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A Note on Mirror Symmetry for Manifolds with Spin(7) Holonomy

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Abstract

Starting from the superconformal algebras associated with G_2 manifolds, I extend the algebra to the manifolds with spin(7) holonomy. I show how the mirror symmetry in manifolds with spin(7) holonomy arises as the automorphism in the extended sperconformal algebra. The automorphism is realized as 14 kinds of T-dualities on the supersymmetric T^4 toroidal fibrations. One class of Joyce's orbifolds are pairwise identified under the symmetry.

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1. Introduction

Mirror symmetry is a beautiful subject both in physics and mathematics. It was first conjectured in [1] that there exists a symmetry which exchanges the complex moduli on one manifold with the Kahler moduli on the dual manifold when we consider the string worldsheet propagation on Calabi-Yau target spaces. The symmetry arises in the sense that the resulting physical spectra of the mirror pair are isomorphic. This requires the Betti numbers of the CY mirror pair satisfy the condition $b_{p,q}(M) = b_{d-p,q}(\tilde{M})$. It was also shown that mirror symmetry could determine non-perturbative effects of worldsheet instantons by counting the number of holomorphic curves in Calabi-Yau spaces [2]. Those who are interested in various aspects of mirror symmetry are referred to [3].

In [4], Strominger, Yau and Zaslow (SYZ) argued that the mirror transformation is equivalent to T-duality on the supersymmetric T^3 fibration in the Calabi-Yau manifolds, by considering the mirror BPS soliton spectra in two theories (IIA/IIB). Some concrete mirror pairs of certain toroidal orbifolds with discrete torsion can be found in [5], where the mirror symmetry is indeed realized as T-duality on toroidal T^3 fibration in the orbifolds. And these examples involving the changes of discrete torsion are related to the main goal of this paper.

In [6][7], Acharya discussed the existence of the mirror symmetry in IIA/IIB string theory compactified on manifolds with exceptional holonomy and argued how the discrete torsion trasforms under the T^4 T-duality. In [8], the authors gave some concrete mirror pairs among Joyce's orbifolds with G_2 holonomy, which are built from resolving or deforming T^7/Z_2^3 orbifolds. They also identified the mirror symmetry as an automorphism in the extended superconformal algebra on manifolds with G_2 holonomy.

Motivated by these known results, I generalize the chiral superconformal algebra to the manifolds with spin(7) holonomy, and identify the corresponding automorphism in the algebra as a combination of a T-duality in 8-direction and a generalized G_2 mirror transformation or a combination of two distinct G_2 mirror transformations. The automorphism could also be understood as T-duality on the supersymmetric T^4 fibrations. In order to make the automorphism clearer, I give an example of one class of Joyce's spin(7) manifolds with discrete torsion. The 14 kinds of T^4 T-dualities are classified into two categories, one of which does flip the discrete torsion and hence lead to a topologically different Joyce's orbifold and the other does not.

The paper is presented as follows. In section 2 I review the mirror symmetry of Calabi-Yau and G_2 manifolds both from the viewpoint of the conformal field theory and the T-duality. In section 3 I will give the construction of spin(7) extended superconformal algebra and identify the automorphism in it as 14 kinds of T-dualities and classify them into two kinds as mentioned above. Section 4 is conclusion and some suggestion for future study.

2. Mirror symmetry for CY and G_2 manifolds

In this section I will give a short review of mirror symmetry on Calabi-Yau orbifolds (T^6/Z_2^2) and Joyce's G_2 manifolds [8][9][10].

2.1. Mirror symmetry of Calabi-Yau threefolds

The generators of the $N=2$ superconformal algebra for string propagation on Calabi-Yau target-space are the stress energy tensor T_{CY}, two supercurrent G_{CY}, G'_{CY} and the U(1) current J_{CY}, along with a complex current Ω_{CY} of conformal weight 3/2 constructed from the worldsheet fermions and its superpartner Ψ_{CY}.

