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Reduction of Charm Quark Mass Scheme Dependence in $\bar{B} \to X_s\gamma$ at the NNLL Level

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Abstract

The uncertainty of the theoretical prediction of the $\bar{B} \to X_s\gamma$ branching ratio at NLL level is dominated by the charm mass renormalization scheme ambiguity. In this paper we calculate those NNLL terms which are related to the renormalization of m_c, in order to get an estimate of the corresponding uncertainty at the NNLL level. We find that these terms significantly reduce (by typically a factor of two) the error on BR($\bar{B} \to X_s\gamma$) induced by the definition of m_c. Taking into account the experimental accuracy of around 10% and the future prospects of the B factories, we conclude that a NNLL calculation would increase the sensitivity of the observable $\bar{B} \to X_s\gamma$ to possible new degrees of freedom beyond the SM significantly.

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1 Introduction

The branching ratio of $\bar{B} \to X_s\gamma$ is a very sensitive probe for new degrees of freedom beyond the standard model (SM) (for a review, see [1]). Within supersymmetric extensions of the SM for example, one can derive stringent bounds on the parameter space of these models [2–8]. Clearly, such bounds will be most valuable when the general nature of the new physics beyond the SM will be identified at the forthcoming LHC experiments.

Because of the heavy mass expansion that is valid for inclusive decay modes, the decay rate of $\bar{B} \to X_s\gamma$ is dominated by the perturbatively calculable partonic decay rate $\Gamma(b \to X_s\gamma)$. QCD corrections to the latter, due to hard-gluon exchange, are the most important perturbative contributions; they were calculated in the past up to the next-to-leading logarithmic (NLL) level [9–18]. Subsequently, also electroweak corrections were calculated [19–22]. After completion of these computations, it was generally believed that the theoretical uncertainty of the branching ratio is below 10%.

However, as first pointed out in 2001 in [23], there is an additional uncertainty in the NLL results for $\Gamma(b \to X_s\gamma)$ which is related to the definition (renormalization scheme) of the charm quark mass. Technically, the charm quark mass depencence enters through the matrix elements $\langle s\gamma|O_{1,2}|b\rangle$ which in the context of a NLL have to be calculated up to $O(\alpha_s)$. As these matrix elements vanish at the lowest order, the charm quark m_c only enters (through the ratio m_c/m_b) at $O(\alpha_s)$. As a consequence, the charm quark mass does not get renormalized in a NLL calculation, which means that the symbol m_c can be identified with $m_{c,\text{pole}}$ or with the $\overline{\text{MS}}$ mass $\bar{m}_c(\mu_c)$ at some scale μ_c or with some other definition of m_c. Formally, all these assignements are equivalent, as they lead to differences which are of order α_s^2.

Note that in contrast to the c-quark mass the b-quark mass does get renormalized in a NLL calculation and we choose to express all the following results in terms of $m_{b,\text{pole}}$. In this respect we do not follow ref. [23], where the $m_{b,1S}$ mass was used. Unless stated otherwise, the symbol m_b stands for $m_{b,\text{pole}}$ in all the formulas in this paper. Numerically, we use $m_b = 4.8$ GeV throughout.

Numerically, it turns out that the NLL result for $\Gamma(b \to X_s\gamma)$ strongly depends on which mass definition of the charm quark mass is used in the NLL expressions. To illustrate this, we first identify m_c with $m_{c,\text{pole}}$ as it was done in all analyses before the paper of Gambino and Misiak [23]. Numerically, we use $m_{c,\text{pole}}/m_{b,\text{pole}} = 0.29$ which is based on the mass difference $m_{b,\text{pole}} - m_{c,\text{pole}} = 3.4$ GeV fixed through the heavy mass expansion of m_B and m_D and $m_{b,\text{pole}} = 4.8$ GeV. The corresponding branching ratio then reads [23]

$$\text{BR}[\bar{B} \to X_s\gamma]_{E_\gamma > m_b/20} \;=\; 3.35 \times 10^{-4}\,. \tag{1}$$

As the charm quarks which are propagating in a loop have a typical virtuality of $m_b/2$, the authors of Ref. [23] suggested to use $\bar{m}_c(\mu_c)$ with $\mu_c \in [m_c, m_b]$ instead of $m_{c,\text{pole}}$. A typical value for the corresponding ratio is $\bar{m}_c(\mu_c)/m_{b,\text{pole}} = 0.22$. Using this value, the branching ratio gets increased w.r.t. (1) by about 11% [23]:

$$\text{BR}[\bar{B} \to X_s\gamma]_{E_\gamma > m_b/20} \;=\; 3.73 \times 10^{-4}\,. \tag{2}$$

In a recent theoretical update of the NLL prediction of this branching ratio, the uncertainty related to the definition of m_c was taken into account by varying m_c/m_b in the

conservative range $0.18 \leq m_c/m_b \leq 0.31$ which covers both, the pole mass (with its numerical error) value and the running mass $\bar{m}_c(\mu_c)$ value with $\mu_c \in [m_c, m_b]$ [24]:

$$\mathrm{BR}[\bar{B} \to X_s\gamma] = (3.70 \pm 0.35|_{m_c/m_b} \pm 0.02|_{\mathrm{CKM}} \pm 0.25|_{\mathrm{param.}} \pm 0.15|_{\mathrm{scale}}) \times 10^{-4}\,. \quad (3)$$

