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Theorems on Estimating Perturbative Coefficients in Quantum Field Theory and Statistical Physics

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ABSTRACT

We present rigorous proofs for several theorems on using Padé approximants to estimate coefficients in Perturbative Quantum Field Theory and Statistical Physics. As a result, we find new trigonometric and other identities where the estimates based on this approach are exact. We discuss hypergeometric functions, as well as series from both Perturbative Quantum Field Theory and Statistical Physics.

I. INTRODUCTION

Recently, we proposed[1-5] a method of estimating coefficients in Perturbative Quantum Field Theory and Statistical Physics with error bars for each estimate. The method makes use of Padé approximants and yields a Padé approximant approximation (PAP). There are many good references for Padé approximants, such as for example Refs. 6-10. We begin by defining the Padé approximant

$$[N/M] = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_N x^N}{1 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots + b_M x^M} \tag{1.1}$$

to the series

$$S = S_0 + S_1 x + \ldots + S_{N+M} x^{N+M} \tag{1.2}$$

where we set

$$[N/M] = S + O(x^{N+M+1}) . \tag{1.3}$$

We have written a computer program that solves Eq. (1.3) numerically and then predicts the coefficient of the next term S_{N+M+1}. It works for arbitrary N and M. Furthermore, we have derived algebraic formulae for the [N/1], [N/2], [N/3], [N/4], [N/5], and [N/6] PAP's, where N is arbitrary.

To illustrate the method, consider the simple example

$$\frac{\ln(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{C} . \tag{1.4}$$

We write the [1/1] Padé as

$$[1/1] = \frac{a_0 + a_1 x}{1 + b_1 x} . \tag{1.5}$$

It is easy to show that $a_0 = 1$, $b_1 = 2/3$, $a_1 = 1/6$, and $C = 9/2$. We can see that the prediction for C is close to the correct value C = 4. For x = 1, we get [1/1] = 7/10, close to the correct result, ln 2 = 0.6931. This is much better than the partial sum

$$1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = 0.8333 . \tag{1.6}$$

By going to higher order, it is easy to show that

$$[1/2] = \frac{1+\frac{x}{2}}{1+x+\frac{x^2}{6}} \tag{1.7}$$

and for x = 1 we obtain

$$m[1/2] = \frac{9}{13} = 0.6923 \tag{1.8}$$

very close to ln 2. The PAP is 7/36 = 0.1944 very close to the correct value of 1/5.

As discussed in our recent article, the error bars are obtained by taking the reciprocals

$$r_n = \frac{1}{S_n} \tag{1.9}$$

and finding the PAP for r_{n+1}, and then taking the reciprocal. Then we consider the differences

$$t_n = r_{n+1} - r_n \tag{1.10}$$

and find the PAP for t_n. We then have

$$r_{n+1} = r_n + t_n \tag{1.11}$$

and then take the reciprocal

$$S_{n+1} = \frac{1}{r_{n+1}} \ . \tag{1.12}$$

Our error bar is calculated from the difference between the results from Eqs. (1.9) and (1.12).

II. THEOREMS

We consider the general series

$$S = \sum_n S_n x^n \ . \tag{2.1}$$

A. Sums of geometric series

If S_n is a sum of M geometric series then the [N/M] PAP for $N \geq M-1$ is exact.

Proof: For M = 2,

$$S_n = ar^n + bs^n \tag{2.2}$$

and

$$S = \sum_{n=0}^{\infty} S_n x^n = \frac{a}{1-rx} + \frac{b}{1-sx} \tag{2.3}$$

so that

$$S = \frac{(a+b)-(as+br)x}{(1-rx)(1-sx)} \tag{2.4}$$

and the [1/2] and higher PAP's are exact.

To prove the general case we use mathematical induction. Assume that the theorem is true for the [M-1/M] PAP. Now for M → M+1, we have

$$\frac{P_{M-1}}{Q_M} + \frac{g}{1-ax} = \frac{P_{M-1}+gQ_M}{Q_M(1-ax)} \tag{2.5}$$

and the [M/M+1] PAP is exact. Here

$$S_n = ar^n + bs^n + \ldots + gw^n \tag{2.6}$$

and P_M and Q_M are polynomials of degree M.

