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Matrix model, Kutasov duality and Factorization of Seiberg-Witten Curves*

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Abstract

We study the duality of $\mathcal{N} = 1$ gauge theories in the presence of a massless adjoint field. We calculate the superpotential using the factorization method and compare with the result obtained by applying Kutasov duality. The latter result is just the leading term of the former, indicating that Kutasov duality is exact only in the IR limit as claimed in the original literature. We also study various checks for the equivalence of the calculational methods developed recently: factorization methods, diagrammatic expansion, loop equations, integrating fluxes.

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1 Introduction

Recently, Dijkgraaf and Vafa discovered a surprising link between effective superpotentials of supersymmetric gauge theories and the effective potential of an associated bosonic matrix model [1]. This link allows one to obtain nonperturbative results by doing a perturbative calculation. It is a powerful method to compute the effective superpotential of $\mathcal{N}=1$ gauge theories with massive matter in tensor and fundamental representations. However, it is difficult to use this method if the gauge theory contains massless adjoint fields.[1] More recently, it is understood why such a simplification can arise as a consequence of (super)symmetries by considering anomalous Ward identities associated to the Konishi anomaly [2] (see also [3]), which turns out to be the same as the loop equation in the matrix model. The loop equation is identical to the minimization of the superpotential, which in turn requires that all periods of the generating 1-form on the Riemann surface defined by the superpotential are integers [4, 5, 6, 7]. By Abel's theorem, the 1-form must be a derivative of a meromorphic function ψ. Finally, the condition that ψ be single valued on the reduced Riemann surface is the factorization of the Seiberg-Witten curve of the original $\mathcal{N}=2$ theory to the reduced curve defined by the superpotential. This completes the recipe for the solution to the problem and corresponds to extending the earlier results of Vafa and his collaborators [8, 9] on the problem without fundamentals.

In a recent paper [11], we utilized Kutasov duality [12] and the gauge theoretic method [13, 14] to find the superpotential of $SU(N_c)$ gauge theory with N_f massive fundamental fields and a massless adjoint field having nontrivial tree level superpotentials. More specifically, for a theory with tree level superpotential

$$W_{tree} = \sum_{l=1}^{2} \mathrm{Tr}(m_l\,\tilde{Q}\Phi^{l-1}Q) + \tfrac{1}{3}\,g\,\mathrm{tr}\Phi^3, \tag{1.1}$$

we have

$$W_{eff} = g\,\Lambda_L^4\,\mathrm{Tr}\left(m_2 {m_1}^{-1}\right), \tag{1.2}$$

To get this result, we first worked out the case where both the adjoint as well as the fundamental fields are massless. Then Kutasov duality was used to map the result to the case where the fundamental fields are massive.

In this paper, we will calculate the superpotential for the same theory using the method of factorizing the Seiberg-Witten curve developed more recently [8, 9, 4, 15, 16]. We work out

[1] See however [17], where the case of massless *fundamental* fields is discussed.

both two-cut as well as one-cut solutions. It turns out that the gauge theory result coincides with the one-cut solution rather than the generic two-cut solution. In that case, what we found is that the result obtained by duality [11] is just the leading term of what is found here. This result can be interpreted as the statement that Kutasov duality holds only in the IR limit where the scale of the theory $\Lambda \to 0$.

The rest of the paper goes as follows: in section 2, we will briefly review the factorization method used in this paper. In section 3, we will calculate the superpotential. In section 4, we give an account for the equivalence of factorization of the Seiberg-Witten curve and the minimization of the superpotential by an explicit calculation. Section 5 will give a summary and the conclusion.

2 Matrix model and factorization of the Seiberg-Witten curve

Here we give a lightning review of the necessary material. We start from the matrix model partition function

$$Z = \frac{1}{vol(G)} \int d\Phi dQ^i d\tilde{Q}^i \exp\left(-\frac{1}{g_s} W(\Phi) - \frac{1}{g_s} \sum_{i=1}^{N_f} \left[\tilde{Q}_i \Phi Q^i - m_i \tilde{Q}_i Q^i \right] \right), \qquad (2.1)$$

where $W(z)$ is a polynomial of order $n+1$. Diagonalizing Φ and integrating over Q and $\tilde{Q}$, we get

$$Z \sim \int \prod_{a=1}^{N} \exp\left(-\frac{1}{g_s} W(\lambda_a) + 2 \sum_{a<b}^{N} \log(\lambda_a - \lambda_b) - \sum_{i=1}^{N_f} \sum_a \log(\lambda_a - m_i) \right) \qquad (2.2)$$

The saddle point equation in the limit $g_s \to 0$ and $N \to \infty$ with $S = g_s N$ fixed, is called the loop equation:

$$S\omega^2(z) + W_0'(z)\omega(z) + \frac{1}{4} f(z) = 0, \qquad (2.3)$$

where $\omega(z)$ is the resolvant defined as

$$\omega(z) = \frac{1}{N} \mathrm{tr} \left(\frac{1}{\Phi - z} \right) = \frac{1}{N} \sum_a \frac{1}{\lambda_a - z} \qquad (2.4)$$

and f is a polynomial of order $n-1$ yet to be determined.