There exists a large number of measurements of the inclusive decay $\bar{B} \to X_s\gamma$ [25–30] and the present experimental accuracy has reached the 10% level [31]:

$$\mathrm{BR}[\bar{B} \to X_s\gamma] = (3.52 \pm 0.30) \times 10^{-4}\,. \quad (4)$$

In the near future, more precise data on this mode are expected from the B factories. Thus, it is mandatory to reduce the present theoretical uncertainty accordingly. A systematic improvement certainly consists in performing a complete NNLL calculation . This is, however, a very complicated task (for discussion and some results see [32–35]) and a certain motivation is needed to enter such an enterprise. In the present paper we try to give such a motivation: By calculating those NNLL terms which are induced by renormalizing the charm quark mass in the NLL expressions, i.e. those terms which are sensitive to the definition of the charm quark mass, we show that the large error at the NLL level related to the m_c definition gets significantly reduced. As this error is the dominant one at the NLL level (see eq. (3)), we conclude that a complete NNLL calculation will drastically improve the theoretical prediction of the branching ratio. We stress here that in the present paper we only make a statement about the reduction of the error at the NNLL level, and not about the central value of the branching ratio; this remains the topic of a complete NNLL calculation!

The remainder of this paper is organized as follows. In section 2 we discuss in some detail how to calculate the NNLL terms induced by renormalizing m_c in the NLL results. In order to make the paper self-contained, we first list in section 3 the structure of the NNL results and then we present the analytical results for the new terms discussed in section 2. Finally, in section 4, we numerically investigate by how much the error related to the definition of m_c gets reduced at the NNLL level.

2 NNLL terms related to m_c renormalization

As already explained in the introduction, the matrix elements $M_{1,2}^{\mathrm{virt}}(m_c) = \langle s\gamma|O_{1,2}(\mu_b)|b\rangle$ only start at order $O(\alpha_s^1)$, or, in other words at the NLL order[2]. As a consequence, the definition of m_c is not fixed at this order, because m_c does not get renormalized. This is also true for the bremsstrahlung contributions $M_{1,2}^{\mathrm{brems}}(m_c) = \langle s\gamma g|O_{1,2}(\mu_b)|b\rangle$, which are needed up to $O(g_s)$ for a NLL calculation. In this section we concentrate on the virtual terms $M_{1,2}^{\mathrm{virt}}(m_c)$, as the extension to the bremsstrahlung contributions $M_{1,2}^{\mathrm{brems}}(m_c)$ is straightforward.

[2]In the present paper we use the operator basis as first introduced in ref. [11]. μ_b denotes the renormalization scale of $O(m_b)$.

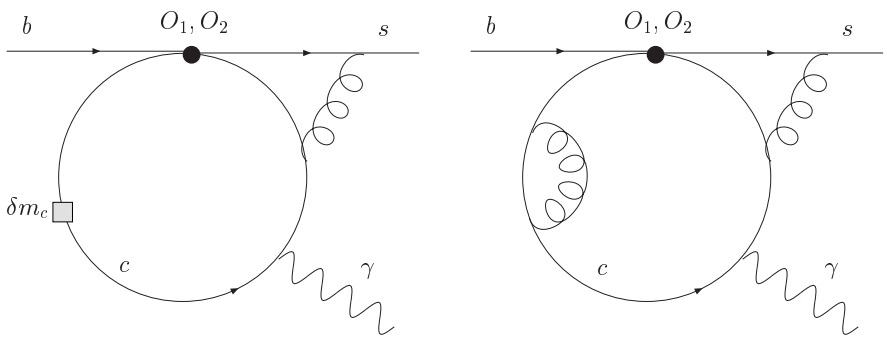

Figure 1: Left frame: Typical δm_c insertion diagram (see text). Right frame: Typical diagram with a charm quark self-energy insertion (see text).

When going to NNLL precision, the matrix elements $M_{1,2}^{\rm virt}(m_c)$ are needed to $O(\alpha_s^2)$. At this level, there are – among many other diagrams – counterterm contributions to these matrix elements, induced by the renormalization of m_c (see the left frame of Fig. 1). The complete set of such diagrams is generated by inserting the operator $-i\delta m_c \bar{\psi}_c \psi_c$ in the $O(\alpha_s)$ diagrams of $O_{1,2}$ in all possible ways. The sum $\delta M_{1,2}^{{\rm virt}(\epsilon)}(m_c) \cdot \delta m_c$ of all these insertions can be obtained by replacing $m_c \to m_c + \delta m_c$ in the $O(\alpha_s^1)$ results $M_{1,2}^{{\rm virt}(\epsilon)}(m_c)$, followed by expanding in δm_c up to linear order:

$$M_{1,2}^{{\rm virt}(\epsilon)}(m_c + \delta m_c) = M_{1,2}^{{\rm virt}(\epsilon)}(m_c) + \delta M_{1,2}^{{\rm virt}(\epsilon)}(m_c) \cdot \delta m_c + O((\delta m_c)^2)\,. \tag{5}$$

As δm_c is ultraviolet divergent, the matrix elements $M_{1,2}^{{\rm virt}(\epsilon)}(m_c)$ are needed in our application up to order ϵ^1, as indicated by the notation in eq. (5).