B. Signs of geometric series

For

$$S_n = (-1)^{{}^nC_m} ar^n \tag{2.7}$$

the [m-1/m] PAP is exact.

Proof: The proof is easily obtained by recognizing that the series in Eq. (2.7) is just a sum of geometric series and by next using Theorem A.

C. A sufficient condition for PAP's to be accurate

If we define

$$g(n) = \frac{d^2 \ln S_n}{dn^2} \tag{2.8}$$

then a sufficient (but not necessary) condition for the PAP's to be accurate is

$$\lim_{n \to \infty} g(n) = 0 \ . \tag{2.9}$$

Proof: We define

$$A_n \equiv 1 + \epsilon_n \equiv \frac{S_n S_{n+2}}{(S_{n+1})^2} \tag{2.10}$$

and hence

$$A_n = 1 + \epsilon_n = e^{g(n)} \quad . \tag{2.11}$$

The % error is expressible in terms of the ε_n and if $g(n) \to 0$ then $\varepsilon_n \to 0$ and the PAP is accurate.

D. A generalization of Theorem C

If, in addition to the conditions of Theorem C for S_n we generalize to a series

$$T = \sum_n T_n x^n \tag{2.12}$$

where

$$T_n = (-1)^{{}^nC_m} S_n \tag{2.13}$$

then the [m-1/m] and higher PAP's will be accurate.

<u>Proof:</u>

$$A_n' = 1 + \epsilon_n' = (-1)^{{}^nC_{m-2}} A_n \tag{2.14}$$

where A_n is given by Eq. (24). Then we use Theorems B and C to prove Theorem D. For further details, see Ref. 2.

E. Polynomials of the n^{th} degree

For $S_n = P_n$ where P_n is a polynomial of degree n, the [N/M] PAP's are exact, where M = n+1 and N ≥ M-1.

Proof: By differentiating

$$S = \sum_n x^n = (1-x)^{-1} \tag{2.15}$$

and multiplying by x, we obtain

$$\sum_n (an+b)x^n = \frac{(a-b)x+b}{(1-x)^2} \tag{2.16}$$

and the [N/2] is exact for N ≥ 1. Now by induction we can easily obtain the desired result.

It should be emphasized that in all cases once the [N/M] PAP's are exact for N ≥ M-1, the results remain exact for all higher order PAP's [N'/M'] for M' > M and N' > M'-1.

III. SOME NEW TRIGONOMETRIC IDENTITIES

If we consider the series given by

$$S_n = \sin[(n+1)\theta + \delta] \tag{3.1}$$

where θ and δ are arbitrary we will prove that the [N/M] PAP's are exact for M ≥ 2 and N ≥ M-1. This leads to new trigonometric identities corresponding to each of the [N/2], [N/3], [N/4], etc., PAP's.

From Eq. (3.1) it can easily be shown that

$$S = \sum_{n=0}^{\infty} S_n x^n = \frac{x\cos(\theta+\delta)\sin\theta + \sin(\theta+\delta)(1 - x\cos\theta)}{1 - 2x\cos\theta + x^2} . \tag{3.2}$$

Hence the [N/M] PAP's are exact for M ≥ 2, N ≥ M-1. With δ = 0, Eq. (3.2) becomes

$$S = \frac{\sin\theta}{1 + x^2 - 2x\cos\theta} \tag{3.3}$$

and, hence, the [0/2] PAP is exact.

Similarly for

$$S_n = \cos[(n+1)\theta + \delta] \tag{3.4}$$

we can obtain

$$S = \frac{\cos(\theta+\delta)(1 - x\cos\theta) - x\sin(\theta+\delta)\sin\theta}{1 - 2x\cos\theta + x^2} . \tag{3.5}$$

In this case, however, if we set δ = 0, we obtain

$$S = \frac{\cos\theta - x}{1 - 2x\cos\theta + x^2} , \qquad (3.6)$$

and the [0/2] PAP is not exact!