The large N can be expressed in terms of density of eigenvalues $\rho(\lambda) = \frac{1}{N}\sum_a \delta(\lambda - \lambda_a)$ normalized by $\int \rho(\lambda) = 1$. In terms of it, $\omega(z) = \int d\lambda \frac{\rho(\lambda)}{\lambda - z}$ and

$$\rho(\lambda) = \frac{1}{2\pi i}[\omega(\lambda + i\epsilon) - \omega(\lambda - i\epsilon)] = \frac{1}{2\pi i}\mathrm{Disc}[\omega(z)]. \quad (2.5)$$

The loop equation can be rewritten as a defining equation of a hyperelliptic Riemann surface

$$y^2 = W_0'(z)^2 + f(z), \quad \text{with} \quad y(z) = 2S\omega(z) + W'(z). \quad (2.6)$$

The curve has n cuts in the z plane. The eigenvalues of Φ are distributed along the cuts according to f. Let N_i be the number of eigenvalues along the i-th cut: $N_i = N\int d\lambda \rho(\lambda)$, and let $S_i := g_s N_i$ be finite in the limit $N \to \infty$, $g_s \to 0$. Using the eq.(2.5), the latter can be rewritten as $S_i = -\frac{1}{4\pi i}\oint_{A_i} y(z)dz$, where A_i is a contour encircling the i-th cut. Following [9, 7], we denote by P, Q the point $z = \infty$ on the two sheets of the hyperelliptic curve such that $y(P) \sim W_0'(P)$. The hyperelliptic curve can be given canonical homology cycles $A_i (i = 1, ..., n-1)$ and $B_i = \hat{B}_i - \hat{B}_n (i = 1, ..., n-1)$. When $n = N_c$, f and the Seiberg-Witten curve were determined in [7] following the work in [9]. For $n < N_c$, this problem is solved in [7, 5]. The result is

$$W_{eff} = \sum_{l=1}^{n} N_{c,i}\frac{\partial F_s}{\partial S_i} + F_d + 2\pi i \tau_0 \sum_{l=1}^{n} S_i. \quad (2.7)$$

One can easily express the relevant quantities in terms of the integral of y's over the infinite cycles,

$$\frac{\partial F_s}{\partial S_i} = -\frac{1}{2}\int_{\hat{B}_i} ydz = -\int_{e_i}^{P} ydz, \quad F_d = \frac{1}{2}\int_{m_i}^{P} ydz, \quad (2.8)$$

up to the integral constants cancelling the divergences, which are independent of S. Here, the e_i's are the boundaries of the cuts. Finally we get

$$W_{eff} = -\sum_{l=1}^{n} N_{c,i}\left(\int_{e_i}^{P} ydz - W(P)\right) + \frac{1}{2}\left(\int_{m_i}^{P} ydz - W(P) + W(m_i)\right) + 2\pi i\tau_0 \sum_{l=1}^{n} S_i. \quad (2.9)$$

What is shown in [7, 5] is that the minimization of this superpotential is equivalent to the integer periodicity along A_i and B_i cycles. By Abel's theorem, the 1-form $\omega(z)$ must be a derivative of a meromorphic function ψ, i.e., $\omega(z)dz = d\psi$. For $N_f < 2N_c$, $\psi = P(z) + \sqrt{P^2(z) - \alpha B(z)}$. Finally, the condition that ψ be single valued on the reduced Riemann surface $y^2 = W'^2 + f$ is the factorization of the Seiberg-Witten curve of the original $\mathcal{N} = 2$ theory to the reduced curve defined by the superpotential.

$$\begin{aligned} P_N^2(z) - \alpha B(z) &= F_{2m}(z)H_{N-m}^2(z), \\ W_{n+1}'^2(z) + f_{n-1}(z) &= F_{2m}(z)Q^2(z). \end{aligned} \quad (2.10)$$

The sub-indices of the polynomials are their orders and the number of cuts m in eigenvalue space is related to the order of F_{2m}. This completes the recipe for the solution to the problem and corresponds to extending the earlier result of Vafa and his collaborators [8, 9] on the problem without fundamentals.

3 Cubic potential in the presence of fundamentals

We now evaluate the superpotential using the factorization method. We consider the simplest nontrivial case: $N_c = 3$ and $N_f = 2$. We take the superpotential of the adjoint field Φ to be

$$W = \frac{g}{3}\mathrm{tr}\Phi^3 + \frac{m_\phi}{2}\mathrm{tr}\Phi^2 + \lambda\mathrm{tr}\Phi, \tag{3.1}$$

such that

$$W'(z) = gz^2 + m_\phi z + \lambda. \tag{3.2}$$

Since W' is of degree two, $y = \sqrt{W'^2 + f_1}$ will have at most two cuts and the relevant Riemann surface is of genus one, allowing an explicit study in terms of well known technology on the torus.