The explicit shift δm_c depends of course on the renormalization scheme. When aiming at expressing the results for $M_{1,2}^{{\rm virt}(\epsilon)}(m_c)$ in terms of $\bar{m}_c(\mu_b)$, the shift reads ($C_F = 4/3$)

$$\delta \bar{m}_c(\mu_b) \;=\; -\frac{\alpha_s(\mu_b)}{4\pi}\, C_F\, \frac{3}{\epsilon}\, \bar{m}_c(\mu_b)\,.$$

On the other hand, when the result is expressed in terms of $m_{c,{\rm pole}}$, the shift reads

$$\delta m_{c,{\rm pole}} \;=\; -\frac{\alpha_s(\mu_b)}{4\pi}\, C_F \left(\frac{3}{\epsilon} + 3\ln\frac{\mu_b^2}{m_c^2} + 4\right) m_{c,{\rm pole}}\,.$$

The infinities induced by the $1/\epsilon$ terms in δm_c get cancelled in a full NNLL calculation, in particular by self-energy diagrams depicted in the right frame of Fig. 1. As we do not perform a full NNLL calculation, we suggest to consider self-energy insertions, where the self-energy $\Sigma(p^2)$ is replaced by $\Sigma_1(p^2 = m_c^2)$.

The Σ_1-part of the self-energy $\Sigma_1(p^2)$ is defined through the decomposition of the full unrenormalized self-energy $\Sigma(p^2)$ as

$$\Sigma(p^2) \equiv \Sigma_2(p^2)(\not{p} - m_c) + \Sigma_1(p^2). \tag{6}$$

At the one-loop level, the corresponding pieces Σ_1^R and Σ_2^R of the renormalized self-energy are

$$\Sigma_2^R(p^2) = \Sigma_2(p^2) + \delta Z_c, \qquad \Sigma_1^R(p^2) = \Sigma_1(p^2) + \delta m_c\,, \tag{7}$$

where $Z_c = 1 + \delta Z_c$ denotes the wave function renormalization constant of the charm quark. Eq. (7) implies that the sigularities in $\delta M_{1,2}^{\mathrm{virt}(\epsilon)}(m_c)\cdot\delta m_c$ cancel when combined with the diagrams with $\Sigma_1(p^2)$ insertions. However, for general p^2, the function $\Sigma_1(p^2)$ depends on the gauge parameter ξ:

$$\Sigma_1(p^2) = \frac{\alpha_s(\mu_b)}{4\pi} C_F m_c \left\{ 3\left[\frac{1}{\epsilon} + \ln\frac{\mu_b^2}{m_c^2}\right] + 4 + \left(1 - \frac{m_c^2}{p^2}\right)\left[\xi - \left(3 - \xi\frac{m_c^2}{p^2}\right)\ln\left(1 - \frac{p^2}{m_c^2}\right)\right]\right\}.$$

As for $p^2 = m_c^2$ the self-energy piece $\Sigma_1(p^2 = m_c^2)$ is gauge-independent, we add $\Sigma_1(p^2 = m_c^2)$ insertions to $\delta M_{1,2}^{\mathrm{virt}(\epsilon)}(m_c)\cdot\delta m_c$.

These momentum independent $\Sigma_1(p^2 = m_c^2)$ insertions can be straightforwardly aborbed into $\delta m_c^{\mathrm{eff}}$ insertions:

$$\begin{aligned} \delta m_{c,\mathrm{pole}}^{\mathrm{eff}} &= \Sigma_1(p^2 = m_c^2) + \delta m_{c,\mathrm{pole}} = 0\,, \\ \delta \bar{m}_c^{\mathrm{eff}}(\mu_b) &= \Sigma_1(p^2 = m_c^2) + \delta \bar{m}_c(\mu_b) = \frac{\alpha_s(\mu_b)}{4\pi} C_F \left(3\ln\frac{\mu_b^2}{m_c^2} + 4\right)\bar{m}_c(\mu_b)\,. \end{aligned} \tag{8}$$

Finally, if we wish to express the matrix elements $M_{1,2}^{\mathrm{virt}(\epsilon)}(m_c)$ in terms of $\bar{m}_c(\mu_c)$, the shift reads

$$\delta \bar{m}_c^{\mathrm{eff}}(\mu_c) = \frac{\alpha_s(\mu_b)}{4\pi} C_F \left(3\ln\frac{\mu_c^2}{m_c^2} + 4\right)\bar{m}_c(\mu_c)\,. \tag{9}$$

3 Analytical results

Before turning to the contributions induced through the renormalization of the charm quark mass, which are NNLL terms, we first summarize the structure of the NLL result for the branching ratio for $b \to X_s\gamma$. We write the decay width for $b \to X_s\gamma$ using a photon energy cut $E_0 = \frac{m_b}{2}(1-\delta) = E_{max}(1-\delta)$ as

$$\Gamma(b \to X_s\gamma)_{E_\gamma \geq E_0} = \Gamma(b \to s\gamma) + \Gamma(b \to s\gamma g)_{E_\gamma \geq E_0}\,, \tag{10}$$

where the two parts are defined as follows:

$$\Gamma(b \to s\gamma) = \frac{G_F^2}{32\pi^4}|V_{ts}^* V_{tb}|^2 \alpha_{em} m_{b,\mathrm{pole}}^5\,|D|^2,$$