Now, for each [N/M] PAP that is exact we will find a trigonometric identity. We begin with M = 2. The [1/2] PAP is given by $S_4 = N/D$ where

$$N = 2S_1S_2S_3 - S_0S_3^2 - S_2^3 \qquad (3.7)$$

and

$$D = S_1^2 - S_0S_2 \qquad (3.8)$$

with the S_n given by Eq. (3.1). Thus we have the identity

$$N - S_4D \equiv 0 \qquad (3.9)$$

which becomes

$$\begin{aligned} &2\sin(2\theta+\delta)\sin(3\theta+\delta)\sin(4\theta+\delta) \\ &-\sin(\theta+\delta)\sin^2(4\theta+\delta) - \sin^3(3\theta+\delta) \\ &-\sin(5\theta+\delta)[\sin^2(2\theta+\delta) - \sin(\theta+\delta)\sin(3\theta+\delta)] = 0 \quad . \end{aligned} \qquad (3.10)$$

Now one can step up in n, $S_n \to S_{n+1}$, and obtain another identity. This procedure can be repeated indefinitely. One can also step down in n, $S_n \to S_{n-1}$, where we set $S_{-1} = 0$. This gives a simpler identity, which can be obtained from simple known identities, for the [0/2] PAP. But we must set $\delta = k\pi$, k = 0, 1, 2, ... , yielding

$$2\sin\theta\sin(2\theta)\sin(3\theta) - \sin^3(2\theta) - \sin(4\theta)\sin^2\theta = 0 \ . \tag{3.11}$$

One can also use Eq. (31) to obtain the same identities for $\cos[(n+1)\theta+\delta]$. However, in this case there is no [0/2] identity for $\delta = 0$.

We now turn to M = 3. The [2/3] PAP is given by

$$S_6 = A/B \tag{3.12}$$

where

$$\begin{aligned} A = {} & 2S_2^2S_3S_5 - 2S_1S_3^2S_5 + 2S_0S_3S_4S_5 \\ & - 2S_1S_2S_4S_5 + S_1^2S_5^2 - S_0S_2S_5^2 + S_2^2S_4^2 \\ & - 3S_2S_3^2S_4 + 2S_1S_3S_4^2 - S_0S_4^3 + S_3^4 \end{aligned} \tag{3.13}$$

and

$$B = S_2^3 - 2S_1S_2S_3 + S_0S_3^2 - S_0S_2S_4 + S_1^2S_4 \ . \tag{3.14}$$

Again we use Eqs. (3.1) and (3.4), but this time there are two identities in each case

$$A = 0 \quad \text{and} \quad B = 0 \; . \tag{3.15}$$

The first part of Eq. (3.15) yields new identities, but the second part gives only previous identities for M = 2. This is in accordance with a theorem presented in an earlier paper.[2] We can again step up in n, $S_n \rightarrow S_{n+1}$, and obtain more identities for both $\sin[(n+1)\theta+\delta]$ and $\cos[(n+1)\theta+\delta]$. This process may be repeated as many times as desired. Now we may very easily also step down in n, $S_n \rightarrow S_{n-1}$, where we set $S_{-1} = 0$. This gives identities for the [1/3]. In this case the identity obtained results from

$$A - BS_6 = 0 \tag{3.16}$$

for both sine and cosine, with $\delta \neq n\pi$. For the sine case, if $\delta = k\pi$, we obtain

$$A = 0 \quad \text{and} \quad B = 0 \; . \tag{3.17}$$

If we step down once more to the [0/3] PAP and set $\delta = k\pi$, then we obtain an identity for sine, but not for cosine.