3.1 One-cut case

First we solve the one-cut condition: $W_3'^2 + f = F_2 \cdot Q_1^2$. It is easy to see that number of unknowns is bigger than number of equations by 1. For a cubic potential, Q must be linear. Hence we put $Q = x - x_3$. Then

$$(x^2 + \frac{m_\phi\, x}{g} + \frac{\lambda}{g})^2 + \frac{f_1\, x + f_0}{g^2} = (x - x_1)\,(x - x_2)\,(x - x_3)^2 \tag{3.3}$$

We introduce the variables $\Delta := (x_1 - x_2)/2$ and $T := (x_1 + x_2)/2$. After some algebra, we can show that x_3 can be determined by Δ from the equation

$$W'(x_3) + \frac{1}{2}g\Delta^2 = 0, \tag{3.4}$$

and T, f_0, f_1 can be determined in terms of Δ, x_3 as follows;

$$T = -m_\phi/g + x_3,\ \ f_1 = (g\Delta)^2(2x_3 + \frac{m_\phi}{g}),\ \ f_0 = g\Delta^2(m_\phi x_3 + 2\lambda + \frac{3}{4}g\Delta^2). \tag{3.5}$$

To determine Δ, we must use the factorization condition,

$$P^2(x) - 4\Lambda^{2N_c-N_f}\prod_{i=1}^{N_f}(x-m_i) = F_2(x)H^2(x). \tag{3.6}$$

For $N_c = 3, N_f = 2$, and setting $m_1 = m_2 = m$, we have the factorized version

$$\begin{aligned} P(x) - 2\,\epsilon\,\Lambda^2\,(x-m) &= (x-x_1)\,(x-h_1)^2, \\ P(x) + 2\,\epsilon\,\Lambda^2\,(x-m) &= (x-x_2)\,(x-h_2)^2. \end{aligned} \tag{3.7}$$

Since we are mainly interested in the $SU(3)$ case, we impose the traceless condition explicitly:

$$x_1 + 2h_1 = 0, \quad x_2 + 2h_2 = 0. \tag{3.8}$$

Then eq. (3.7) gives

$$3T\Delta = 4\epsilon\Lambda^2, \quad 3T^2\Delta + \Delta^3 + 8\epsilon\Lambda^2 m = 0, \tag{3.9}$$

and Δ is determined by

$$\Delta^4 + 8\epsilon m\Lambda^2\Delta + \frac{16}{3}\Lambda^4 = 0. \tag{3.10}$$

Similarly, T is determined by

$$T^4 + 2mT^3 + \frac{16}{27}\Lambda^4 = 0. \tag{3.11}$$

For $\Lambda \ll m$, we can solve for Δ and T in terms of a power series in Λ. There are two real solutions: One solution is of integer powers in Λ;

$$\begin{aligned} \Delta &= -\frac{2}{3}\frac{\Lambda^2}{m} - \frac{2\Lambda^6}{81m^5} - \frac{8\Lambda^{10}}{2187m^9} - \frac{44\Lambda^{14}}{59049m^{13}} + O(\Lambda^{18}), \\ T &= -2m + \frac{2\Lambda^4}{27m^3} + \frac{2\Lambda^8}{243m^7} + \frac{10\Lambda^{12}}{6561m^{11}} + O(\Lambda^{16}). \end{aligned} \tag{3.12}$$

It turns out that this solution does not correspond to the result obtained in gauge theory. The other solution is of fractional power in Λ: $T \sim -(2/3)(\Lambda^4/m)^{1/3}$ and $\Delta \sim -2(\epsilon m\Lambda^2)^{1/3}$. In order to understand this latter solution, we introduce ${\Lambda_L}^3 = m\Lambda^2$. In fact Λ_L is precisely the low energy quantum scale appearing in the field theory analysis that is defined by

$$\Lambda^{2N_c-N_f}\det m = \Lambda_L^{2N_c}. \tag{3.13}$$

In terms of Λ_L, the second solution is analytic in Λ_L and given by

$$\begin{aligned} T = -\frac{2\,\Lambda_L^2}{3\,m} - \frac{2\,\Lambda_L^4}{27\,m^3} - \frac{2\,\Lambda_L^6}{81\,m^5} &- \frac{70\,\Lambda_L^8}{6561\,m^7} - \frac{308\,\Lambda_L^{10}}{59049\,m^9} - \frac{2\,\Lambda_L^{12}}{729\,m^{11}} \\ &- \frac{21736\,\Lambda_L^{14}}{14348907\,m^{13}} - \frac{111826\,\Lambda_L^{16}}{129140163\,m^{15}} + O(\Lambda_L^{20}), \end{aligned} \tag{3.14}$$

$$\Delta/\epsilon = -2\,\Lambda_L + \frac{2}{9\,m^2}\,\Lambda_L^3 + \frac{4}{81\,m^4}\,\Lambda_L^5 + \frac{40}{2187\,m^6}\,\Lambda_L^7 + \frac{2}{243\,m^8}\,\Lambda_L^9 + \frac{728}{177147\,m^{10}}\,\Lambda_L^{11} + \frac{10472}{4782969\,m^{12}}\,\Lambda_L^{13} + \frac{8}{6561\,m^{14}}\,\Lambda_L^{15} + O(\Lambda_L^{17}). \quad (3.15)$$

3.1.1 massless adjoint

Finally, we can calculate the superpotential for the massless adjoint field,

$$W_{eff} = gu_3 = \frac{g}{4}(T+\Delta)^3 - 2g\epsilon\Lambda_L^3, \quad (3.16)$$

as a power series in Λ_L:

$$W_{eff} = -\frac{2\,g}{m}\,{\Lambda_L}^4 + \frac{4\,g}{27\,m^3}\,{\Lambda_L}^6 + \frac{2\,g}{81\,m^5}\,{\Lambda_L}^8 + \frac{16\,g}{2187\,m^7}\,{\Lambda_L}^{10} + \frac{2\,g}{729\,m^9}\,{\Lambda_L}^{12} + \frac{208\,g}{177147\,m^{11}}\,{\Lambda_L}^{14} + \mathrm{O}({\Lambda_L}^{16}). \quad (3.17)$$

Notice the absence of a term proportional to Λ_L^3 and the independence of the superpotential of the choice of $\epsilon = \pm 1$. In fact, there is no term with an odd power of Λ_L. Considering the structure of eq. (3.16) and the Λ dependence of T, Δ in eqs. (3.14),(3.15), this result seems to be rather nontrivial.