In T^6/Z_2^2 orbifolds, they can be expressed as, [8][11][12]

$$
\begin{aligned}
&T_{CY} = \frac{1}{2}\sum_{j=1}^{6} :\partial x^j \partial x^j: - \frac{1}{2}\sum_{j=1}^{6} :\psi^j \partial \psi^j: \ , \\
&G_{CY} = \sum_{j=1}^{6} :\psi^j \partial x^j: \ , \ G'_{CY} = \sum_{j=1}^{3} (\psi^{2j-1}\partial x^{2j} - \psi^{2j}\partial x^{2j-1}) \ , \ J_{CY} = \sum_{j=1}^{3} \psi^{2j-1}\psi^{2j} \ , \\
&\Omega_{CY} = \psi^1\psi^3\psi^5 - \psi^1\psi^4\psi^6 - \psi^2\psi^3\psi^6 - \psi^2\psi^4\psi^5 + i(\psi^1\psi^3\psi^6 + \psi^1\psi^4\psi^5 + \psi^2\psi^3\psi^5 - \psi^2\psi^4\psi^6) \ , \\
&\Psi_{CY} := \{G_{CY}, \Omega_{CY}\} \ .
\end{aligned}
\tag{2.1}
$$

There exists an automorphism in the superconformal algebra or OPE, which leave invariant the $N=1$ superconformal subalgebra generated by T_{CY} and G_{CY}.

$$
G'_{CY} \rightarrow -G'_{CY}, \quad J \rightarrow -J, \ \ \Omega \rightarrow \Omega^*, \quad \Psi \rightarrow \Psi^*. \tag{2.2}
$$

Calabi-Yau mirror symmetry is to apply the above automorphism to one of the chiralities of the algebra, for instance, $\tilde{G}'_{CY}$, $\tilde{J}_{CY}$, $\tilde{\Omega}_{CY}$, and $\tilde{\Psi}_{CY}$. Recall that the T-duality in ith direction will leave ∂x_i and ψ_i invariant but reverse $\bar{\partial} x_i$ and $\tilde{\psi}_i$. Therefore, we can easily see that the T-duality on T^3 fibrations in the following directions (which appear in the indices of Ω_{CY}) also generates the mirror symmetry.

$$\{(1,3,5),(1,4,6),(2,3,6),(2,4,5),(1,3,6),(1,4,5),(2,3,5),(2,4,6)\} \tag{2.3}$$

Some concrete examples of these T-dualities acting on the T^3 fibration and changing the discrete torsion can be found in [5][8].

2.2. Compact orbifolds with G_2 holonomy

In this and the following sections, I will first give an example of Joyce's orbifolds which were constructed by disingularising T^7/Z_2^3 and how the choices in resolving (deforming) the singularities can result in topologically different spaces. After that, I will write down the G_2 extended chiral superconformal algebra and look for the automorphism in it. We will see that applying the automorphism transformation to one of two chiralities is equivalent to applying a T-duality on certain T^3 toroidal fibration.

Consider the orbifolds of T^7/Γ, where $x_i = x_i + 1$ and Γ is generated by three Z_2,[9]

$$\begin{aligned}
\alpha &= (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7) \ , \\
\beta &= (-x_1, 1/2 - x_2, x_3, x_4, -x_5, -x_6, -x_7) \ , \\
\gamma &= (-x_1, x_2, -x_3, x_4, -x_5, x_6, -x_7) \ .
\end{aligned} \tag{2.4}$$

In order to desingularize the orbifolds, one has to know, for instance, how the 16 α fixed T^3s get identified under the group generated by β and γ. What we found in this example is the 16 T^3s fixed by α or β are reduced to 4 orbits of order 4 by the free-acting of the $<\beta,\gamma>$ or $<\gamma,\alpha>$. In the γ-fixed T^3 sector, the group $<\alpha,\beta>$ only reduce 16 T^3 to 8 orbits of order 2 since $\alpha\beta$ acts trivially on them.