$$\Gamma(b \to s\gamma g)_{E_\gamma \geq E_0} = \frac{G_F^2}{32\pi^4}|V_{ts}^* V_{tb}|^2 \alpha_{em} m_{b,\mathrm{pole}}^5\,A,$$

$$D = C_7^{(0)\mathrm{eff}}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi}\left(C_7^{(1)\mathrm{eff}}(\mu_b) + \sum_{i=1}^{8} C_i^{(0)\mathrm{eff}}(\mu_b)\left[r_i + \gamma_{i7}^{(0)\mathrm{eff}}\ln\left(\frac{m_b}{\mu_b}\right)\right] - \frac{16}{3}C_7^{(0)\mathrm{eff}}(\mu_b)\right), \tag{11}$$

$$A = \left(e^{-\alpha_s(\mu_b)\ln(\delta)[7+2\ln(\delta)]/(3\pi)} - 1\right) |C_7^{(0)\rm eff}(\mu_b)|^2 + \frac{\alpha_s(\mu_b)}{\pi} \sum_{i,j=1,i\leq j}^{8} C_i^{(0)\rm eff}(\mu_b) C_j^{(0)\rm eff}(\mu_b) f_{ij}(\delta)\,. \tag{12}$$

The expressions for the Wilson coefficients $C_i(\mu_b)$ can be found in [36]. Their numerical values we take from table 5.1 in ref. [37]. Writing the results in this specific form, the functions $f_{ij}(\delta)$ and r_i are understood to be taken from [11] and not from the original paper [10] where the results were parametrized differently.

Following common practice, we write the branching ratio (without taking into account non-perturbative corrections) as

$$BR(b \to X_s\gamma)_{E_\gamma \geq E_0} = \frac{\Gamma(b \to X_s\gamma)_{E_\gamma \geq E_0}}{\Gamma(b \to X_c e \bar{\nu})}\, {\rm BR}_{sl}^{exp}\,, \tag{13}$$

where the semileptonic decay rate is given by

$$\Gamma(b \to X_c e^- \bar{\nu}) = \frac{G_F^2\, m_{b,\rm pole}^5}{192\pi^3}\, |V_{cb}|^2\, g\left(\frac{m_c^2}{m_b^2}\right)\, K\left(\frac{m_c^2}{m_b^2}\right)\,. \tag{14}$$

$g(z) = 1 - 8\,z + 8\,z^3 - z^4 - 12\,z^2\,\ln(z)$ is the phase-space factor and the function

$$K(z) = 1 - \frac{2\alpha_s(m_b)}{3\pi}\,\frac{f(z)}{g(z)},$$

with

$$\begin{aligned} f(z) = &-(1-z^2)\left(\frac{25}{4} - \frac{239}{3}\,z + \frac{25}{4}\,z^2\right) + z\,\ln(z)\left(20 + 90\,z - \frac{4}{3}\,z^2 + \frac{17}{3}\,z^3\right) \\ &+ z^2\,\ln^2(z)\,(36 + z^2) + (1-z^2)\left(\frac{17}{3} - \frac{64}{3}\,z + \frac{17}{3}\,z^2\right)\ln(1-z) \\ &-4\,(1 + 30\,z^2 + z^4)\,\ln(z)\ln(1-z) - (1 + 16\,z^2 + z^4)\left(6\,{\rm Li}(z) - \pi^2\right) \\ &-32\,z^{3/2}(1+z)\left[\pi^2 - 4\,{\rm Li}(\sqrt{z}) + 4\,{\rm Li}(-\sqrt{z}) - 2\ln(z)\,\ln\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)\right]\,. \end{aligned}$$

accounts for $O(\alpha_s)$ QCD corrections. We note that m_c is understood to be the pole mass in eq. (14).

We now turn to that part of NNLL corrections which is responsible for the reduction of the charm quark mass renormalization scheme dependence, as explained in section 2. We first turn to terms $\delta M_{1,2}^{{\rm virt}(\epsilon)}$ induced by m_c renormalization in the matrix elements $M_{1,2}^{{\rm virt}(\epsilon)}$. To this end, we need $M_{1,2}^{{\rm virt}(\epsilon)}$ up to oder ϵ^1. In [10] have calculated these matrix elements up to terms ϵ^0, using Mellin-Barnes representations for generalized propogator to obtain analytic results in the form of the series in $z = (m_c/m_b)^2$ and $L = \ln(z)$. As in these calculations the expansion in ϵ was the last step, it is straightforward to calculate $M_{1,2}^{{\rm virt}(\epsilon)}$ up to order ϵ^1.

In order to get finite results for these matrix elements, we add counterterms related to operator mixing as in ref. [10], adapted however, to the operator basis defined in ref. [11]. This step leads to $M_{1,2}^{\rm virt,ren}$, which we decompose as in ref. [10]:

$$M_2^{\rm virt,ren} = \langle s\gamma|O_7|b\rangle\,\frac{\alpha_s}{4\pi}\left(\frac{416}{81}\ln\frac{m_b}{\mu_b} - \frac{784}{81}\epsilon\ln^2\frac{m_b}{\mu_b} - 4\epsilon\ln\frac{m_b}{\mu_b} r_2^{(0)} + r_2^{(0)} + r_2^{(1)}\epsilon\right)\,. \tag{15}$$