3.1.2 massive adjoint

For the case when the adjoint field is massive in the $SU(3)$ theory, all the calculations are the same except that

$$W_{eff} = gu_3 + m_\phi u_2. \quad (3.18)$$

Using

$$u_2 := 3(T+\Delta)^2/4 - 2\epsilon\Lambda_L^3/m, \quad (3.19)$$

the superpotential is calculated to be

$$W_{eff} = 3\,m_\phi\,{\Lambda_L}^2 + (-\frac{2\,g}{m} - \frac{1}{3}\,\frac{m_\phi}{m^2})\,{\Lambda_L}^4 + (\frac{4\,g}{27\,m^3} - \frac{1}{27}\,\frac{m_\phi}{m^4})\,{\Lambda_L}^6 + (\frac{2\,g}{81\,m^5} - \frac{7}{729}\,\frac{m_\phi}{m^6})\,{\Lambda_L}^8 + (\frac{16\,g}{2187\,m^7} - \frac{22}{6561}\,\frac{m_\phi}{m^8})\,{\Lambda_L}^{10} + (\frac{2\,g}{729\,m^9} - \frac{1}{729}\,\frac{m_\phi}{m^{10}})\,{\Lambda_L}^{12} + (\frac{208\,g}{177147\,m^{11}} - \frac{988}{1594323}\,\frac{m_\phi}{m^{12}})\,{\Lambda_L}^{14} + \mathrm{O}({\Lambda_L}^{16}). \quad (3.20)$$

3.2 Generic two-cut case

We can consider an $\mathcal{N}=1$ theory with superpotential W as a deformation of an $\mathcal{N}=2$ theory with particular moduli parameters for which the following equation holds:

$$P_{N_c}(x;u_p)^2 - 4\Lambda^{2N_c-N_f}\prod_{i=1}^{N_f}(x-m_i) = H_{N-n}^2(x)\frac{1}{g_{n+1}^2}(W'^2+f), \tag{3.21}$$

where $P_N(x)=\det(x-\Phi)$ for $N_f<N_c$. [2] This corresponds to the $Q=1$ case in (2.10). This equation dictates that, apart from the n photons, there should be $N-n$ extra massless fields like monopoles and dyons. Notice that if we let the coefficient of highest power be 1, there are $N_c+(N_c-n)+n$ parameters and $2N_c$ equations. Therefore, given W' and the matter part $4\Lambda^{2N_c-N_f}\prod_{i=1}^{N_f}(x-m_i)$, both the moduli of the Seiberg-Witten curve, P_N, as well as those of the monopoles, H_{N-n}, are uniquely fixed. For the parameters satisfying this factorization, the glueball fields S_i are also determined in terms of the classical data of W, m_i and the quantum scale Λ.

For our case $N_f=2, N_c=3$, we carry out the factorization under the assumption that all fundamental hypermultiplets have the same mass m :

$$P_3^2 - 4\Lambda^4(x-m)^2 = (x-a_1)^2(W'^2+f)/g^2. \tag{3.23}$$

Notice that the left hand side is factorized canonically, one of $(P_3 \pm 2\Lambda^2(x-m))$ must have the factor $(x-a_1)^2$. Let

$$P_3 - 2\epsilon\Lambda^2(x-m) = (x-a_1)^2(x-a_2), \tag{3.24}$$

with $\epsilon=\pm 1$. Then,

$$P_3 + 2\epsilon\Lambda^2(x-m) = (x-a_1)^2(x-a_2) + 4\epsilon\Lambda^2(x-m). \tag{3.25}$$

Therefore we can identify

$$(W'^2+f)/g^2 = (x-a_1)^2(x-a_2)^2 + 4\epsilon\Lambda^2(x-m)(x-a_2). \tag{3.26}$$

[2] For $N_f>N_c$, the eq.(3.21) should be replaced by:

$$(P_N(x;u_p) + \frac{1}{4}\Lambda^{2N_c-N_f}Q_{N_f-N_c})^2 - 4\Lambda^{2N_c-N_f}\prod_{i=1}^{N_f}(x-m_i) = H_{N-n}^2(x)\frac{1}{g_{n+1}^2}(W'^2+f), \tag{3.22}$$

with $Q=\sum_i^{N_f-N_c} x^{N_f-N_c-i}t_i(m)$.