The choices of blowing-up or deforming also come from this γ-fixed sector. From a discrete torsion analysis based on the requirement of modular invariance [8], we know that blowing-up (deforming) corresponds to discrete torsion in the γ-fixed sector $\epsilon_{\gamma;\tilde{f}} = 1$ (-1) and the even (odd) $\alpha\beta$ parity. By virtue of the correspondence between the RR ground states and the cohomology, we can write down the RR ground states in γ-fixed sector.

For $\alpha\beta$ parity even case, we have,

$$\begin{aligned}
&\epsilon_{\gamma;\tilde{f}} = 1 \ , \\
&|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{2+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{4+}\psi^{6+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{2+}\psi^{4+}\psi^{6+}|0,0;\tilde{f}\rangle_\gamma \ ,
\end{aligned} \tag{2.5}$$

where $\tilde{f} = 1,..8$ labelling the γ-fixed points after α or β identification.

For $\alpha\beta$ parity odd case, the RR ground states are,

$$\begin{aligned} &\epsilon_{\gamma;\tilde{f}} = -1 \ , \\ &\psi^{4+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{6+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{2+}\psi^{4+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{2+}\psi^{6+}|0,0;\tilde{f}\rangle_\gamma \ . \end{aligned} \tag{2.6}$$

One should regard $|0,0;\tilde{f}\rangle_\gamma$ as the harmonic two form associated with the exceptional divisors of the blowing-up (deformation). Therefore, blowing-up contributes 1 to b_2 and 1 to b_3 while the deformation increases b_3 by 2.

For the RR ground states in γ-fixed sector, the operation $\alpha\beta$ reverses the 4th and 6th directions. Therefore, we can express it as,

$$\alpha\beta = \frac{1}{4}\psi_0^4\psi_0^6\tilde{\psi}_0^4\tilde{\psi}_0^6\epsilon_{\gamma;\tilde{f}} \ . \tag{2.7}$$

We denote X_l Joyce's manifold with l blow-ups and $8-l$ deformations. After summing up all Betti numbers from various sectors, we have,

$$(b_0, ..., b_7) = (1, 0, 8+l, 47-l, 47-l, 8+l, 0, 1) \ . \tag{2.8}$$

2.3. G_2 extended superconformal algebra

The algebra on manifolds with G_2 holonomy is generated by appending a spin 3/2 operator Φ_{G_2} and its superpartner X_{G2} to the $N=1$ superconformal subalgebra spanned by T_{G_2} and G_{G_2} [11][13]. In our basis of coordinates, they are,

$$\begin{aligned} T_{G_2} &= \frac{1}{2}\sum_{j=1}^{7} :\partial x^j \partial x^j: - \frac{1}{2}\sum_{j=1}^{7} :\psi^j \partial \psi^j: \ , \quad G_{G_2} = \sum_{j=1}^{6} :\psi^j \partial x^j: \ , \\ \Phi_{G_2} &= \psi^1\psi^3\psi^6 + \psi^1\psi^4\psi^5 + \psi^2\psi^3\psi^5 - \psi^2\psi^4\psi^6 + \psi^1\psi^2\psi^7 + \psi^3\psi^4\psi^7 + \psi^5\psi^6\psi^7 \ , \\ X_{G2} &= -\psi^2\psi^4\psi^5\psi^7 - \psi^2\psi^3\psi^6\psi^7 - \psi^1\psi^4\psi^6\psi^7 + \psi^1\psi^3\psi^5\psi^7 - \psi^3\psi^4\psi^5\psi^6 \\ &\quad - \psi^1\psi^2\psi^5\psi^6 - \psi^1\psi^2\psi^3\psi^4 - \frac{1}{2}\sum_{j=1}^{7} :\psi^j \partial \psi^j: \ . \end{aligned} \tag{2.9}$$

The extended superconformal algebra has one obvious automorphism [11][13].

