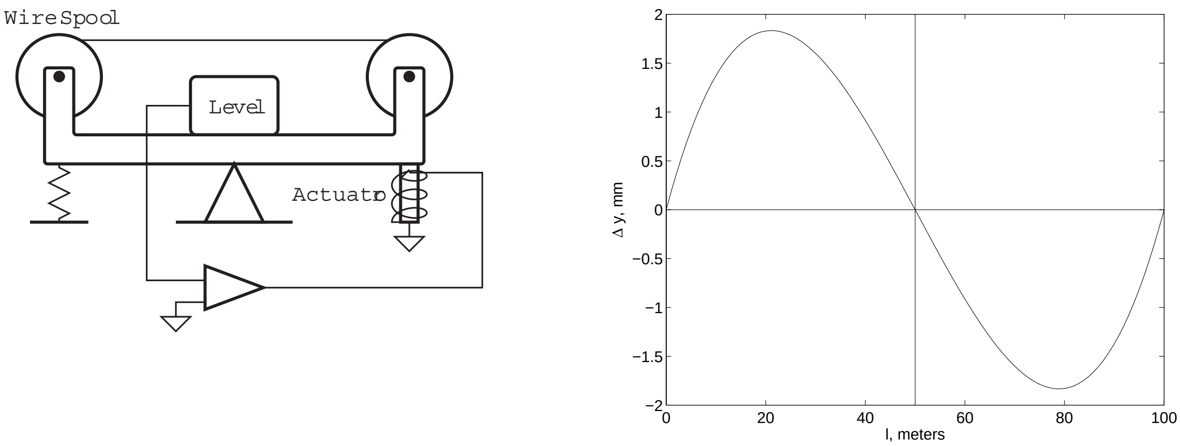

Figure 1: Left: wire weighing balance. Right: deflection difference Δ y between wire with linear increasing weight $w(z) = (.9 + .2z/l)w_0$ and uniform wire weight $w_0 = 1.6$ gram/m. Tension $T = 14$ kg and length $l = 100$ m.

2.2 Elastic catenary

The word catenary is from the Latin for chain. Derivation of the catenary assumes that the 'cable', 'wire' or chain can not transmit bending or torsional moments. It is a structure made up of inextensible links joined with complete angular flexibility. The catenary formula depends only on tension and weight/length. Neither elastic modulus nor wire diameter appears in the equations. No elastic energy is stored. For real wires, stretch and bending stiffness modify the catenary form, even for thin wires. The case of the stretchable elastic catenary is covered by Irvine [6]. The main effect of stretching is a reduction of the weight/length w. This reduces the wire sag. Stress across the wire section area A_0 is $\sigma = T / A_0$. Wire strain $\varepsilon \equiv \Delta l / l = \sigma / E$ for wire elastic modulus *E*. When tensioned, wire stretches by $\varepsilon * l$. This reduces the weight/length to $w = w_0 /(1+\varepsilon)$. The parabolic approximation for the mid span sag of a stretchable wire then becomes:

$$y(l/2) = -\frac{1}{8}\left(\frac{1}{1+\varepsilon}\right)\frac{w_0}{T}l^2 \quad (6)$$

The effect is easy to visualize in a thought experiment where an inextensible wire is tensioned by a weight over a pulley at one end. When the wire is allowed to stretch, the weight pulls $\varepsilon * l$ of the wire over the pulley, reducing the weight of the wire remaining between the supports. For a 100 meter .5mm diameter steel wire with $E = 2.1\times10^3\,\mathrm{kg/mm^2}$ (207GPa), tensioning to 14 kg (1/3 of yield) stretches the wire by $\varepsilon = .003$ or .3 meters. This strain reduces the deflection by about 0.5 mm from 142.9 mm to142.4 mm, a small but possibly measurable effect.

3. ELASTIC RODS

Larger deviations from a pure catenary are observed if the bending stiffness of the wire is considered. Wires that have bending and torsional stiffness are considered ‘rods’. The mechanics of long slender rod-like structures is a currently active branch of applied mathematics. Applications range from the tangling of undersea cables to the guiding of catheters through arteries to the equilibrium forms of the DNA helix. Because their transverse dimensions are so small compared to their length, rods and wires can experience large displacements without large stress or strain. This leads to spatially nonlinear differential equations and the mathematics of differential geometry.