We obtain for $r_2 = r_2^{(0)} + \epsilon r_2^{(1)}$ (note that $r_1 = -\frac{1}{6}\, r_2$):

$$
\begin{aligned}
r_2^{(0)} = & -\frac{1666}{243} - \frac{8}{27}\left(-48 - 3\,L^2 - L^3 + 5\,\pi^2 + 9\,L\,(-4+\pi^2) + 36\,\zeta(3)\right) z + \frac{32}{27}\,\pi^2 z^{3/2} \\
& + \frac{8}{27}\left(L^3 - 6\,L(-2+\pi^2) + 18 + 2\,\pi^2\right) z^2 - \frac{4}{81}\left(9 - 182\,L + 126\,L^2 + 14\,\pi^2\right) z^3 \\
& - \frac{8\,i\,\pi}{27}\left[\frac{10}{3} + 2\left(-15 - 3\,L - 3\,L^2 + \pi^2\right) z + 2\left(-3\,L^2 + \pi^2\right) z^2 + \frac{8}{3}\,(-7 + 3\,L)\, z^3\right]
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
r_2^{(1)} = & -\frac{19577}{729} + \frac{184}{243}\,\pi^2 - \frac{2}{405}\Big(-18180 + 75\,L^4 + 3240\,\pi^2 + 46\,\pi^4 - 30\,L^2(-24 + 7\,\pi^2) \\
& + 9000\,\zeta(3) + 120\,L(-66 + 14\,\pi^2 + 27\,\zeta(3))\Big) z - \frac{32}{81}\,\pi^2\,(-49 + 6\,L + 24\,\ln(2))\, z^{3/2} \\
& + \frac{2}{81}\Big(48\,L^3 - 15\,L^4 + 24\,L^2(-3+\pi^2) - 24\,L(3 + 5\,\pi^2) + 1116 + 36\,\pi^2 \\
& + 40\,\pi^4 + 432\,\zeta(3)\Big) z^2 - \frac{1120}{81}\,\pi^2 z^{5/2} + \frac{1}{729}\Big(22705 - 2484\,L^2 + 4536\,L^3 - 6036\,\pi^2 \\
& + 6\,L(-1783 + 192\,\pi^2) + 8208\,\zeta(3)\Big) z^3 \\
& + \frac{8\,i\,\pi}{27}\Big[-\frac{221}{9} + \Big(15\,L^2 - 6\,L^3 - 4\,L(-9+\pi^2) + 186 - 10\,\pi^2 - 36\,\zeta(3)\Big) z \\
& - 2\Big(-3 - 6\,L^2 + 3\,L^3 + 2\,\pi^2 + 2\,L\pi^2 + 18\,\zeta(3)\Big) z^2 \\
& + \frac{4}{9}\,(-67 + 66\,L + 9\,L^2 + 12\,\pi^2) z^3\Big]
\end{aligned}
\tag{17}
$$

In these formulas we retained all terms up to order z^3, as higher order terms contribute much less than 1%. Nevertheless, in the numerical evaluations in section 4 all terms up to z^6 were included.

At the level of the decay width, the implementation of the contribution coming from renormalization of the c-quark mass in the virtual contributions is (according to eq. (5)) most easily achieved by replacing $r_{1,2}$ in eq. (11) by $r_{1,2}^{(0)} + \Delta r_{1,2}$, where

$$
\Delta r_{1,2} = \delta m_c \frac{d}{dm_c}\left(r_{1,2}^{(0)} + \epsilon\, r_{1,2}^{(1)}\right) . \tag{18}
$$

At the NLL order, the bremsstrahlung corrections to the decay width are encoded in the quantities $f_{ij}(\delta)$ (see eq. (12)), which correspond to the interference terms (O_i, O_j). In the following, we calculate the shifts Δf_{ij} to these quantities induced by the renormalization of the charm quark mass. In principle, we calculate the decay width using a photon energy cut $\delta = 0.9$ (see eq. (10)). However, as all bremsstrahlung contributions which contain charm quark loops are finite for $\delta \to 1$, we can approximate these terms by putting $\delta = 1$. Numerically the relative error is of order 10^{-4}.

We first calculate the shift Δf_{27}. To this end, we shift the charm quark mass in the matrix element of $\langle s\gamma g|O_2|b\rangle$ as in eq. (5) and then work out the interference with $\langle s\gamma g|O_7|b\rangle$. Because of the $1/\epsilon$ term in δm_c, the result is ultraviolet singular. In a full NNLL calculation this singularity gets cancelled when combined with self-energy

insertions in the charm quark lines in the matrix element of O_2. We therefore do the phase space integrations involved in the derivation of f_{27} (or Δf_{27}) in d=4 dimensions. As only the matrix element of O_2 depends on m_c, the shift Δf_{27} can be constructed by first considering the quantity f_{27} itself. Using the integral representation for the building block for photon and gluon emission from the c-quark loop [10], one obtains after integration over all but one of the phase space parameters

$$f_{27} = -\frac{8}{9}\left(\frac{\mu_b}{m_b}\right)^{2\epsilon} \int dx\, dy\, du\, u^2(1-u)(1-x)y\, \frac{[x(1-x)]^{-\epsilon}\,\Gamma(2+\epsilon)\,\mathrm{e}^{\epsilon\gamma+i\epsilon\pi}}{\left[uy - \frac{z}{x(1-x)} + i\eta\right]^{(1+\epsilon)}}\,. \tag{19}$$