$$\Phi_{G2} \to -\Phi_{G_2}; \quad K_{G2} \to -K_{G2}; \quad T_{G2} \ , \ G_{G2} \ , \ X_{G2} \ , \ M_{G2} \ \textit{unchanged.} \tag{2.10}$$

If the G_2 manifolds are of the form $(CY_3 \times S^1)/Z^2$ as the Joyce G_2 manifolds, we can also reformulate the superconformal generators in terms of the Calabi-Yau ones.

$$
\begin{aligned}
T_{G_2} &= T_{CY} + \frac{1}{2} : \partial x^7 \partial x^7 : - \frac{1}{2} : \psi^7 \partial \psi^7 : , \quad G_{G_2} = G_{CY} + : \psi^7 \partial x^7 : , \\
\Phi_{G_2} &= Im(\Omega_{CY}) + : J_{CY} \psi^7 : , \\
X_{G_2} &= \ : Re(\Omega_{CY}) \psi^7 : + \frac{1}{2} : J_{CY} J_{CY} : - \frac{1}{2} : \psi^7 \partial \psi^7 : , \\
K_{G2} &= Im(\Psi_{CY}) + : J_{CY} \partial x^7 : + : G'_{CY} \psi^7 : , \\
M_{G2} &= \ : Re(\Psi_{CY}) \psi^7 : - : Re(\Omega_{CY}) \partial x^7 : + : \partial x^7 \partial \psi^7 : + : J_{CY} G'_{CY} : - \frac{1}{2} \partial G_{CY} \ .
\end{aligned}
\tag{2.11}
$$

Similarly, the generalized mirror symmetry for manifolds with G_2 holonomy is to apply the above automorphism to one of the two chiralities. On the other hand, the T-duality in the following (i_1, i_2, i_3) directions can obviously realize the automorphism.

$$
\begin{aligned}
&(i_1, i_2, i_3) \in I_3^+ \cup I_3^- \ , \\
&I_3^+ = \{(2,4,6), (2,3,5), (1,2,7)\} \ , \\
&I_3^- = \{(1,3,6), (1,4,5), (3,4,7), (5,6,7)\} \ .
\end{aligned}
\tag{2.12}
$$

If we combine any two different T-dualities listed above, we obtain another set of T-dualities acting on toroidal T^4, which also leave the extended chiral algebra invariant. Hence, they are mirror symmetry which take IIA (IIB) to IIA (IIB).

$$
\begin{aligned}
&(i_1, i_2, i_3, i_4) \in I_4^+ \cup I_4^- \ , \\
&I_4^+ = \{(1,3,5,7), (1,4,6,7), (3,4,5,6)\} \ , \\
&I_4^- = \{(2,4,5,7), (2,3,6,7), (1,2,5,6), (1,2,3,4)\} \ .
\end{aligned}
\tag{2.13}
$$

Recall that T-duality in ith direction will give $\tilde{\psi}_0^i$ a minus sign. It's not hard to see that I_3^+ (I_4^+) does not change the discrete torsion while I_3^- (I_4^-) does. We can summarize the action of the T-dualities as follows.

$$
\begin{aligned}
&IIA(IIB)/X_l \longleftrightarrow IIB(IIA)/X_{8-l} \ , \quad under \ I_3^- \ , \\
&IIA(IIB)/X_l \longleftrightarrow IIA(IIB)/X_l \ , \quad under \ I_3^+ \ .
\end{aligned}
\tag{2.14}
$$

3. Mirror symmetry for spin(7) manifolds

3.1. Joyce's construction of spin(7) manifolds

There are many known examples of Joyce spin(7) orbifolds [14]. For simplicity, I will take one orbifold for example in which we have choices in desingularizing T^8/Z_2^4 as before. The generators are,

$$\begin{aligned}
\alpha &= (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7, x_8)\ ,\\
\beta &= (x_1, x_2, x_3, x_4, -x_5, -x_6, -x_7, -x_8)\ ,\\
\gamma &= (1/2 - x_1, -x_2, x_3, x_4, 1/2 - x_5, -x_6, x_7, x_8)\ ,\\
\delta &= (-x_1, x_2, 1/2 - x_3, x_4, 1/2 - x_5, x_6, 1/2 - x_7, x_8)\ .
\end{aligned} \tag{3.1}$$

Again, the periodicity of x_i is unity. In general, the singularities arises in five different types and the corresponding desingularization is following.