3.1. Influence of bending stiffness on the form of a suspended wire

A.E.H.Love’s Treatise on the Mathematical Theory of Elasicity, Section 273A [7] covers the effects of bending stiffness on a highly tensioned wire. Axial stretching of the wire is ignored as well as internal shear stress. Only bending stresses are added to the forces of gravity and tension acting on the simple chain model. The problem is assumed symmetric with uniform wire weight w gram/m and end points at equal elevation. The coordinate system chosen has its origin centered at mid-span. The wire is assumed to leave the end supports horizontal. Bending stiffness EI adds a 2nd term to the equilibrium of the vertical forces. For wire of diameter d, the wire section inertia $I = (\pi/64)d^4$. The slope $\theta(s)$ at position s along the wire is related to the bending stiffness EI, tension T, and specific weight w by:

$$EI\frac{d^2\theta}{ds^2} = T\sin\theta - \cos(ws) \qquad \text{or} \qquad EI\frac{d^2\theta}{ds^2} = T\theta - ws \qquad \theta << 1 \quad (7)$$

This 2nd order ODE can be solved in small angle approximation to give:

$$\theta = \beta\sinh(\lambda s) + \frac{w}{T}s, \quad \text{where} \quad \lambda \equiv \sqrt{\frac{\mathrm{T}}{EI}}, \quad \beta \equiv -\frac{w}{T}\frac{l/2}{\sinh(\lambda l/2)} \quad (8)$$

Without trying to get an explicit relation between vertical y and horizontal z, both slope components, $\sin\theta$ and $\cos\theta$ can be simultaneously integrated to get the wire trajectory $z(s), y(s)$:

$$\frac{d}{ds}\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos(\beta \sinh(\lambda s) + \frac{w}{T} s) \\ \sin(\beta \sinh(\lambda s) + \frac{w}{T} s) \end{bmatrix}. \qquad (9)$$

Eq. (9) can be numerically integrated with any ODE code such as *Matlab* ode45 but for the thin steel wire used in this note, evaluation of $\beta \sinh(\lambda s)$ is numerically near enough to $0 * \infty$ to cause trouble. This can be avoided by the approximation for

$$\beta \sinh(\lambda s) = -\frac{wl}{2T}\left(\frac{\sinh \lambda s}{\sinh(\lambda l/2)}\right) \quad \text{approximating} \quad \frac{\sinh \lambda s}{\sinh(\lambda l/2)} \cong e^{\lambda\left(s-\frac{l}{2}\right)} - e^{-\lambda\left(s+\frac{l}{2}\right)}. \qquad (10)$$

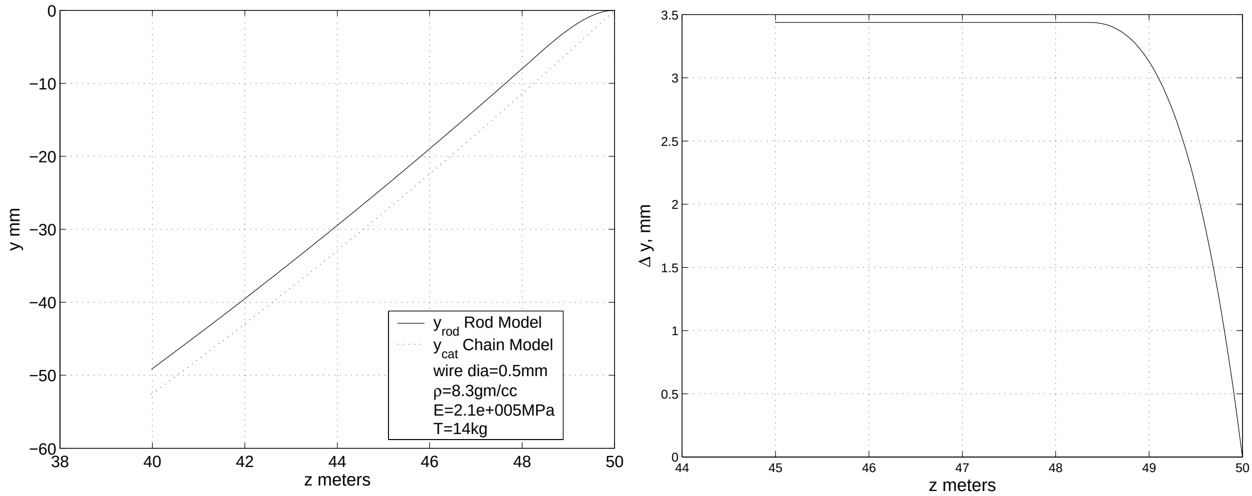

Figure 2: Wire with bending stiffness compared to equivalent catenary. Δy is the difference between pure catenary and actual wire with bending stiffness.

