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A Differential Algebraic Integration Algorithm for Symplectic Mappings in Systems with Three-Dimensional Magnetic Field

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Abstract

A differential algebraic integration algorithm is developed for symplectic mapping through a three-dimensional (3-D) magnetic field. The self-consistent reference orbit in phase space is obtained by making a canonical transformation to eliminate the linear part of the Hamiltonian. Transfer maps from the entrance to the exit of any 3-D magnetic field are then obtained through slice-by-slice symplectic integration. The particle phase-space coordinates are advanced by using the integrable polynomial procedure. This algorithm is a powerful tool to attain nonlinear maps for insertion devices in synchrotron light source or complicated magnetic field in the iteraction region in high energy colliders.

Key words:
Differential algebras, Symplectic integration, particle tracking, transfer maps
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1 Introduction

In most storage rings, magnetic field can usually be well approximated by dominant transverse components. Thus, a single longitudinal component of the magnetic vector potential $A_s(x, y, s)$ is sufficient to describe the system. Because the Hamiltonian can be separated into a term that depends only on coordinates and another on momenta, the phase-space coordinates can be advanced with the conventional symplectic integrators (1).

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However, a general 3-D magnetic field can not be represented by a vector potential A_s alone. The vector potential needs at least two transverse components, e.g. A_x and A_y. The Hamiltonian can no longer be separated into the coordinate-dependent-only and the momenta-dependent-only terms. This paper presents a new algorithm to obtain self-consistent symplectic maps for general 3-D magnetic field. In Sec. 2, a differential algebraic Drift-kick-kick-drift (DKKD) procedure is presented to obtain the self-consistent reference orbit and the Hamiltonian expanded around the reference orbit. In Sec. 3, we discuss sympelctic mapping and particle tracking. The conclusion is given in Sec. 4.

2 The Hamiltonian and the determination of reference orbit

After being normalized to the mechanical momentum magnitude of a reference synchronous particle, the dimensionless Hamiltonian for a charged particle in a 3-dimensional magnetic field can be given, in Frenet-Serret coordinate system, by

$$H = -A_s - \left(1 + \frac{x}{\rho}\right)\left[(1+\delta)^2 - (p_x - A_x(x,y,s))^2 - (p_y - A_y(x,y,s))^2\right]^{1/2}, \tag{1}$$

where ρ is the radius of curvature, δ is the fractional momentum deviation, p_x and p_y are the canonical conjugate momenta, A_x, A_y, and A_s are the magnetic vector potential components that are functions of the coordinates, x, y, s, where s is the time-like coordinate along the longitudinal direction while x, y are the transverse coordinates. Note that eliminating the $\frac{x}{\rho}$ term would simply yield the the Hamiltonian in the local Cartesian coordinate system. The mixing of conjugate momenta (p_x, p_z) with the vector potential A_x and A_y that are functions of the coordinates (x, y) poses the challenge of obtaining symplectic Taylor maps for particle tracking. This problem is tackled by using a differential algebraic drift-kick-kick-drift (DKKD) algorithm to derive the self-consistent reference orbit and obtain Taylor map with respect to the reference orbit.

We consider a region of 3D magnetic field, longitudinally located between s_i and s_f with $L = s_f - s_i$. The system is divided into N-slices such that the length of each slice is $\Delta s = L/N$. The entry, mid-point, and exit of the k-th slice are $s_k = s_i + (k-1)\Delta s$, $\bar{s}_k = s_i + (k-1/2)\Delta s$, and $s_{k+1} = s_i + k\Delta s$. The differential algebraic DKKD algorithm is used to obtain the particle's reference trajectory: $\vec{z}_0(\delta, s) \equiv (x_0(\delta, s), p_{x0}(\delta, s), y_0(\delta, s), p_{y0}(\delta, s))$. The Hamiltonian is expanded around this reference orbit.

(1) We evaluate the Hamiltonian $H(x, p_x, y, p_y, \delta; s_k)$, $dx/ds = \partial H/\partial p_x$ and

$dy/ds = \partial H/\partial p_y$ at the slice-interface s_k. The coordinates of reference orbit are advanced to the mid-point $\bar{s}_k$ by

$$x_0(\delta, \bar{s}_k) = x_0(\delta, s_k) + (\Delta s/2) \cdot (\partial H/\partial p_x)|_{\vec{z}_0(\delta, s_k)}\,, \tag{2}$$
$$y_0(\delta, \bar{s}_k) = y_0(\delta, s_k) + (\Delta s/2) \cdot (\partial H/\partial p_y)|_{\vec{z}_0(\delta, s_k)}\,. \tag{3}$$

