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QCD and a Holographic Model of Hadrons

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We propose a five-dimensional framework for modeling low energy properties of QCD. In the simplest three parameter model we compute masses, decay rates and couplings of the lightest mesons. The model fits experimental data to within 5%. The framework is a holographic version of the QCD sum rules, motivated by the AdS/CFT correspondence. The model naturally incorporates properties of QCD dictated by chiral symmetry, which we demonstrate by deriving the Gell-Mann-Oakes-Renner relationship for the pion mass.

Introduction.—QCD has eluded an analytic solution, despite extensive efforts applied to this problem in the past 30 years. Recently, the gravity/gauge, or AdS/CFT correspondence [1] has revived the hope that QCD can be reformulated as a solvable string theory. So far, theories which can be solved using AdS/CFT techniques differ substantially from QCD, most notably by the strong coupling in the ultraviolet (UV) regime and the lack of asymptotic freedom. Nevertheless, certain important properties of QCD such as confinement and chiral symmetry breaking are present in many of these theories, and the gravity/gauge duality provides a new approach to studying the resulting dynamics. An important development in the prototypical example of $\mathcal{N}=4$ super Yang-Mills (SYM) theory has been the introduction of fundamental quarks using probe D7 branes [2]. The mesons that appear in these theories behave in many ways similarly to the mesons in QCD [3, 4].

Inspired by the gravity/gauge duality we propose the following complementary approach. Rather than deform the SYM theory to obtain QCD, we start from QCD and attempt to construct its five-dimensional (5D) holographic dual. In this Letter, we present an exploratory study of a simple holographic model of QCD. The field content of the 5D theory is chosen to reproduce holographically the dynamics of chiral symmetry breaking in QCD, the boundary theory. The model has four free parameters, one of which is fixed by the number of colors; the remaining three parameters can be fitted using three well-measured observables, *e.g.* the rho meson mass, the pion mass, and the pion decay constant. The model then predicts other low-energy hadronic observables with surprisingly good accuracy.

Such an approach is similar in spirit to the construction of the QCD moose theory in Ref. [5], where the holographic description arises in the limit of infinitely many hidden local symmetries. Unlike Ref. [5] we start directly in the continuum limit. See also Ref. [6]. We expect the success of our model to diminish above roughly the scale given by the mass of the lightest isospin-carrying spin-2 resonance, namely the a_2 (1318 MeV [12]). In particular, we are completely neglecting stringy physics which becomes important at higher energies, and we have not included in our description any modes with spin larger than one. We also neglect running of the QCD coupling, which we have no reason to expect to be a consistent assumption for a larger range of energies. While our model is too simple to provide a complete dual description of QCD, its success seems to suggest that there is a quantitatively useful reformulation of QCD as a string theory in a higher-dimensional curved space.

Field content.—Table I illustrates the field content of our model. The choice of the 5D fields is dictated by a principle of the AdS/CFT correspondence: each operator $\mathcal{O}(x)$ in the 4D field theory corresponds to a field $\phi(x,z)$ in the 5D bulk theory. The 5D theory dual to QCD should, therefore, contain an infinite number of fields corresponding to the infinite number of operators in QCD. There is, however, a small number of operators that are important in the chiral dynamics: the left- and right-handed currents corresponding to the $SU(N_f)_L\times SU(N_f)_R$ chiral flavor symmetry, and the chiral order parameter (see Table I). We shall include in our model only the 5D fields which correspond to these operators and neglect all other fields.

TABLE I:

4D: $\mathcal{O}(x)$	5D: $\phi(x,z)$	p	Δ	$(m_5)^2$
$\bar{q}_L\gamma^\mu t^a q_L$	$A^a_{L\mu}$	1	3	0
$\bar{q}_R\gamma^\mu t^a q_R$	$A^a_{R\mu}$	1	3	0
$\overline{q}^\alpha_R q^\beta_L$	$(2/z)X^{\alpha\beta}$	0	3	-3

The 5D masses m_5 of the fields $A^a_{L\mu}$, $A^a_{R\mu}$, and X are determined via the relation [7, 8] $(\Delta+p)(\Delta+p-4)=m_5^2$, where Δ is the dimension of the corresponding p-form operator – see Table I. We assumed here that these operators keep their canonical dimensions, which is true only for the conserved currents. However, for the field X we could easily incorporate corrections to its classical dimension. The factor $1/z$ in Table I is dictated by the dimension of the operator $\bar{q}q$, while the factor of 2 is of no physical significance and is chosen for later convenience.