Although this process can be continued indefinitely for M = 4, 5, 6, ..., the formulae become increasingly complicated, as will soon be seen. So we present results for only one more value of M, namely M = 4. The [3/4] PAP is given by

$$S_8 = C/D \tag{3.18}$$

where

$$\begin{aligned} C = {} & 2(2S_2S_3S_4^2S_7 - S_3^3S_4S_7 - S_1S_4^3S_7 \\ & - S_2^2S_4S_5S_7 + S_1S_3S_4S_5S_7 - S_3^2S_4^2S_6 + 2S_1S_4^2S_5S_6 \\ & + S_2^2S_4S_6^2 - S_1S_3S_4S_6^2 - S_2S_3S_4S_5S_6 - S_2S_4^3S_6 \\ & + S_3^2S_4S_5^2 + S_2S_4^2S_5^2 - 2S_3S_4^3S_5 - S_1S_4S_5^3 + S_2S_3^2S_5S_7 \\ & + S_0S_4^2S_5S_7 + S_1S_2S_5^2S_7 - S_0S_3S_5^2S_7 - S_2^2S_4S_5S_7 \\ & - S_1S_3S_4S_5S_7 - S_2^2S_3S_6S_7 - S_0S_3S_4S_6S_7 - S_1^2S_5S_6S_7 \\ & + S_0S_2S_5S_6S_7 + S_1S_2S_4S_6S_7^2 + S_1S_3^2S_6S_7 - S_1S_2S_3S_7 \\ & - S_3^3S_5S_6 - S_1S_3S_4S_6^2 - S_1S_2S_5S_6^2 + S_1S_3S_5^2S_6 \\ & + S_0S_3S_5S_6^2 - S_2S_3S_5^3) - 3S_0S_4S_5^2S_6 + S_2^3S_7^2 + S_0S_3^2S_7^2 \\ & + S_1^2S_4S_7^2 - S_0S_2S_4S_7^2 + S_2S_3^2S_6^2 + S_1^2S_6^3 - S_0S_2S_6^3 \\ & + S_3^2S_4S_5^2 + S_2^2S_5^2S_6 + S_4^5 + S_2S_4^2S_5^2 + S_3^2S_4^2S_6 \\ & - S_2S_4^3S_6 + S_0S_4^2S_6^2 + S_0S_5^4 \end{aligned} \tag{3.19}$$

and

$$D = 2(S_1 S_3^2 S_5 + S_1 S_2 S_4 S_5 - S_1 S_3 S_4^2$$

$$- S_2^2 S_3 S_5 - S_1 S_2 S_3 S_6 - S_0 S_3 S_4 S_5) + 3\, S_2 S_3^2 S_4 \tag{3.20}$$

$$- S_3^4 - S_1^2 S_5^2 + S_1^2 S_4 S_6 + S_0 S_4^3 + S_0 S_2 S_5^2$$

$$+ S_0 S_3^2 S_6 - S_0 S_2 S_4 S_6 + S_2^3 S_6 - S_2^2 S_4^2 \ .$$

The identities for the [3/4] PAP are

$$C = 0 \quad \text{and} \quad D = 0 \ . \tag{3.21}$$

for arbitrary δ in both the sine and cosine cases. Again we may step up $S_n \to S_{n+1}$, etc., and obtain new identities. We may also step down to the [2/4] PAP. For the [1/4] PAP the identity is obtained for arbitrary δ from

$$C - DS = 0 \tag{3.22}$$

for arbitrary δ. For $\delta = k\pi$ we obtain for the sine case the identities

$$C = 0 \quad \text{and} \quad D = 0 \tag{3.23}$$

For the [0/4], for $\delta = k\pi$, sine works but not cosine.

We believe these identities are new, except for the [0/2] PAP. We would be interested in hearing from anyone who believes any of these identities are already known.

IV. THE GENERALIZED HYPERGEOMETRIC FUNCTION

The hypergeometric function ${}_kF_m$ represents a large number of elementary functions. Thus we can consider PAP's for many functions at once. We will see that the PAP's are accurate for arbitrary k and m and a large number of parameters a, b, c, ... For many examples of how numerous mathematical functions can be expressed in terms of the hypergeometric function ${}_2F_1$ or the confluent hypergeometric function ${}_1F_1$ see, for example, the books by Arfken,[12] by Abramowitz and Stegun,[13] and by Gradshteyn and Rhyzik.[14]

Consider the hypergeometric series given by

$$S_n = \frac{(a)_n (b)_n}{(c)_n \, n!} \tag{4.1}$$

where

$$(a)_n = a(a+1)\ldots(a+n+1) \tag{4.2}$$

and hence

$${}_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} S_n x^n \,. \tag{4.3}$$