Type(1): increase b_2 by 1, b_3 by 4, b_{4+} by 3, and b_{4-} by 3. The singularity type is $T^4 \times (B_\epsilon^4/\{\pm 1\})$, where B_ϵ^4 is defined as an open ball of radius ϵ about 0 in R^4.

Type(2): increase b_2 by 1, b_{4+} by 3, and b_{4-} by 3. The singularity if of the form $(T^4/\{\pm 1\} \times (B_\epsilon^4/\{\pm 1\})$.

Type(3): increase b_{4+} by 1. The singularity is $(B_\epsilon^4/\{\pm 1\} \times (B_\epsilon^4/\{\pm 1\})$.

Type(4A) increase b_2 by 1, b_3 by 2, b_{4+} by 1, and b_{4-} by 1.

Type(4B) increase b_3 by 2, b_{4+} by 2, and b_{4-} by 2.

The singularity of type(4) is an isometric involution σ of $T^4 \times (B_\epsilon^4/\{\pm 1\})$, where $\sigma = (1/2 + x_1, x_2, -x_3, -x_4, y_1, y_2, -y_3, -y_4)$. Namely, the singular set is isomorphic to $(T^4 \times (B_\epsilon^4/\{\pm 1\}))/ < \sigma >$.

Type(5A) increase b_2 by 1, b_{4+} by 1, and b_{4-} by 1.

Type(5B) increase b_{4+} by 2, and b_{4-} by 2.

The singularity of type(5) is isomorphic to $(T^4/\{\pm 1\} \times B_\epsilon^4/\{\pm 1\})/ < \sigma >$.

As a result, one found the singular set of this orbifold contains 2 type(1), 8 type(2), 64 type(3) and 4 type(4). If we choose to have j type(4A) and $4 - j$ type(4B) and add up all the Betti numbers in the twisted sectors as well as the untwisted sector, we have the Joyce's manifolds Y_j with

$$b_2 = 10 + j,\ b_3 = 16,\ b_{4+} = 109 - j,\ b_{4-} = 45 - j, \quad j = 0, ..., 4 \tag{3.2}$$

$$\hat{A} = \frac{1}{24}(-1 + b_1 - b_2 + b_3 + b_{4+} - 2b_{4-}) = 1. \tag{3.3}$$

In fact, the 4 type(4) singularities come from 16 γ-fixed T^4s. Notice that $\alpha\delta$ acts trivially on these T^4s and the group elements α, β, $\alpha\beta$, and $\beta\delta$ act freely on them and reduce the number of T^4s to be 4. Therefore, we have RR ground states $|0,0;\tilde{f}=1,2,3,4\rangle_\gamma$ corresponding to the harmonic two forms of the exceptional divisors. Similarly, the $\alpha\delta$ parity of $|0,0;\tilde{f}\rangle_\gamma$ is also given by the discrete torsion $\epsilon_{\gamma,\tilde{f}}$. Since the action of $\alpha\delta$ inverses direction 4 and 7, we can construct RR ground states accordingly as follows.