Here x, y are Feynman parameters and u is the remaining phase space parameter, $0 \leq x, y, u \leq 1$. To solve the integrals, we use the Mellin–Barnes representation for the generalized propagator

$$\left[uy - \frac{z}{x(1-x)} + i\eta\right]^{-1-\epsilon} = \frac{1}{2i\pi\Gamma(1+\epsilon)} \int_\gamma ds \frac{\mathrm{e}^{i\pi s}\,\Gamma(-s)\,\Gamma(1+\epsilon+s)}{(uy)^{1+\epsilon}} \left[\frac{z}{uyx(1-x)}\right]^s$$

appearing in eq. (19). γ denotes the integration path parallel to imaginary axes which hits the real axes somewhere between $(-1-\epsilon)$ and 0. Closing the integration path in the right s-half plane, one gets an expansion for f_{27} in $z = (m_c/m_b)^2$ and $L = \ln(z)$.

The shift Δf_{27} is then obained as

$$\Delta f_{27} = \delta m_c \frac{dz}{dm_c}\frac{df_{27}}{dz} = 2\frac{m_c}{m_b}\frac{\delta m_c}{m_b}\left(f_{27}^{d(0)} + \epsilon\, f_{27}^{d(1)}\right)\,. \tag{20}$$

To summarize, the NNLL contributions due to renormalization of m_c in the (O_2, O_7) interference are taken into account by replacing $f_{27} \to f_{27}^{(0)} + \Delta f_{27}$ in eq. (12). Explicitly, we find:

$$\begin{aligned} f_{27}^{(0)} &= -\frac{8}{9}\left[\frac{1}{12} + \frac{1}{8}(7 - 2\pi^2 + 6L + 2L^2)z + (\pi^2 - 2L - L^2)z^2\right. \\ &+ \frac{1}{4}(-11 - 6\pi^2 - 4L + 6L^2)z^3 + \frac{1}{3}(-6 + 10L)z^4 \\ &+ \left.\frac{1}{24}(13 + 70L)z^5 + \frac{1}{15}(32 + 63L)z^6\right]\,, \end{aligned} \tag{21}$$

$$\begin{aligned} f_{27}^{d(0)} &= -\frac{8}{9}\left[\frac{1}{8}(13 - 2\pi^2 + 10L + 2L^2) + 2(-1 + \pi^2 - 3L - L^2)z\right. \\ &+ \frac{1}{4}(-37 - 18\pi^2 + 18L^2)z^2 + \frac{2}{3}(-7 + 20L)z^3 \\ &+ \left.\frac{5}{24}(27 + 70L)z^4 + \frac{1}{5}(85 + 126L)z^5\right]\,, \end{aligned} \tag{22}$$

$$\begin{aligned} f_{27}^{d(1)} &= -\frac{8}{9}\left[\frac{1}{48}(165 - 52\pi^2 - 4(-15 + \pi^2)L - 18L^2 - 8L^3 - 72\zeta(3))\right. \\ &+ \frac{1}{3}(2(-12 + \pi^2)L - 3L^2 + 4L^3 + 6(-5 + 4\pi^2 + 6\zeta(3)))z \end{aligned}$$

$$
\begin{aligned}
&- \frac{3}{4}(-7+15\pi^2+(15+2\pi^2)L-15L^2+4L^3+36\zeta(3))z^2 \\
&- \frac{5}{27}(235+48\pi^2-204L+36L^2)z^3 \\
&+ \frac{1}{432}(-10076-4200\pi^2+19425L-3150L^2)z^4 \\
&+ \frac{1}{250}(-3554-4200\pi^2+20685L-3150L^2)z^5\Big] \,.
\end{aligned} \tag{23}
$$

Note that f_{27}^0 in eq. (21) is an expanded version in z of the integral expression for f_{27} in ref. [11]. We further note that $f_{28} = -\frac{1}{3}f_{27}$, $f_{17} = -\frac{1}{6}f_{27}$, $f_{18} = \frac{1}{18}f_{27}$; the same relations also hold for the respective Δf_{ij} (see for instance, [38]).

Finally, we turn to the shift Δf_{22} related to the (O_2, O_2)-interference. To derive this quantity, one has to perform the shift $m_c \to m_c + \delta m_c$ only in one of the two interfering one-loop amplitudes $M_2^{\rm brems} = \langle s\gamma g|O_2|b\rangle$. To this end, one writes integral representations for both, $M_2^{\rm brems}$ and $dM_2^{\rm brems}/dm_c$. Δf_{22} is then represented as a five dimensional integral (4 Feynman parameters and one phase space parameter), which can be solved by double Mellin-Barnes techniques (see for instance [39]). Omitting the detail of this calculation, the terms induced by renormalizing m_c in the (O_2, O_2) bremsstrahlung terms are implemented in eq. (12) by replacing $f_{22} \to f_{22}^{(0)} + \Delta f_{22}$, where

$$
\Delta f_{22} = 2\frac{m_c}{m_b}\frac{\delta m_c}{m_b}\left(f_{22}^{d(0)} + \epsilon\, f_{22}^{d(1)}\right) \,. \tag{24}
$$