Comparison of a wire with bending stiffness to the equivalent catenary is plotted in Figure 2. Only the last 10 m near the end support is plotted to better show how the difference begins. The effect of bending stiffness extends 2 meters out from the end supports. Beyond 2 meters, wire with bending stiffness is effectively indistinguishable from a catenary. This implies that the end supports should be placed about 2 meters beyond the last sensor. The wire's bending radius is sharpest at the endpoints. Equation (8) for θ can be differentiated to compute the bend radius:

$$\frac{1}{r} = \frac{d\theta}{ds} = \frac{w}{T}\left[1 - \frac{s\lambda}{2\tanh(\lambda s)}\right], \quad r(l/2) \cong \frac{\sqrt{TEI}}{wl/2} \qquad (11)$$

Here, the approximation for end point radius $r(l/2)$ is accurate when $\lambda l/2 >> 1$ and $\tanh(\lambda l/2) \cong 1$. A 100 meter .5 mm dia wire stretched to 14 kg has an end point bend radius of $r = 369$ mm.

3.2. Spooling Helix

Deviations from a pure catenary so far considered lie only in the vertical plane and do not disturb the wire's horizontal straightness. If the wire is capable of transmitting torque as well as bending moments, 3D distortions are possible. The only practical storage for 100 meters of wire is on spools. Coiling wire on spools can leave the wire permanently bent if the spool radius is small enough. A wire's maximum elastic bending stress σ and its elastic modulus E are in the same ratio as the wire diameter d and the diameter D of the spool it is wound on:

$$\frac{\sigma}{E} = \frac{d}{D} \tag{12}$$

For 0.5 mm steel wire (E=207GPa) with yield stress $\sigma = 690$ MPa, winding on a 150 mm diameter spool stresses the wire to yield. Once off the spool and under tension, most of this curvature will be gone. The problem can be thought of as the tensioning of a helical spring as shown in Figure 3.

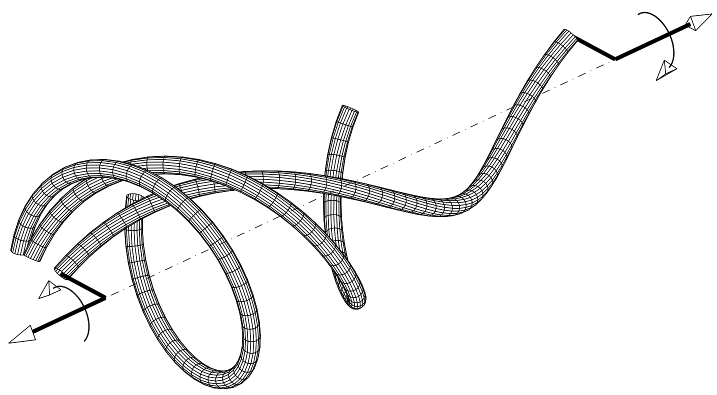

Figure 3: Uncoiling of a single-turn coil spring with increasing axial tension. Moments are applied through each end crank to maintain 1 twist revolution.

A helix is an intermediate geometry between the limits of a closed circle and a straight line. Circles, helixes and straight lines are the only 3D geometries with constant curvature and torsion. Pulling out the wire lengthens the helix pitch and reduces the helix radius as illustrated in Figure 3 for a single turn helical 'spring'. How does the helix radius decrease as the wire is tensioned? What residual helix is left in the wire when tension approaches the yield strength of the wire? A.E. H. Love treats the theory of spiral springs in Section 271 of his Treatise [6]. There, a wire is considered to have torsional and bending stiffness but can not stretch. Love relates the axial tension and end moments to the radius and pitch of the spring in his eq.(40). If the end moments are set to hold the initial twist/turn $= 2\pi$ as tension is increased, these equations are solvable for the helix radius r. For steel wire of 0.5 mm diameter starting from a coil of 150 mm diameter, the helix radius decreases with increasing tension as plotted in Figure 4. As tension is increased, the helix radius becomes inversely proportional to tension. At a tension of 14 kg (about half the breaking strength for steel wire), a helix radius of 64 microns still remains. The amount of helix remaining in the wire at tension depends on the initial curvature at zero tension. The sharper the initial zero-tension curvature, the larger the final remaining helix at tension: short kinks in the wire are almost impossible to remove by tension without exceeding elastic limits. Large gentle initial curvatures disappear nearly completely with tension.