Since f is at most linear in x, we can determine f and a_i in terms of classical data and Λ by the identification

$$\begin{aligned} W'/g &= (x-a_1)(x-a_2)+2\epsilon\Lambda^2, \\ f/g^2 &= 4\epsilon\Lambda^2(a_1-m)(x-a_2)-4\Lambda^4. \end{aligned} \tag{3.27}$$

Therefore a_1, a_2 are solutions of $W'/g - 2\epsilon\Lambda^2 = 0$:

$$a_1, a_2 = -\frac{m_\phi}{2g} \pm \sqrt{\frac{m_\phi^2}{4g^2} - \frac{\lambda}{g} + 2\epsilon\Lambda^2}. \tag{3.28}$$

Since $f = -4gSx + f_0$, the factorization determines the exact value of gluino condensate:

$$S = \epsilon g \Lambda^2 \left(-\frac{m_\phi}{2g} - m + \sqrt{\frac{m_\phi^2}{4g^2} - \frac{\lambda}{g} + 2\epsilon\Lambda^2} \right). \tag{3.29}$$

To calculate the superpotential in this case, we use the method of [8].

$$P_3(x) = (x-a_1)^2(x-a_2) + 2\epsilon\Lambda^2(x-m) := \sum_{i=0}^{N_c=3} s_{N_c-i} x^i. \tag{3.30}$$

Using Newton's relation, $ks_k + \sum_r r u_r s_{k-r} = 0$,

$$u_1 = -s_1, \quad u_2 = -s_2 + \frac{1}{2}s_1^2, \quad u_3 = -s_3 + s_2 s_1 - \frac{1}{3}s_1^3. \tag{3.31}$$

And we get

$$\begin{aligned} u_1 &= 2a_1 + a_2, \\ u_2 &= \frac{1}{2}(2a_1^2 + a_2^2) - 2\epsilon\Lambda^2, \\ u_3 &= \frac{1}{3}(2a_1^3 + a_2^3) - 2\epsilon\Lambda^2(2a_1 + a_2 - m). \end{aligned} \tag{3.32}$$

Therefore

$$\begin{aligned} W_{eff} &= g u_3 + m_\phi u_2 + \lambda u_1 \\ &= 2W(a_1) + W(a_2) - 2\epsilon\Lambda^2(m_\phi + g(2a_1 + a_2 - m)). \end{aligned} \tag{3.33}$$

Notice that the minimization of W_{eff} w.r.t. a_i gives

$$W'(a_i) = 2g\epsilon\Lambda^2, \quad \text{for } \; i = 1, 2. \tag{3.34}$$

This is consistent with the factorization result (3.27). After some calculation using (3.28), we get

$$W_{eff} = -\frac{3}{2}\frac{\lambda m_\phi}{g} + \frac{1}{4}\frac{m^3}{g^2} + (m_\phi + 2gm)\epsilon\Lambda^2 + \left(\frac{2}{3}\lambda - \frac{m_\phi^2}{6g} - \frac{4}{3}g\epsilon\Lambda^2\right)\sqrt{\frac{m_\phi^2}{4g^2} - \frac{\lambda}{g} + 2\epsilon\Lambda^2} \quad (3.35)$$

This result contains both the disk as well as the sphere contribution as argued in [15].

In the case of $SU(3)$, we have to impose the traceless condition: $u_1 = dW_{eff}/d\lambda = 2a_1 + a_2 = 0$. Together with $a_1 + a_2 = -m_\phi/g$, this gives us

$$a_1 = m_\phi/g, \quad a_2 = -2m_\phi/g, \quad \lambda = 2g\epsilon\Lambda^2 - 2m_\phi^2/g. \quad (3.36)$$

Notice that a_1, a_2 became the same as the classical solution of W'. Then, the gluino condensate is given by

$$S = \epsilon\Lambda^2 g\left(\frac{m_\phi}{g} - m\right), \text{ for } SU(3). \quad (3.37)$$

The superpotential for $SU(3)$ can be simplified to

$$W_{eff} = \frac{m_\phi^3}{g^2} - 2\epsilon\Lambda^2(m_\phi - gm). \quad (3.38)$$

4 Factorization vs. minimization

Here we give an account for the equivalence of the two methods by explicit computations.

4.1 Factorization

Consider an $\mathcal{N} = 2$ SYM with $N_c = 2$ and $N_f = 2$ and deformed by a quadratic superpotential for the adjoint field Φ;

$$W = \frac{1}{2}m_\phi \mathrm{tr}\Phi^2 + \lambda \mathrm{tr}\Phi. \quad (4.1)$$

Hence

$$W'(x) = m_\phi x + \lambda, \ f = -4Sm_\phi, \quad (4.2)$$

where S is yet to be determined. We consider the case where the two m_i are equal to m. Using the Seiberg-Witten curve for $N_c = N_f$ given in [18], the factorization condition is

$$y^2 = P^2 - 4\Lambda^2(x-m)^2 = (x-a)^2 F_2, \quad (4.3)$$

where $P = P_2 + \delta\Lambda^2$ and δ is usually 1/4 but 1/8 for $N_c = 2$. Therefore

$$P - 2\epsilon\Lambda(x - m) = (x - a)^2. \tag{4.4}$$

Then

$$\begin{aligned} F_2 &= (x-a)^2 + 4\epsilon\Lambda(x-m) \\ &= (W'^2 + f)/m_\phi^2 = (x + \frac{\lambda}{m_\phi})^2 - \frac{4S}{m_\phi}, \end{aligned} \tag{4.5}$$

which leads us to the relations $4\epsilon\Lambda - 2a = 2\lambda/m_\phi$, and $a^2 - 4m\epsilon\Lambda = \frac{\lambda^2}{m_\phi^2} - \frac{4S}{m_\phi}$. From this we get $a = 2\epsilon\Lambda - \frac{\lambda}{m_\phi}$, and more importantly for our purpose, we determine the gaugino condensate

$$S = \epsilon\lambda\Lambda - m_\phi\Lambda^2 + \epsilon m m_\phi\Lambda. \tag{4.6}$$