We shall choose the simplest possible metric for our model, namely a slice of the Anti-de Sitter (AdS) metric.

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$$ds^2 = \frac{1}{z^2}(-dz^2 + dx^\mu dx_\mu), \quad 0 < z \le z_m. \qquad (1)$$

The fifth coordinate z corresponds to the energy scale, as higher energy (or momentum transfer Q^2) QCD physics is reflected by the behavior of the fields closer to the AdS boundary $z = 0$: $Q \sim 1/z$. By virtue of the conformal isometry of the AdS space, in such a model the running of the QCD gauge coupling is neglected in a window of scales until an infrared (IR) scale $Q_m \sim 1/z_m$. To make the theory confining, one introduces an IR cutoff in the metric at $z = z_m$ where spacetime ends, in analogy with the case of the cascading gauge theory studied in [9]. We shall call $z = z_m$ the "infrared brane" and impose certain boundary conditions on the fields at $z = z_m$. Certainly, this is only a crude model of confinement. Indeed, our model requires two dimensionful parameters related to chiral symmetry breaking, whereas in QCD there is only one. In addition, an UV cutoff can be provided by setting the boundary to $z = \epsilon$ instead of $z = 0$. Below we shall frequently use such a cutoff as a mathematical tool, but we shall always imply the limit of $\epsilon \to 0$.

5D action and chiral symmetry breaking.—The action of the theory in the bulk is:

$$S = \int d^5x \sqrt{g}\, \mathrm{Tr}\left\{|DX|^2 + 3|X|^2 - \frac{1}{4g_5^2}(F_L^2 + F_R^2)\right\} \quad (2)$$

where $D_\mu X = \partial_\mu X - iA_{L\mu}X + iXA_{R\mu}$, $A_{L,R} = A^a_{L,R}t^a$, $F = dA$. As usual in AdS/CFT correspondence, the gauge invariance in the 5D bulk theory corresponds to the conservation of the global symmetry current in the 4D theory.

At the IR brane we must impose some gauge invariant boundary conditions, and we make a natural choice: $(F_L)_{z\mu} = (F_R)_{z\mu} = 0$. The QCD dynamics does not *a priori* fix this boundary condition, and to some extent allowing the IR scale z_m to vary in the model compensates for a variation in the IR boundary condition. In the following we shall be using the gauge $A_z = 0$. In this case our boundary conditions are simply Neumann.

The expectation value of the field X is determined by the classical solution satisfying the UV boundary condition $(2/\epsilon)X(\epsilon) = M$ for quark mass matrix M:

$$X_0(z) = \frac{1}{2}Mz + \frac{1}{2}\Sigma z^3, \qquad (3)$$

The matrix Σ is determined by the IR boundary condition on X. Instead of specifying this condition we shall choose Σ as an input parameter of the model. The meaning of Σ in QCD can be found by calculating the variation of the vacuum energy w.r.t. M [10]: $\Sigma^{\alpha\beta} = \langle \bar{q}^\alpha q^\beta \rangle$. We shall assume, as usual, $\Sigma = \sigma\mathbf{1}$ and take $M = m_q\mathbf{1}$.

At this stage the model has four free parameters: m_q, σ, z_m and g_5. The gauge coupling g_5 will be fixed by the QCD operator product expansion (OPE) for the product of currents, leaving three adjustable parameters.

In this Letter we will focus on $N_f = 2$ lightest flavors and neglect effects of $O(m_q^2)$. Therefore, in Table I, $\alpha, \beta = 1, 2$; $a, b = 1, 2, 3$ and $t^a = \sigma^a/2$, where σ^a are the Pauli matrices.

Matching the 5D gauge coupling.—We will use the holographic duality to relate of the 5D coupling g_5 in (2) to the number of colors N_c in QCD. The precise sense of the holographic correspondence is the equivalence between the generating functional of the connected correlators in the 4D theory $W_{4D}[\phi_0(x)]$ and the effective action of the 5D theory $S_{5D,\mathrm{eff}}[\phi(x,\epsilon)]$, with UV boundary values of the 5D bulk fields set to the value of the sources in 4D theory:

$$W_{4D}[\phi_0(x)] = S_{5D,\mathrm{eff}}[\phi(x,\epsilon)] \quad \text{at} \quad \phi(x,\epsilon) = \phi_0(x). \ (4)$$

QCD Green's functions can therefore be obtained by differentiating the 5D effective action with respect to the sources. In the case that stringy effects can be neglected, $S_{5D,\mathrm{eff}}$ is simply given by Eq. (2). The action is evaluated on solutions to the 5D equations of motion subject to the condition that the value of each bulk field at the boundary $z = \epsilon \to 0$ be given by the source ϕ of the corresponding 4D operator $\mathcal{O}$ (see Table I).