(2) We now evaluate H with the updated new reference coordinates, $dp_x/ds = -\partial H/\partial x$ and $dp_y/ds = -\partial H/\partial y$ at the mid-point of the slice. The conjugate momenta of reference orbit at the mid-point are advanced by

$$p_{x_0}(\delta, \bar{s}_k) = p_{x_0}(\delta, s_k) - (\Delta s/2) \cdot (\partial H/\partial x)|_{\vec{z}_0(\delta, \bar{s}_k)}\,, \tag{4}$$
$$p_{y_0}(\delta, \bar{s}_k) = p_{y_0}(\delta, s_k) - (\Delta s/2) \cdot (\partial H/\partial y)|_{\vec{z}_0(\delta, \bar{s}_k)}\,. \tag{5}$$

(3) The Hamiltonian $H(x, p_x, y, p_y, \delta; \bar{s}_k)$, $dp_x/ds = -\partial H/\partial x$, and $dp_y/ds = -\partial H/\partial y$ at the mid-point can be evaluated. Changing the conjugate phase-space coordinates ($X = x - x_0, p_X = p_x - p_{x_0}, Y = y - y_0, p_Y = p_y - p_{y_0}$) at the mid-point $\bar{s}_k$, one can evaluate and save the Hamiltonian, $H(\delta, \bar{s}_k) = h_k + f_k$. Which describes the dynamics and is Taylor-expanded around the parameterized reference orbit. It is integrated with the previous concatenated map by a second-order symplectic integrator for obtaining the section-map.

(4) With the $dp_x/ds = -\partial H/\partial x$ and $dp_y/ds = -\partial H/\partial y$ obtained at this mid-point, the conjugate momenta of reference orbit are then advanced to the end-point s_{k+1} of the k-th slice by

$$p_{x_0}(\delta, s_{k+1}) = p_{x_0}(\delta, \bar{s}_k) - (\Delta s/2) \cdot (\partial H/\partial x)|_{\vec{z}_0(\delta, \bar{s}_k)}\,, \tag{6}$$
$$p_{y_0}(\delta, s_{k+1}) = p_{y_0}(\delta, \bar{s}_k) - (\Delta s/2) \cdot (\partial H/\partial y)|_{\vec{z}_0(\delta, \bar{s}_k)}\,. \tag{7}$$

(5) The H at the mid-point is re-evaluated with the updated new reference momenta. The $dp_x/ds = -\partial H/\partial x$ and $dp_y/ds = -\partial H/\partial y$ are also obtained. The coordinates of reference orbit are advanced to the end-point s_{k+1} of the k-th slice by

$$x_0(\delta, s_{k+1}) = x_0(\delta, \bar{s}_k) + (\Delta s/2) \cdot (\partial H/\partial p_x)|_{\vec{z}_0(\delta, \bar{s}_k)}\,, \tag{8}$$
$$y_0(\delta, s_{k+1}) = y_0(\delta, \bar{s}_k) + (\Delta s/2) \cdot (\partial H/\partial p_y)|_{\vec{z}_0(\delta, \bar{s}_k)}\,. \tag{9}$$

Using conjugate phase-space coordinates at the end-point s_{k+1}, one evaluates the Hamiltonian $H(x, p_x, y, p_y, \delta; s_{k+1})$, $dp_x/ds = -\partial H/\partial x$ and $dp_y/ds = -\partial H/\partial y$. The integration procedure repeats for the next slice.

In this integration process, an important constraint is the continuity of the vector potential at the interface of each slice. Since this process uses differential algebras, the dependence of the reference orbit on δ is included up to a desired order. The transfer map is given by $\exp\{-\Delta s : \tilde{H}(x, p_x, y, p_y, \delta, \bar{s}_k) :\}$, where the canonical phase-space coordinates are with respect to the δ-dependent

reference orbit. Note that the reference orbit may also be obtained by solving the tedious differential equations that turn out to obey the Lorentz force law after making a canonical transformation with the generating function

$$F_2 = (x - x_0)(p_X + p_{x_0}) + (y - y_0)(p_Y + p_{y_0}),$$

such that the linear part of the transformed Hamiltonian vanishes (2).

The DKKD integration algorithm does not require symplecticity. It is a natural method to derive the self-consistent reference orbit and obtain the Hamiltonian for symplectic mapping using Lie operator method. One can also use the Runge-Kutta integration methods to solve the reference orbit and derive the Hamiltonian. However, the continuity condition must be carefully implemented in order to obtain a proper Hamiltonian for transfer map.

3 Map Concatenation

For convenience, let the state vector $\vec{z}$ represent the transverse particle canonical phase-space coordinates with respect to the reference orbit, i.e. $\vec{z} \equiv (x, p_x, y, p_y)^\dagger$. The Hamiltonian $H(\vec{z}, \delta, \bar{s})$ is a polynomial with a minimum order of 2 and the transfer map that advances the particle phase-space coordinates (with respect to the reference orbit) from the entrance to the exit for the k^{th} slice is given by $\exp\{-\Delta s : H(z, \delta, \bar{s}_k) :\}$.