For $\alpha\delta$ parity even case, we have,

$$\begin{aligned}
&\epsilon_{\gamma;\tilde{f}} = 1 \ , \\
&|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{3+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{8+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{3+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{4+}\psi^{7+}|0,0;\tilde{f}\rangle_\gamma \ , \\
&\psi^{3+}\psi^{4+}\psi^{7+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{4+}\psi^{7+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{3+}\psi^{4+}\psi^{7+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma \ .
\end{aligned} \tag{3.4}$$

For $\alpha\delta$ parity odd case, the RR ground states are,

$$\begin{aligned}
&\epsilon_{\gamma;\tilde{f}} = -1 \ , \\
&\psi^{4+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{7+}|0,0;\tilde{f}\rangle_\gamma \ , \\
&\psi^{3+}\psi^{4+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{3+}\psi^{7+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{4+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{7+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma \ , \\
&\psi^{3+}\psi^{4+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma \ , \ \psi^{3+}\psi^{7+}\psi^{8+}|0,0;\tilde{f}\rangle_\gamma \ .
\end{aligned} \tag{3.5}$$

Obviously, we obtain $\Delta b_2 = 1, \Delta b_3 = 2, \Delta b_4 = 2$ in parity even case, and $\Delta b_3 = 2, \Delta b_4 = 4$ in parity odd case, which agrees with the mathematical analysis in [14].

3.2. spin(7) extended superconformal algebra

Consider a direct product space $M \times S^1$, where M is a manifold with G_2 holonomy. It is always possible to define a spin(7) structure. And the Cayley 4-form ϕ_4 in this manifold with spin(7) structure can be written as,

$$\phi_4 = *\phi_3 + \phi_3 \wedge dx^8, \tag{3.6}$$

where ϕ_3 is the calibrated three form in the G_2 manifold.

It is true that in the example in the previous section $T^7/\langle\alpha,\gamma,\delta\rangle$ gives rise to a Joyce's 7-manifold of G_2 holonomy, if we forget about 8-direction. In fact, we can have a more generic statement which is $T^7/\langle\alpha,\gamma,\delta\rangle$ is *always* a manifold with G_2 holonomy for any choices of the constants c_i and d_i [15], where

$$
\begin{aligned}
\alpha &= (-x_1, -x_2, -x_3, -x_4, x_5, x_6, x_7) \ , \\
\gamma &= (c_1 - x_1, c_2 - x_2, x_3, x_4, c_5 - x_5, c_6 - x_6, x_7) \ , \\
\delta &= (c_1 - x_1, x_2, c_3 - x_3, x_4, c_5 - x_5, x_6, c_7 - x_7) \ .
\end{aligned}
\tag{3.7}
$$

If we reformulate the action of β in the previous section, we will find that β acts as,

$$
\beta : x^8 \to -x^8, \ \beta^*(\phi_3) = -\phi_3, \ \beta^*(*\phi_3) = *\phi_3. \tag{3.8}
$$

In this example, β indeed turns the spin(7) structure into the spin(7) holonomy. However, it is not clear that we can always form manifolds with spin(7) holonomy by modding out this kind of Z_2 involution on $G_2 \times S^1$.

Therefore, at least in Joyce's orbifolds, the relation (3.6) enables us to write down the expression of the stress energy tensor $T_{spin(7)}$ and the supercurrent $G_{spin(7)}$ in terms of the corresponding quantities in G_2 manifolds [13].

$$
\begin{aligned}
T_{spin(7)} &= T_{G_2} + \frac{1}{2} : \partial x^8 \partial x^8 : - \frac{1}{2} : \psi^8 \partial \psi^8 : \ , \\
G_{spin(7)} &= G_{G_2} + : \psi^8 \partial x^8 : \ , \\
X_{spin(7)} &= X_{G_2} + \Phi_{G_2} \psi^8 + \frac{1}{2} \psi^8 \partial \psi^8 \ , \\
M_{spin(7)} &= [G_{spin(7)}, X_{spin(7)}] \\
&= \partial x^8 \Phi_{G_2} - K_{G2} \psi^8 - M_{G_2} + \frac{1}{2} \partial^2 x^8 \psi^8 - \frac{1}{2} \partial x^8 \partial \psi^8 \ .
\end{aligned}
\tag{3.9}
$$