We may now fix the 5D gauge coupling by comparing the result for the vector current two-point function obtained from the above prescription with that of QCD. Introducing the vector field as $V = (A_L + A_R)/2$, one finds, in the $V_z(x,z) = 0$ gauge, the equation of motion for the transverse part of the gauge field:

$$\left(\partial_z\left(\frac{1}{z}\partial_z V^a_\mu(q,z)\right) + \frac{q^2}{z}V^a_\mu(q,z)\right)_\perp = 0. \qquad (5)$$

Here $V^a_\mu(q,z)$ is the 4D Fourier transform of $V^a_\mu(x,z)$. The equations of motion are linearized, as is appropriate for determination of two-point functions. Evaluating the action on the solution leaves only the boundary term

$$S = -\frac{1}{2g_5^2}\int d^4x \left(\frac{1}{z}V^a_\mu \partial_z V^{\mu a}\right)_{z=\epsilon}. \qquad (6)$$

If $V_0^{\mu a}(q)$ is the Fourier transform of the source of the vector current $J^a_\mu = \bar{q}\gamma_\mu t^a q$ at the boundary then letting $V^\mu(q,z) = V(q,z)V_0^\mu(q)$, we require that $V(q,\epsilon) = 1$. Differentiating twice with respect to the source V_0, we arrive at the vector current two-point function,

$$\int_x e^{iqx}\langle J^a_\mu(x)J^b_\nu(0)\rangle = \delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\Pi_\mathrm{V}(Q^2), \quad (7a)$$

$$\Pi_\mathrm{V}(-q^2) = -\frac{1}{g_5^2 Q^2}\frac{\partial_z V(q,z)}{z}\bigg|_{z=\epsilon}, \quad (7b)$$

where $Q^2 = -q^2$. For large Euclidean Q^2 we only need to know $V(q,z)$ near the boundary,

$$V(Q,z) = 1 + \frac{Q^2 z^2}{4}\ln(Q^2 z^2) + \cdots \qquad (8)$$

which up to contact terms gives

$$\Pi_{\rm V}(Q^2) = -\frac{1}{2g_5^2}\ln Q^2. \quad (9)$$

On the other hand, we can compute Π_V from QCD by evaluating Feynman diagrams [11]. The leading-order diagram is the quark bubble,

$$\Pi_{\rm V}(Q^2) = -\frac{N_c}{24\pi^2}\ln Q^2. \quad (10)$$

This leads to the identification

$$g_5^2 = \frac{12\pi^2}{N_c}\,, \quad (11)$$

which completes the definition of the action (2).

Hadrons.—The hadrons of QCD correspond to the normalizable modes of the 5D fields. These normalizable modes satisfy the linearized equation of motion and decay sufficiently rapidly near the boundary $z \to 0$ so as to have a finite action. The IR boundary condition gives rise to a discrete tower of normalizable modes. The eigenvalue a normalizable mode is the squared mass of the corresponding meson, and the derivative of the mode near the UV boundary yields the decay constant.

To illustrate the above, consider the tower of the ρ mesons. A ρ wavefunction, $\psi_\rho(z)$, is a solution to Eq. (5) for an arbitrary component of V_μ with $q^2 = m_\rho^2$, subject to $\psi_\rho(\epsilon) = 0$, $\partial_z\psi_\rho(z_m) = 0$ and normalized as $\int(dz/z)\,\psi_\rho(z)^2 = 1$. Consider the Green's function corresponding to Eq. (5) for an arbitrary component of V^μ:

$$G(q;z,z') = \sum_\rho \frac{\psi_\rho(z)\psi_\rho(z')}{q^2 - m_\rho^2}. \quad (12)$$

One can show that $V(q,z')$ of Eq.(7b) is given by $-(1/z)\partial_z G(q;z,z')$ at $z = \epsilon$. Now from (7b) we find:

$$\Pi_V(-q^2) = -\frac{1}{g_5^2}\sum_\rho \frac{[\psi_\rho{}'(\epsilon)/\epsilon]^2}{(q^2 - m_\rho^2)m_\rho^2}. \quad (13)$$

This allows us to extract the weak decay constants F_ρ:

$$F_\rho^2 = \frac{1}{g_5^2}[\psi_\rho{}'(\epsilon)/\epsilon]^2 = \frac{1}{g_5^2}[\psi_\rho{}''(0)]^2, \quad (14)$$

where F_ρ is defined by $\langle 0|J_\mu^a|\rho^b\rangle = F_\rho\delta^{ab}\varepsilon_\mu$ for a ρ meson with polarization ε_μ. Eqs. (9), (13) are the holographic version of the QCD sum rules.