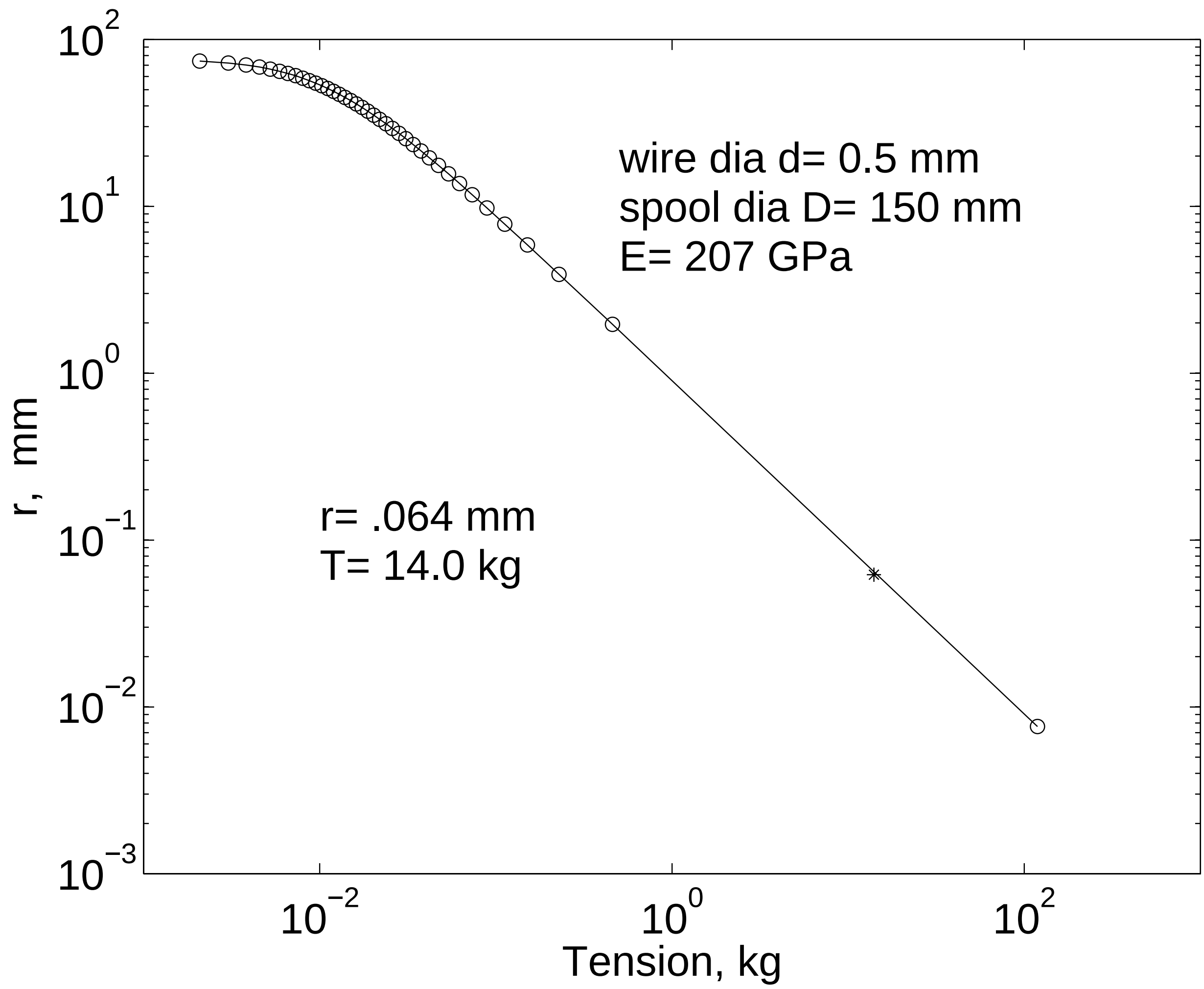

Figure 4: Wire coil helix radius r starting from 150 mm diameter vs axial tension for .5mm diameter steel wire.