From these generators for the extended supercomformal algebra, it is not difficult to see that the combination of the G_2 automorphism (2.10) and the T-duality in 8-direction is an automorphism in the algebra. In addition, the T-duality in (2.13) is also an automorphism in the algebra. Therefore, we have a list of 14 T-dualities on T^4 toroidal fibrations which generate the mirror symmetry,

$$
\begin{aligned}
&\{(2,4,6,8), (2,3,5,8), (1,2,7,8), (1,3,6,8), (1,4,5,8), (3,4,7,8), (5,6,7,8), \\
&(1,2,5,7), (1,4,6,7), (3,4,5,6), (2,4,5,7), (2,3,6,7), (1,2,5,6), (1,2,3,4)\} \ .
\end{aligned}
\tag{3.10}
$$

The first line consists of T-dualities in directions in (2.12) and 8-direction. The second line is the same as the directions listed in (2.13). In this spin(7) case, we don't have the similar relation like (2.10). Therefore, in order to visualize the automorphism in the algebra, we have to express the spin(7) generators and the algebra in terms of G_2 generators and construct our desirable mirror transformation from G_2 automorphism (2.10)(2.12)(2.13). Finally, the expression of $\alpha\delta$ in γ-fixed sector is,

$$\alpha\delta = \frac{1}{4}\psi_0^4\psi_0^7\tilde{\psi}_0^4\tilde{\psi}_0^7\epsilon_{\gamma;\tilde{f}} \quad . \tag{3.11}$$

By the same reasoning, the 14 T-dualities are divided into two sets $J_4^{\pm}$. Their action is also summarized as follows.

$$\begin{aligned}
&(i_1, i_2, i_3, i_4) \in J_4^+ \cup J_4^- \ , \\
&J_4^+ = \{(2,3,5,8),(1,3,6,8),(3,4,7,8),(1,4,6,7),(2,4,5,7),(1,2,5,6)\} \ , \\
&J_4^- = \{(2,4,6,8),(1,2,7,8),(1,4,5,8),(5,6,7,8),(1,2,5,7),(3,4,5,6), \\
&\qquad\quad (2,3,6,7),(1,2,3,4)\} \ .
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&IIA(IIB)/Y_j \longleftrightarrow IIA(IIB)/Y_{4-j}, \quad under\ J_4^- \ , \\
&IIA(IIB)/Y_j \longleftrightarrow IIA(IIB)/Y_j, \quad under\ J_4^+ \ .
\end{aligned} \tag{3.13}$$

4. Conclusion

In this paper I have generalized the construction of [8] to the Joyce's manifolds with spin(7) holonomy and shown how the mirror symmetry is realized in the superconformal algebra as a combination of a T-duality in 8-direction and a G_2 mirror symmetry transformation, or a combination of 2 distinct G_2 mirror transformations. The spin(7) mirror transformation contains 14 different kinds of T-dualities on the T^4 fibrations. By an analysis on the change of discrete torsion, one can classify these 14 T-dualities into 2 kinds, one of which changes the discrete torsion and the other does not.

In [16], the authors completed a cycle of the dualities by explicitly performing the T-duality on T^3 fibration and a G_2 flop in M-theory. It would be interesting to generalize the computation to a duality cycle involving spin(7) and G_2 manifolds and understand how the generalized mirror symmetry lies in this picture [17].

In order to understand the G_2/spin(7) mirror symmetry better, one may try to T-dualize the known various non-compact metric solutions with G_2/spin(7) holonomy [18] and see how they are connected through mirror symmetry. In the Calabi-Yau case, NS-NS fluxes can turn the CY target space into half-flat [19]. The generalized mirror symmetry for G_2 and spin(7)in the presence of the background fluxes also begs some further study.

Finally, it would also be interesting to see how we can fit the G_2 or spin(7) mirror symmetry into the correspondence of heterotic(G_2)/ IIA(G_2 orientifold)/M-theory(spin(7)) [20].

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