In the axial sector (a_1 and π mesons), the action to quadratic order is

$$S = \int d^5x\,\left[-\frac{1}{4g_5^2 z}F_A^a F_A^a + \frac{v(z)^2}{2z^3}(\partial\pi^a - A^a)^2\right], \quad (15)$$

where we have defined $v(z) = m_q z + \sigma z^3$, $A = (A_L - A_R)/2$, and $X = X_0\exp(i2\pi^a t^a)$. In the $A_z = 0$ gauge, the resulting equations of motion in 4D momentum space are ($A_\mu = A_{\mu\perp} + \partial_\mu\varphi$):

$$\left(\partial_z\left(\frac{1}{z}\partial_z A_\mu^a\right) + \frac{q^2}{z}A_\mu^a - \frac{g_5^2 v^2}{z^3}A_\mu^a\right)_\perp = 0; \quad (16)$$

$$\partial_z\left(\frac{1}{z}\partial_z\varphi^a\right) + \frac{g_5^2 v^2}{z^3}(\pi^a - \varphi^a) = 0; \quad (17)$$

$$-q^2\partial_z\varphi^a + \frac{g_5^2 v^2}{z^2}\partial_z\pi^a = 0. \quad (18)$$

The a_1, being a spin-1 particle, is the solution to Eq.(16) with $\psi_{a_1}(0) = \partial_z\psi_{a_1}(z_m) = 0$. The a_1 decay constant, F_{a_1}, is given by an expression similar to Eq.(14), but with ρ replaced by a_1.

Our theory has all the consequences of chiral symmetry built in. Let us derive the Gell-Mann–Oakes–Renner (GOR) relation,

$$m_\pi^2 f_\pi^2 = (m_u + m_d)\langle\bar{q}q\rangle = 2m_q\sigma. \quad (19)$$

Since $\langle 0|A_\mu|\pi\rangle = if_\pi q_\mu$, the axial current correlator in the $m_\pi = 0$ limit has a singularity at $q^2 = 0$: $\Pi_{\rm A}(-q^2) \to -f_\pi^2/q^2$. Using the holographic recipe [cf. Eq. (7)],

$$f_\pi^2 = -\frac{1}{g_5^2}\left.\frac{\partial_z A(0,z)}{z}\right|_{z=\epsilon}, \quad (20)$$

where $A(0,z)$ is the solution to Eq.(16) with $q^2 = 0$, satisfying $A'(0,z_m) = 0$, $A(0,\epsilon) = 1$. The pion is the solution to Eqs. (17) and (18), subject to $\varphi'(z_m) = \varphi(\epsilon) = \pi(\epsilon) = 0$. We may construct such a solution perturbatively in m_π by letting $\varphi(z) = A(0,z) - 1$. Then, from Eq. (18), to leading order in m_π^2,

$$\pi(z) = m_\pi^2\int_0^z du\frac{u^3}{v(u)^2}\cdot\frac{1}{g_5^2 u}\partial_u A(0,u)\,. \quad (21)$$

The function $u^3/v(u)^2$ has a significant support only for $u \sim z_c \equiv \sqrt{m_q/\sigma}$. The function $\partial_u A(0,u)/(g_5^2 u)$ for such small values of u can be replaced by its value at $u = \epsilon$, which is related to f_π via (20). Performing the integral one finds that $\pi = -m_\pi^2 f_\pi^2/(2m_q\sigma)$ for $z \gg z_c$. Equations (16) and (17) are solved by $\varphi = A(0,z) - 1$ and $\pi = $ const for $z \gg z_c$ only if $\pi = -1$, hence $m_\pi^2 f_\pi^2 = 2m_q\sigma + \mathcal{O}(m_q^2)$.