For the [N/2] PAP the percentage error is given by 100p where

$$p = \frac{\epsilon_N^2/\epsilon_{N-1} - \epsilon_{N+1}(1+\epsilon_N)^2}{(1+\epsilon_N)^2(1+\epsilon_{N+1})};\ N \geq 1\ . \tag{4.4}$$

It can be shown for the ${}_2F_1$ hypergeometric function that

$$p \sim \frac{-2B(1+B)}{N^4} \tag{4.5}$$

where

$$B = c + 1 - a - b \tag{4.6}$$

and, hence, the PAP's quickly become accurate as $N \to \infty$. For ${}_1F_1(a,c;x)$

$$p \sim +\frac{2}{N^2} \tag{4.7}$$

and for ${}_2F_0(a,b,c;x)$

$$p \sim -\frac{2}{N^2}\ . \tag{4.8}$$

For the general case ${}_kF_m$, if $k \neq m+1$,

$$p \sim -\frac{2A}{N^2} \tag{4.9}$$

where

$$A = k-(m+1) \ , \tag{4.10}$$

and if k = m+1, then

$$p \sim \frac{-2B(1+B)}{N^4} \tag{4.11}$$

where

$$B = 2+k^2-2k+m-km+\sum_{i=1}^{m} c_i - \sum_{i=1}^{k} a_i \ . \tag{4.12}$$

In general if

$$\varepsilon_n \sim A/N \tag{4.13}$$

$$p \sim -2A^2/N^2 \ , \tag{4.14}$$

and if

$$\varepsilon_n \sim B/N^2 \ , \tag{4.15}$$

$$p \sim -2B(1+B)/N^4 \ . \tag{4.16}$$

To check the behavior for M ≠ 2 we have written a computer program that scans over a, b, c values (skipping over integers) and evaluates the corresponding PAP's. The parameters a, b, and c vary from -5.0 to 5.0 in steps of 0.125. For each [N/M] PAP, the fractional error p is evaluated, and the maximum and minimum values of p listed as TESTMAX and TESTMIN, respectively. The results for ${}_2F_1$, ${}_1F_1$, and ${}_2F_0$ are presented in Tables I, II, and III, respectively. It can be seen that the TESTMIN and TESTMAX values decrease rapidly in going to progressively higher order. We have listed only diagonal PAP's, but other Padé's were also computed and gave very good results.

V. OTHER EXAMPLES OF EXACT PAP's

Other examples can be found in which PAP's are exact. Any series whose sum is a rational fraction of two polynomials

$$S = \sum_{n=0}^{\infty} S_n x^n = \frac{P_{N_0}(x)}{Q_{M_0}(x)} \tag{5.1}$$

will be exact for the [N/M] PAP where $N \geq N_0$ and $M \geq M_0$. Some examples include

$$S_n = (2n+1); \qquad N_0 = 1\ ,\ M_0 = 2 \tag{5.2}$$

$$S_n = (n+1)^2; \qquad N_0 = 1 \ , \ M_0 = 3 \tag{5.3}$$

$$S_n = (2n+1)^2; \qquad N_0 = 2 \ , \ M_0 = 3 \tag{5.4}$$

$$S_n = (a+nd); \qquad N_0 = 1 \ , \ M_0 = 2 \tag{5.5}$$

$$S_n = (n+1); \qquad N_0 = 0 \ , \ M_0 = 2 \tag{5.6}$$

$$S_{n+1} - S_n = (n+2)\,, S_0 = 1; \qquad N_0 = 0 \ , \ M_0 = 3 \tag{5.7}$$

and

$$S_n = 1; \qquad N_0 = 0 \ , \ M_0 = 1 \ . \tag{5.8}$$

VI. NON-SINGLET MOMENTS OF DEEP INELASTIC STRUCTURE FUNCTIONS IN QCD

In this section we make use of some recent results of Larin, van Rittergen, and Vermeseren.[15] They have calculated the next-to-next leading QCD approximations for non-singlet moments of deep inelastic structure functions, in the leading twist approximation, for the moments N = 2, 4, 6, 8 of the non-singlet deep inelastic structure function F_L. They have calculated the three-loop anomalous dimensions of the corresponding non-singlet operators and the three-loop coefficient functions of the structure factor F_L, in the leading twist and massless quark approximation.