Explicity, we get:

$$
\begin{aligned}
f_{22}^{(0)} &= 0.04938272 + (16.64197 + 1.887290L - 0.4444444L^2 - 0.09876543L^3)z \\
&+ (57.92026 + 47.67037L + 1.185185L^2 + 3.134737L^3 + 0.05925926L^5)z^2 \\
&+ (-93.12628 + 32.36078L - 12.95977L^2 + 1.777778L^3 - 0.2962963L^4)z^3 \\
&+ (11.92082 - 11.21491L + 2.074074L^2 - 0.5925926L^3)z^4 \\
&+ (0.6482797 - 4.160089L + 0.1810700L^2 - 0.3292181L^3)z^5 \\
&+ (-1.125313 - 4.320604L - 0.2444444L^2 - 0.3456790L^3)z^6 \,,
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
f_{22}^{d(0)} &= 18.52926 + 0.9984013L - 0.7407407L^2 - 0.09876543L^3 \\
&+ (163.5109 + 97.71112L + 11.77458L^2 + 6.269473L^3 + 0.2962963L^4 + 0.1185185L^5)z \\
&+ (-247.0180 + 71.16280L - 33.54596L^2 + 4.148148L^3 - 0.8888889L^4)z^2 \\
&+ (36.46839 - 40.71149L + 6.518519L^2 - 2.370370L^3)z^3 \\
&+ (-0.9186906 - 20.43831L - 0.08230453L^2 - 1.646091L^3)z^4 \\
&+ (-11.07248 - 26.41251L - 2.503704L^2 - 2.074074L^3)z^5 \,,
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
f_{22}^{d(1)} &= 41.24600 - 7.794263L - 0.7525535L^2 + 0.3950617L^3 + 0.07407407L^4 \\
&+ (234.4505 + 44.95451L - 64.68047L^2 - 0.5200208L^3
\end{aligned}
$$

$$
\begin{aligned}
&- \quad 4.498135L^4 - 0.05925926L^5 - 0.08559671L^6)z \\
&+ \quad (-368.8104 + 245.2526L - 95.81857L^2 + 28.84099L^3 - 3.851852L^4 + 0.5728395L^5)z^2 \\
&+ \quad (-3.708986 - 93.16023L + 34.50854L^2 - 7.670782L^3 + 1.283951L^4)z^3 \\
&+ \quad (-45.10910 - 54.73258L + 13.39517L^2 - 4.073160L^3 + 0.8779150L^4)z^4 \\
&+ \quad (-92.60340 - 74.33726L + 10.32328L^2 - 4.786008L^3 + 1.094650L^4)z^5 \,.
\end{aligned} \tag{27}
$$

We decided to give the expansion coefficients in these equations in numerical form, because the exact results are somewhat lenghty. We note that f_{22}^0 in eq. (25) is an expanded version in z of the integral expression for f_{22} in ref. [11]. We further note that $f_{11} = \frac{1}{36} f_{22}$ and $f_{12} = -\frac{1}{3} f_{22}$; the same relations also hold for the respective Δf_{ij}.

These analytical results are defined parts of the complete NNLL contribution which can be used within a future NNLL calculation.

4 Numerical results

In the following analysis we show that the NNLL terms, induced through the renormalization of m_c, drastically reduce the error related to the definition of the charm quark mass in $\mathrm{BR}(b \to X_s\gamma)$. To illustrate this feature as clearly as possible, we take the fixed values shown in Table 1 for the input parameters. In particular, we use the fixed ratio

$m_b = 4.8$ GeV	$m_{c,\mathrm{pole}}/m_b = 0.29$	$m_Z = 91.187$ GeV

$\alpha_s(m_Z) = 0.119$	$\alpha_{\mathrm{em}} = 1/137.036$	$\lvert V_{ts}^* V_{tb}/V_{cb}\rvert^2 = 0.95$	$BR_{\mathrm{sl}} = 10.49\%$

Table 1: Input parameters used in the numerical analysis.

$m_{c,\mathrm{pole}}/m_{b,\mathrm{pole}} = 0.29$. Furthermore, we always leave the semileptonic decay width, which enters the branching ratio for $b \to X_s\gamma$ through eq. (13), expressed in terms of $m_{c,\mathrm{pole}}$ as given in eq. (14). In this way the m_c definition dependence of the $\mathrm{BR}(b \to X_s\gamma)$ only comes from the numerator in eq. (13). For our studies, we neglect electroweak corrections and non-perturbative effects. As already mentioned, in the bremsstrahlung contribution we use $\delta = 0.9$ for the lower cut in the photon energy (see eq. (10)).

Starting from $m_{c,\mathrm{pole}} = 0.29 \cdot 4.8$ GeV $= 1.392$ GeV, we first calculate $\bar{m}_c(m_{c,\mathrm{pole}})$, using the one-loop expression

$$
\bar{m}_c(m_{c,\mathrm{pole}}) = m_{c,\mathrm{pole}} \left[1 - \frac{\alpha_s(m_{c,\mathrm{pole}})}{\pi} C_F \right] . \tag{28}
$$

To get $\bar{m}_c(\mu_c)$ for an arbitrary scale (typically between 1.25 GeV and 5 GeV), we use two-loop running (with 5 flavours) according to

$$
\bar{m}_c(\mu_c) = \bar{m}_c(\mu_0) \left(\frac{\alpha_s(\mu_c)}{\alpha_s(\mu_0)} \right)^{\frac{\gamma_m^{(0)}}{2\beta_0}} \left[1 + \left(\frac{\gamma_m^{(1)}}{2\beta_0} - \frac{\beta_1 \gamma_m^{(0)}}{2\beta_0^2} \right) \frac{\alpha_s(\mu_c) - \alpha_s(\mu_0)}{4\pi} \right] \tag{29}
$$

with $\mu_0 = m_{c,\mathrm{pole}}$. Numerically, we get the values shown in table 2. In Figure 2 our results are given for three different values of μ_b, where μ_b represents the usual renormalization