Now let's move to determine the superpotential.

$$P_2 = (x + \frac{\lambda}{m_\phi})^2 - \frac{4S}{m_\phi} - 2\epsilon\Lambda(x - m) - \delta\Lambda^2. \tag{4.7}$$

Hence

$$\begin{aligned} u_1 &= 2(\epsilon\Lambda - \frac{\lambda}{m_\phi}), \\ u_2 &= \left(\frac{\lambda}{m_\phi}\right)^2 - 2\Lambda^2 + 2\epsilon m\Lambda - \delta\Lambda^2. \end{aligned} \tag{4.8}$$

Now,

$$\begin{aligned} W_{eff} &= m_\phi u_2 + \lambda u_1 \\ &= -\frac{\lambda^2}{m_\phi} + 2\epsilon(\lambda\Lambda + m m_\phi\Lambda) - (2+\delta)m_\phi\Lambda^2. \end{aligned} \tag{4.9}$$

For comparison with previous literature we also consider $U(2)$ with $\lambda = 0$;

$$a = 2\epsilon\Lambda, \quad S = -m_\phi\Lambda^2 + \epsilon m m_\phi\Lambda, \tag{4.10}$$

and

$$W_{eff} = 2\epsilon m m_\phi\Lambda - (2+\delta)m_\phi\Lambda^2 = 2S - \delta m_\phi\Lambda^2. \tag{4.11}$$

For $SU(2)$, by imposing $u_1 = 0$, λ is determined and a and S are simplified so that we have

$$\lambda = \epsilon m_\phi\Lambda, \quad a = \epsilon\Lambda, \quad S = \epsilon m m_\phi\Lambda. \tag{4.12}$$

The sphere contribution to the effective potential is

$$W_{eff} = 2\epsilon m m_\phi\Lambda - (1+\delta)m_\phi\Lambda^2. \tag{4.13}$$

4.2 Minimization

We apply the loop equation to the case where the exact result is known. $W_0 = \frac{1}{2}m_\phi \mathrm{tr}\Phi^2$. So the tree-level superpotential is

$$W = W_0 + \sum_{i=1}^{N_f}(\tilde{Q}_i \Phi Q_i + \tilde{Q}_i m Q_i). \tag{4.14}$$

Then the surface equation is given by

$$y^2 = W_0'^2 + f = (m_\phi z)^2 - 4Sm_\phi. \tag{4.15}$$

So,

$$\begin{aligned} W_{eff}^{(s)} = -N_c \int_e^P y dz &= -N_c m_\phi \int_e^P \sqrt{z^2 - e^2}, \\ &= -\frac{1}{2}N_c m_\phi \left[\Lambda_0 \sqrt{\Lambda_0^2 - e^2} - \log(\Lambda_0 + \sqrt{\Lambda_0^2 - e^2})\right] \\ &= -\frac{1}{2}N_c \Lambda^3 + N_c S(1 - \log \frac{S}{\Lambda^3}), \end{aligned} \tag{4.16}$$

where $e^2 = 4S/m_\phi$ and $\Lambda^3 = m_\phi \Lambda_0^2$. Similarly,

$$\begin{aligned} W_{eff}^{(d)} &= \frac{1}{2}\sum_{i=1}^{N_f}\int_{m_i}^P y dz \\ &= N_f S\left[\frac{\Lambda_0^2}{e^2} - \frac{1}{2} - \log\frac{2\Lambda_0}{e}\right] - N_f S\left[\frac{m}{e}\sqrt{\frac{m^2}{e^2} - 1} - \log\left(\frac{m}{e} + \sqrt{\frac{m^2}{e^2} - 1}\right)\right] \\ &= N_f \frac{m_\phi}{4}\left(\Lambda_0^2 - m^2\right) + N_f S \log\left(\frac{m}{\Lambda_0}\right) \\ &\quad - N_f S\left(\frac{1}{2} + \frac{\sqrt{1-4\alpha S} - 1}{4\alpha S} - \log\frac{1 + \sqrt{1 - 4\alpha S}}{2}\right), \end{aligned} \tag{4.17}$$

with $\alpha = 1/(m^2 m_\phi)$.

By minimizing the total superpotential $d(W_{eff}^{(s)} + W_{eff}^{(d)})/dS = 0$, we get an equation to determine S;

$$\left(\frac{S}{m_\phi \Lambda^2}\right)^{N_c} = \left(\frac{m}{\Lambda}\frac{1 + \sqrt{1 - 4\alpha S}}{2}\right)^{N_f}. \tag{4.18}$$

For $U(2)$ with trace coupling, $W = \frac{1}{2}m_\phi \mathrm{tr}\Phi^2 + \lambda \mathrm{tr}\Phi$, one can easily verify that the result can be obtained by shifting

$$m \to m + \lambda/m_\phi, \quad \text{and} \quad \alpha \to \alpha/(1 + \frac{\lambda}{m m_\phi})^2. \tag{4.19}$$

Then the equation (4.18) with $N_f = N_c$ will produce

$$S = \lambda\Lambda - m_\phi \Lambda^2 + m m_\phi \Lambda, \tag{4.20}$$

which is precisely the result in eq. (4.6) with $\epsilon = 1$. The $SU(2)$ case can be treated by regarding λ as a Lagrange multiplier and we leave this as an exercise. The value of the superpotential is $N_c S + N_f \frac{m_\phi}{4}(\Lambda_0^2 - m^2)$. One should notice that the $S \log S$ term in this case canceled between W^s and W^d.