One can choose to track particles slice by slice. But this would be quite CPU time consuming. Instead, one can choose to concatenate all the slice transfer maps into one for fast tracking. To do so, we first reform each of the slice transfer maps into two major terms, one for the linear motion and the other for the nonlinear motion. For example, the k^{th}-slice transfer map would be reformed as

$$\exp\left\{-\Delta s : \sum_{j=2} H_j(\bar{s}_k) :\right\} = \exp\{: h_k + f_k :\}$$

where the linear part $h_k \equiv -\Delta s \cdot H_2(\bar{s}_k)$ and the nonlinear part $f_k \equiv -\Delta s \cdot \sum_{j=3} H_j(\bar{s}_k)$ can be separated by a symplectic integrator. Using the second order symplectic integrator, one obtains

$$\exp\{: h_k + f_k :\} = \exp\left\{: \frac{1}{2} h_k :\right\} \exp\{: f_k :\} \exp\left\{: \frac{1}{2} h_k :\right\}, \tag{10}$$

and the whole map as

$$M = \prod_{k=1}^{N} \left\{ e^{:\frac{1}{2}h_k:} e^{:f_k:} e^{:\frac{1}{2}h_k:} \right\} = e^{:h:} e^{:f:}, \tag{11}$$

where $e^{:h:}$ is the concatenated linear map. The nonlinear map $e^{:f:}$ can be evaluated by the method of integrable polynomials (3). Therefore, the initial phase-space coordinates $\vec{Z}(s_i)$ at the entrance of a subsystem can be directly mapped to the final coordinates $\vec{Z}(s_f)$ at the exit of the subsystem. One must keep in mind the continuity condition of mechanical momenta required at the entrance and the exit. If the transverse vector potentials are zero at both the entry s_i and the exit s_f, then the transverse canonical momenta is equal to their corresponding mechanical momenta. Otherwise, one should make additional transformation between mechanical momenta and canonical momenta at the entrance and the exit.

We use a quadrupole to illustrate and check the validity of our algorithm. Conventionally, the vector potential of a quadrupole field is represented by a single longitudinal component $\vec{A} \equiv (A_x, A_y, A_s) = \left(0, 0, \frac{B_1}{2}(y^2 - x^2)\right)$, where $B_1 = \partial B_y/\partial x$. To check the slice integration algorithm, the Quadrupole vector potential is represented by two transverse components, $\vec{A} = (B_1 xs, -B_1 ys, 0)$ for a valid comparison of the slice-by-slice integrated numerical transfer matrix to the cooresponding analytic one. The parameters in our calculation are $K = B_1/B_0\rho = 2.870480\ \mathrm{m}^{-2}$, the quadrupole-length $L = 0.35$ m, and $B_0\rho = -5.00346$ T-m.

We set the entry position $s_i = 0$ such that particle's mechanical momenta are the same as the canonical momenta. However, at the exit position, $s_f = L^-$, an additional transformation has to be made for the transfer map from canonical momenta to mechanical momenta that become the canonical momenta at $s = L^+$ provided that the vector potential outside the quadrupole is 0. This is because a hard-edge quadrupole prevents us from making vector potential continuous on both boundaries. If the fringe fields have been included to achieve vector potential continuity at both boundaries, the additional transformation at $s_f = L$ would not be necessary.

The linear transfer matrix of the quadrupole agrees with the analytic formula to better than 10^{-4}, provided that $N \geq 20$ longitudinal slices are used. One may gain precision by taking higher order symplectic integrators and a larger number of slices. For the second order symplectic integrator, we find the error is proportional to $1/N^2$.

4 Conclusion and Discussion

We have developed a slice-by-slice symplectic transfer mapping in a 3D magnetic field where the vector potential can not be described by A_s alone. We divide the system longitudinally into N-slices and implement a differential algebraic drift-kick-kick-drift (DKKD) procedure to advance the parameterized

($\delta - dependent$) reference orbit phase-space coordinates and simultaneously obtain the Hamiltonian with respect to the reference orbit. Note that the reference orbit is chosen such that the Taylor expanded Hamiltonian is without the first order so that all feed-downs from higher-order multipoles are automatically included in the transfer maps. For fast particle tracking, one may concatenates slice transfer maps into one with a symplectic integrators. That the reference orbit is actually Taylor expanded in momentum deviation δ offers the path length difference and the dispersion functions. Requirement of mechanical momenta continuity can be automatically satisfied if one imposes the continuity condition of the vector potential.

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