$\bar{m}_c(1.25)/m_b = 0.257$	$\bar{m}_c(2.5)/m_b = 0.214$	$\bar{m}_c(5.0)/m_b = 0.187$

Table 2: $\bar{m}_c(\mu_c)/m_b$ for $\mu_c = 1.25, 2.5, 5$ GeV using $m_{c,\mathrm{pole}}/m_b = 0.29$ as input.

scale of the effective field theory. We compare the branching ratio for $b \to X_s\gamma$ within the pole and the $\overline{\mathrm{MS}}$ scheme for the charm quark mass. Within each vertical string the solid dot represents the branching ratio using $m_{c,\mathrm{pole}}$, while the open symbols correspond to $\bar{m}_c(\mu_c)$ for $\mu_c = 1.25$ GeV (triangle), $\mu_c = 2.5$ GeV (quadrangle) and $\mu_c = 5.0$ GeV (pentagon), respectively.

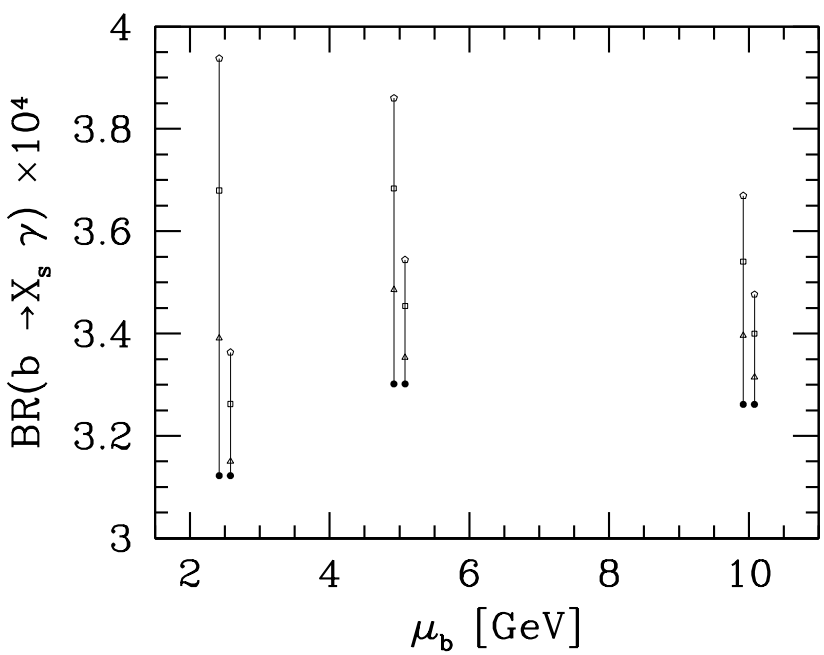

Figure 2: $\mathrm{BR}(b \to X_s\gamma)$ for three values of μ_b. For each value of μ_b the left string shows the NLL results for $m_{c,pole}$ (solid dot) and for $\bar{m}_c(\mu_c)$ with $\mu_c = 1.25; 2.5; 5.0$ GeV (open symbols). The right strings show the corresponding NLL results supplemented by the δm_c mass insertions and the $\Sigma_1(p^2 = m_c^2)$ insertions (see text for more details).

For each μ_b the left string shows the value of the branching ratio at the NLL level, while the right string shows the corresponding value where in addition δm_c mass insertions and $\Sigma_1(p^2 = m_c^2)$ insertions were taken into account, as explained in detail in section 2. Because the combination of these insertions is zero by construction for the pole scheme (see eq. (8)), the solid dots are at the same place in the left and the right string for a

given value of μ_b.

From Figure 2 we see that the error related to the charm quark mass definition gets significantly reduced when taking into account NNLL terms connected with mass insertions. Taking as an example the results for $\mu_b = 5$ GeV, we find that at the NLL level the branching ratio evaluated for $\bar{m}_c(2.5\text{ GeV})$ is 12.6% higher than the one based on $m_{c,\text{pole}}$, in agreement with ref. [23]. Including the new contributions, these 12.6% get reduced to 5.1%.

A remark concerning the remaining NNLL terms is in order: As these terms give contributions to the branching ratio which (up to terms of order α_s^3) do not depend on charm quark mass definition, the error related to m_c in the full NNLL result is expected to stay essentially the same as estimated in the present paper.

However, to obtain a NNLL prediction for the central value of the branching ratio, it is of course necessary to calculate all NNLL terms.

Summing up, we have shown that the relatively large error related to the definition of the charm quark mass in the NLL result for $\text{BR}(b \to X_s\gamma)$ gets significantly reduced (typically by a factor of 2) at the NNLL level. Taking into account the present experimental accuracy of around 10% and the future prospects of the B factories and also of possible Super-B factories [40, 41], we conclude that a future NNLL QCD calculation of the $b \to X_s\gamma$ branching ratio will significantly increase the sensitivity of this observable to possible new physics.

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