This confirms, in the simplest possible case, that the method of factorizing Seiberg-Witten curve is equivalent to that of minimizing the full superpotential. This equivalence is highly nontrivial to directly check in general.

4.3 Integral evaluation of the effective superpotential for cubics

We now calculate the effective superpotential given by the prescription [5]. For pure SYM, the calculation is done in [8]. Here we give a treatment with fundamentals for the second simplest case $N_f = 2$, $N_c = 3$. The values of the physical parameters were already determined by the factorization method. The superpotential is shown to be

$$W_{eff} = -\sum_{i=1}^{n} N_i \Pi_i + \sum_{j=1}^{N_f} D_j + 2\pi i(\tau_0 S + \sum_{i=1}^{n-1} b_i S_i) - i(N_c - N_f/2)S + C, \tag{4.21}$$

where [3]

$$\Pi_i = \frac{1}{2}\int_{e_i}^{\Lambda} y(z)dz, \quad D_j = \frac{1}{2}\int_{m_i}^{\Lambda} y(z)dz, \quad C = \frac{1}{2}\left((2N_c - N_f)W(\Lambda) + \sum_{I=1}^{N_f} W(z_I) \right). \tag{4.22}$$

Let

$$W'^2 + f(x) = g^2(x-e_1)(x-e_2)(x-e_3)(x-e_4), \quad e_1 > e_2 > e_3 > e_4. \tag{4.23}$$

The integrals are evaluated in terms of elliptic functions;

$$S_1 = \int_{e_2}^{e_1} y(x)dx = \left[2(e_2-e_3)^2 \beta \alpha_1^4\right] \cdot \int_0^{K_1} \frac{\mathrm{sn}^2 u \, \mathrm{dn}^2 u \, \mathrm{cn}^2 u}{(1-\alpha_1^2 \, \mathrm{sn}^2 u)^4} := J_1 \cdot I_1, \tag{4.24}$$

[3]Notice the sign difference in Π_i integral from [5] since we perform the integral in first sheet. If we change $m \to -m$ as in the usual literature then we need to change the sign of the integral to confirm the known result, as can be shown by examining the quadratic potential.

$$S_2 = \int_{e_4}^{e_3} y(x)dx = \left[2(e_2-e_3)^2\beta\alpha_2^4\right] \cdot \int_0^{K_2} \frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,\mathrm{cn}^2 u}{(1-\alpha_2^2\,\mathrm{sn}^2 u)^4} := J_2 \cdot I_2, \tag{4.25}$$

$$\Pi_1 = \int_{e_1}^{\Lambda} y(x)dx = \left[2(e_1-e_2)^2\beta\alpha_3^4\right] \cdot \int_0^{u_\Lambda} \frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,\mathrm{cn}^2 u}{(1-\alpha_3^2\,\mathrm{sn}^2 u)^4} := J_3 \cdot I_3, \tag{4.26}$$

$$\Pi_2 = \int_{-\Lambda}^{e_4} y(x)dx = \left[2(e_3-e_4)^2\beta\alpha_4^4\right] \cdot \int_{u_{(-\Lambda)}}^{0} \frac{\mathrm{sn}^2 u\,\mathrm{dn}^2 u\,\mathrm{cn}^2 u}{(1-\alpha_4^2\,\mathrm{sn}^2 u)^4} := J_4 \cdot I_4, \tag{4.27}$$

$$D_j = \Pi_1(\Lambda) - \Pi_1(m_j),\ \ for\ \ j = 1, ..., N_f, \tag{4.28}$$

where $\beta = \sqrt{(e_1-e_3)(e_2-e_4)}$ and

$$\alpha_1^2 = \frac{e_1-e_2}{e_1-e_3} < 1,\ \alpha_2^2 = \frac{e_3-e_4}{e_2-e_4} < 1,\ \alpha_3^2 = \frac{e_1-e_4}{e_2-e_4} > 1,\ \alpha_4^2 = \frac{e_1-e_4}{e_1-e_3} > 1, \tag{4.29}$$

and $\mathrm{sn}, \mathrm{dn}, \mathrm{cn}$ are Jacobi's elliptic functions [19] and u_i (=u in I_i), is defined by

$$\mathrm{sn}^2 u_1 = \frac{1}{\alpha_1^2}\frac{x-e_2}{x-e_3},\ \mathrm{sn}^2 u_2 = \frac{1}{\alpha_2^2}\frac{e_3-x}{e_2-x},\ \mathrm{sn}^2 u_3 = \frac{1}{\alpha_3^2}\frac{x-e_1}{x-e_2},\ \mathrm{sn}^2 u_4 = \frac{1}{\alpha_4^2}\frac{e_4-x}{e_3-x}. \tag{4.30}$$

and in each case the k's in the elliptic functions are given by

$$k_1^2 = \alpha_1^2\frac{e_3-e_4}{e_2-e_4},\ k_2^2 = \alpha_2^2\frac{e_1-e_2}{e_1-e_3},\ k_3^2 = \alpha_3^2\frac{e_2-e_3}{e_1-e_3},\ k_4^2 = \alpha_4^2\frac{e_2-e_3}{e_2-e_4}. \tag{4.31}$$