We present our results in Tables IV-XI. In each case, we estimate the $O(\alpha_s^3)$ coefficient and compare our estimate with the now-known Larin *et al.* result. We neglect the term that depends on the sum of the quark charges $\Sigma\, q_f$, since the term is small in all cases of interest. We present our results for N_f = 3, 4, 5, where N_f is the number of quark flavors. We then present our estimates for the next (unknown) $O(\alpha_s^4)$ coefficients, in each case.

In Table IV we present results for $C_{L,2}$. It is seen that, for N_f = 3, 4, 5, our estimates are within the error-bars, for the $O(\alpha_s^3)$ terms and we estimate the next (unknown) $O(\alpha_s^4)$ terms. Table V shows the results for $C_{L,4}$, Table VI for $C_{L,6}$, and Table VII for $C_{L,8}$.

In Tables VIII-XI we present our results for the anomalous dimensions γ_2, γ_4, γ_6, and γ_8. Here again, in each case, our estimates are within the error bars of the Larin *et al.* results for $O(\alpha_s^3)$ and we estimate the next (unknown) $O(\alpha_s^4)$ term. For further details on how we obtain our error bars, see Ref. 11.

VII. EXAMPLES FROM STATISTICAL PHYSICS

In this section we consider two examples[16] from statistical physics. They are the low temperature ferromagnetic susceptibility coefficients in the Ising Model. Table XII gives the results for the honey-combed (hc) lattice and the results for the square (sq) lattice are shown in Table XIII.

It can be seen that the results are excellent and that the % error decreases in going to higher order. In all cases the estimates are within 2σ of the exact results for the known coefficients and the next (unknown) coefficient is predicted. For the hc lattice the estimate is 538,596,000 ± 10,500 and for the square lattice it is 185,000,000 ± 55,000,000.

VIII. CONCLUSIONS

We have proved several theorems on using Padé approximants to estimate coefficients in Perturbative Quantum Field Theory and Statistical Physics. These theorems give sufficient conditions for the PAP method of estimating the next term in a series expansion to work. In addition, we have presented new trigonometric identities which we obtained as a result of the PAP being exact. We have also considered the generalized hypergeometric function, for which the method works. As a result, many series are dealt with at the same time, since the hypergeometric function can represent many elementary functions merely by changing the parameters.

We have considered several series from QCD. These are for the non-singlet moments of deep inelastic structure functions. We have also considered two series from statistical physics. These are the low temperature ferromagnetic susceptibility coefficients in the Ising model.

In all cases, the method works beautifully! Thus the information needed for estimating the next term in perturbative series is, in fact, contained in the lower-order results.

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TABLE I. Padé estimates for ${}_2F_1$.

[N/M]	TESTMIN	TESTMAX
[4/4]	0.649×10^{-8}	0.649
[5/5]	0.105×10^{-9}	34.0
[6/6]	0.103×10^{-11}	2.57
[7/7]	0.100×10^{-13}	7.0
[8/8]	0.107×10^{-15}	0.494
[9/9]	0.136×10^{-17}	0.687×10^{-2}
[10/10]	0.173×10^{-19}	0.332×10^{-3}
[11/11]	0.284×10^{-21}	0.151×10^{-4}
[12/12]	0.442×10^{-23}	0.215×10^{-5}
[13/13]	0.757×10^{-25}	0.294×10^{-6}
[14/14]	0.151×10^{-26}	0.386×10^{-7}
[15/15]	0.652×10^{-30}	0.492×10^{-8}
[16/16]	0.197×10^{-29}	0.611×10^{-9}
[17/17]	0.705×10^{-30}	0.720×10^{-10}
[18/18]	0.192×10^{-29}	0.847×10^{-11}

TABLE II. Padé estimates for ${}_1F_1$.