3.3. Twisted Catenary

Twisting the end of a perfectly straight wire stretched in the absence of gravity does not deflect the wire so long as the tension is sufficient to prevent instability and the wire tangling into loops called hockles. But if the wire already hangs under gravity as a catenary, twisting can cause the wire to deflect to the side as illustrated in Figure 5.

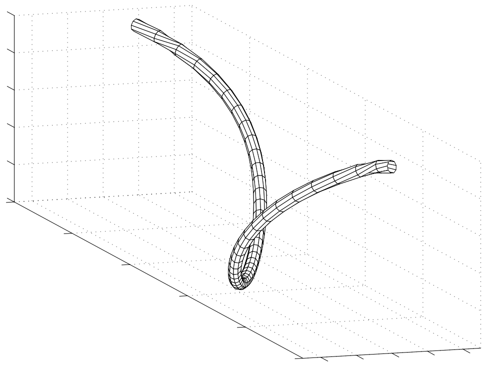

Figure5: Hanging catenary with twisting moment applied at each end.

Several recent papers [8], [9] have considered the effect of gravity on twisted rods. For the case of long small diameter wires at high tension, numerical estimates of this effect are difficult to obtain. In general though, the solutions indicate that as the wire length/diameter and tension increase, the deviations from catenary shape tend to confine themselves near the ends of the wire leaving the bulk of the span hanging in an undistorted planer catenary. In effect, the wire launch points move slightly to the left and right of the untwisted catenary. The zone at each end affected by the twist is extremely short for thin wire. According to reference [8], the fraction of the span distorted by the twist is $\varepsilon = \sqrt{EI / wL^3} = 2\times10^{-4}$ for 0.5mm steel wire 100 meter long. Moving the first and last measurement stations off from the end points by several meters should completely avoid distortion caused by twist.

4. WIRE SIZE AND MATERIAL

When comparing wires, all tensioned to the same percentage of their breaking strength, mid-span deflection is a material property independent of wire diameter (Wire diameter only sets the magnitude of forces which must be resisted at the end mounts.). The ratio of material yield strength to material density then determines both deflection and transverse vibration wave velocity independent of wire diameter. In the case of ductile metals like stainless or beryllium copper, tensile strength increases as the wire is drawn down to smaller diameters due to work hardening. The tabulated strengths are for fine wire.

Material	Density ρ gram/cc	Tensile Strength GPa	Deflection at break $\Delta l / L^2$, m^{-1}	Wave Velocity m/sec
Stainless steel	8.4	2.1	5.0×10^{-6}	495
Be Cu	8.6	1.3	7.8×10^{-6}	395
Carbon fibre	1.8	3.8	0.58×10^{-6}	1453

Carbon fibre has by far the lowest gravitational deflection. A 100 meter long fibre sags only $\left(0.58\times10^{-6}\ \mathrm{m}^{-1}\right)\left(100\mathrm{m}\right)^{2}=5.8\,\mathrm{mm}$ at breaking tension. Most carbon threads are composed of ~1000 small filaments, 7 microns in diameter. Because of this construction, a carbon fibre has the potential for avoiding many of the elastic distortions considered in this paper. Substituting a great number of small filaments for a larger solid wire greatly reduces the bending stiffness EI over that of an equivalent solid wire. Residual distortions disappear much more easily during tensioning. Unfortunately the brittle nature of carbon filaments makes them prone to breakage leaving the filament bundle or 'tow' with a fuzzy surface and reduced strength. By twisting filaments into threads and twisting thread into cord, tensile stress can be more evenly distributed reducing the fuzz of broken filaments and increasing tensile strength. Such cord is now available (www.fibraplex.com). Carbon fibres have much higher electrical resistance than metals and this limits position detection technologies to capacitive or optical. To use carbon fibres with electrical pulse or rf techniques, carbon fibres need either a conductive sheath or core.

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