Meson interactions and $g_{\rho\pi\pi}$—The meson interactions can be read from the non-bilinear terms in the 5D action,

$$S_{\pi\rho} = \int d^5x\,\frac{v(z)^2}{2z^3}(\partial\pi^a - A^a + \epsilon^{abc}V_b\pi_c)^2. \quad (22)$$

The π-ρ coupling is therefore given by:

$$g_{\rho\pi\pi} = g_5\int dz\,\frac{v^2(z)}{z^3}\,(\pi - \varphi)(z)\,\pi(z)\,\psi_\rho(z), \quad (23)$$

where the pion wavefunction π is normalized such that the pion kinetic term is canonically normalized, *i.e.* the integral in (23) equals 1 if we substitute 1 for $\psi_\rho(z)$.

TABLE II: Results of the model for QCD observables. Model A is a fit of the three model parameters to m_π, f_π and m_ρ (see asterisks). Model B is a fit to all seven observables.

Observable	Measured (MeV)	Model A (MeV)	Model B (MeV)
m_π	139.6±0.0004 [12]	139.6*	140
m_ρ	775.8±0.5 [12]	775.8*	793
m_{a_1}	1230±40 [12]	1363	1256
f_π	92.4±0.35 [12]	92.4*	86.5
$F_\rho^{1/2}$	345±8 [13]	329	337
$F_{a_1}^{1/2}$	433±13 [5, 14]	452	449
$g_{\rho\pi\pi}$	6.03±0.07 [12]	5.43	6.05

Predictions.—The model described above in some ways mimics the features of QCD sum rules, and we naturally expect our model to work about as well, at roughly the 10–20% level.

The ρ wavefunction is the lowest resonance satisfying (5). The solutions to (5) are Bessel functions and the eigenstates have masses determined by zeroes of $J_0(qz_m)$. Hence, $m_\rho = 2.405/z_m$. The experimental value of the rho mass, $m_\rho = 775.8 \pm 0.5$ MeV [12], then fixes $z_m = 1/(323 \text{ MeV})$. The remaining parameters, m_q and σ, are fit to the experimental values $m_\pi = 139.6 \pm 0.0004$ MeV [12] and $f_\pi = 92.4 \pm 0.35$ MeV [12], yielding $m_q = 2.29$ MeV and $\sigma = (327 \text{ MeV})^3$. With these values of the parameters (Model A), the remaining four observables which we have calculated in this model are given in the third column of Table II. As a measure of the success of our model we used the RMS error, $\varepsilon_{\rm RMS} = \left(\sum_O (\delta O/O)^2/n\right)^{1/2}$, where $\delta O/O$ is the fractional error of an observable O and $n = 4$ equals number of observables minus number of parameters. The RMS error for Model A is 8%.

We also performed a global fit of the model to the central values of all seven observables simultaneously by minimizing the RMS error. The best fit is given by $z_m = 1/(330 \text{ MeV})$, $m_q = 2.34$ MeV and $\sigma = (311 \text{ MeV})^3$. The values of the observables with this fit (Model B) are shown in the last column of Table II. The RMS error of this fit is a remarkably small 4%.

Discussion and outlook.—The holographic model of QCD studied here is quite crude and depends on only three free parameters, but it agrees surprisingly well with the seven experimentally measured observables which we have studied. There are several ways in which we may attempt to extend and improve the model. (i) The glueball spectrum can be calculated from the gravitational and dilaton modes in the theory, which were not included in this study. (ii) It is straightforward to describe power corrections in the current correlators [15]. Here we matched the gauge coupling g_5 in our model to the leading term – the unit operator – in the OPE of the product of currents. Higher dimension operators also appear in the OPE, suppressed by powers of the Euclidean momentum Q. These corrections can be calculated in QCD [11]. In the holographic model, these corrections arise from trilinear and higher terms in the 5D action, such as $\int d^5x \sqrt{g} X^2 F^2$. Matching the QCD OPE coefficients to the coefficients of the 5D action provides a method of building and constraining the effective 5D action. (iii) Including the strange quark into the model with an approximate SU(3)×SU(3) chiral symmetry is a natural extension of the model. (iv) The chiral anomaly can be incorporated via a 5D Chern-Simons term. (v) We can include corrections to the dimensions of the chiral order parameters by varying the mass of the corresponding fields X in the 5D theory, and we can include running of the gauge coupling via logarithmic corrections to the AdS geometry. It is interesting to note in this context, that those results which follow from partial conservation of the axial current, *e.g.* the GOR relation, continue to hold as we vary the 5D mass of X in the model [15].

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