They satisfy $k_i^2 < \alpha_i^2$, $k_i^2 < 1$. K_i is defined to be $K(k_i)$ where

$$K(k) = \frac{\pi}{2}F(\frac{1}{2},\frac{1}{2};1;k^2) = \frac{\pi}{2}(1+\frac{1}{4}k^2+\frac{9}{64}k^4+...) \tag{4.32}$$

satisfies $\mathrm{sn}K_i = 1$. F is the hypergeometric function. $\mathrm{sn}u_\Lambda = \Lambda$. The integrals $I_i(\alpha_i)$ look very similar to one another but according to whether α^2 is bigger or smaller than 1, they have a very different behavior as we will see. This corresponds to the difference of S_i and Π_i, the periods along compact vs. non-compact cycles. We are interested in the leading orders in a small Λ expansion. We need to find the e_i's, the zeroes of y, in terms of m_ϕ, m, g, and the scale of the theory Λ. Let c_1, c_2 be the classical value of the zeros of W', namely, $c_{1,2} = -\frac{m_\phi}{2g} \pm \sqrt{\frac{m_\phi^2}{4g^2} - \frac{\lambda}{g}}$.

1 $m_\phi \neq 0$ case:

$$\begin{aligned}
e_1 &= c_1 + \sqrt{\frac{m-c_1}{c_1-c_2}}\epsilon\Lambda + O(\Lambda^2),\\
e_2 &= c_1 - \sqrt{\frac{m-c_1}{c_1-c_2}}\epsilon\Lambda + O(\Lambda^2),\\
e_3 &= c_2 + \frac{1}{c_1-c_2}\left(4 + \frac{m-c_1}{c_1-c_2}\right)\epsilon\Lambda^2 + O(\Lambda^3),\\
e_4 &= a_2 = c_2 - \frac{2\epsilon\Lambda^2}{c_1-c_2} + O(\Lambda^3).
\end{aligned} \tag{4.33}$$

If $\lambda = 0$ as is often calculated in the gauge theory literature to be compared, $c_1 = 0, c_2 = -m_\phi/g$, and consequently

$$e_{1,2} = \pm\sqrt{mg/m_\phi}, \quad e_3 = -m_\phi/g + g(4+gm/m_\phi)\Lambda^2/m_\phi, \quad e_4 = -m_\phi - 2g\Lambda^2/m_\phi. \tag{4.34}$$

Here we set $\epsilon = 1$. Now, we can evaluate J_i and I_i in leading order in a small Λ expansion.

$$J_1 \sim J-3 \sim 8m\Lambda^2, \quad J_2 \sim J_4 \sim 2g\frac{\Lambda^4}{m_\phi}\left(6+gm/m_\phi\right)^2, I_1 \sim I_2 \sim \frac{5\pi}{16} + O(\Lambda^2), \tag{4.35}$$

so that $S_i \sim J_i$. To obtain I_3, I_4, needs more care.

2 $m_\phi = 0$ case: In this case, $c_{1,2} = \sqrt{2\Lambda^2 - \lambda/2}$, and especially if $\lambda = 0$, $c_{1,2} = \pm\sqrt{2}\Lambda$. For the $SU(3)$ case, we have degenerate $c_{1,2} = 0$ result. In both cases an examination of $W'^2 + f = 0$ tells us that the leading term is of order $m^{1/3}\Lambda^{2/3}$ and the prefactors of S_i and Π_i are all the same order $J_i \sim m\Lambda^2$ and the integrals I_i for S_i are complete elliptic integrals which are of order 1. Therefore $W \sim 8m\Lambda^2$. From this and by dimensional analysis and analyticity of W near $\Lambda = 0$, the structure of the leading and subleading orders are

$$W \sim m\Lambda^2\left(1 + b_1(\Lambda/m) + b_2(\Lambda/m)^2 + ...\right) \sim m\Lambda^2 + b_1\Lambda^3 + b_2\Lambda^4/m + ... \tag{4.36}$$

The first two terms confirm the analysis of the factorization result while the third term is the leading order in gauge theory analysis [14]. The main point in this analysis is that the loop equations say that the first two terms exist and there is no reason why they should vanish.

5 Conclusion

In this paper, we calculated the superpotential for an $SU(N_c)$ gauge theory with N_f massive fundamental fields and a massless adjoint field having nontrivial tree-level superpotentials using the method of factorizing the Seiberg-Witten curve. We worked out both two-cut as well as one-cut solutions. It turns out that the gauge theory result coincides with the one-cut solution rather than the generic two-cut solution. In that case, what we found is that the result obtained by duality [11] is just the leading term of what is found here. This means that Kutasov duality holds only in the IR limit where the scale of the theory $\Lambda \to 0$, which is perfectly consistent with known wisdom.

A few questions remain. First of all, why does the gauge theoretic method correspond to the one-cut solution rather than to a generic two-cut solution? Secondly, is it possible to find a corresponding gauge theoretic result for the two-cut solutions? We will return to these questions in later publications.

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