[N/M]	TESTMIN	TESTMAX
[4/4]	0.536×10^{-5}	0.246×10^{-6}
[5/5]	0.170×10^{-6}	6.721
[6/6]	0.195×10^{-6}	35.4
[7/7]	0.251×10^{-6}	4.56
[8/8]	0.124×10^{-7}	0.456
[9/9]	0.878×10^{-8}	0.102×10^{-1}
[10/10]	0.835×10^{-9}	0.114×10^{-2}
[11/11]	0.168×10^{-8}	0.697×10^{-3}
[12/12]	0.148×10^{-9}	0.260×10^{-3}
[13/13]	0.394×10^{-10}	0.270×10^{-3}
[14/14]	0.419×10^{-11}	0.676×10^{-5}
[15/15]	0.157×10^{-11}	0.119×10^{-5}
[16/16]	0.530×10^{-13}	0.844×10^{-6}
[17/17]	0.393×10^{-13}	0.121×10^{-6}
[18/18]	0.984×10^{-14}	0.201×10^{-7}
[19/19]	0.259×10^{-14}	0.217×10^{-8}
[20/20]	0.158×10^{-14}	0.128×10^{-8}

TABLE III. Padé estimates for ${}_2F_0$.

[N/M]	TESTMIN	TESTMAX
[4/4]	0.146×10^{-4}	127.3
[5/5]	0.118×10^{-4}	45.3
[6/6]	0.458×10^{-6}	0.304
[7/7]	0.259×10^{-5}	3.00
[8/8]	0.986×10^{-8}	0.307×10^{-1}
[9/9]	0.262×10^{-8}	0.247×10^{-1}
[9/10]	0.528×10^{-8}	0.227×10^{-1}
[10/9]	0.572×10^{-9}	0.298×10^{-2}
[10/10]	0.590×10^{-8}	0.152×10^{-1}
[11/11]	0.165×10^{-8}	0.774×10^{-1}
[13/13]	0.132×10^{-10}	0.573×10^{-4}
[13/14]	0.298×10^{-10}	0.227×10^{-3}
[14/13]	0.214×10^{-10}	0.776×10^{-3}
[16/16]	0.136×10^{-11}	0.151×10^{-5}
[17/17]	0.377×10^{-13}	0.105×10^{-6}
[18/18]	0.272×10^{-13}	0.275×10^{-7}
[19/19]	0.349×10^{-14}	0.729×10^{-8}
[20/20]	0.296×10^{-16}	0.198×10^{-8}

TABLE IV. Padé estimates for $C_{L,2}$.

Estimate	Error	Exact	|Estimate - Exact|
$\underline{N_f = 3}$			
1046	1046	2230	1184
82,812	205,498	——	——
$\underline{N_f = 4}$			
837	837	2313	1476
82,233	32,915	——	——
$\underline{N_f = 5}$			
652	652	2420	1768
80,203	13,522	——	——

TABLE V. Padé estimates for $C_{L,4}$.

Estimate	Error	Exact	|Estimate - Exact|
$N_f = 3$			
1376	668	1473	137
56,946	29,021	——	——
$N_f = 4$			
1106	553	1166	60
39,468	19,846	——	——
$N_f = 5$			
897	449	881	16
25,076	4004	——	——

TABLE VI. Padé estimates for $C_{L,6}$.

Estimate	Error	Exact	$\lvert$Estimate - Exact$\rvert$
$\underline{N_f = 3}$			
2305	1153	1433	872
41,989	17,795	——	——
$\underline{N_f = 4}$			
2018	1009	1159	859
27,443	12,654	——	——
$\underline{N_f = 5}$			
1750	875	905	845
15,894	8434	——	——

TABLE VII. Padé estimates for $C_{L,8}$.

Estimate	Error	Exact	|Estimate - Exact|
$\underline{N_f = 3}$			
1437	719	1985	548
124,711	78,967	——	——
$\underline{N_f = 4}$			
1226	613	2043	817
130,095	139,494	——	——
$\underline{N_f = 5}$			
1031	516	2118	1087
133,699	814,125	——	——

TABLE VIII. Padé estimates for γ_2.

Estimate	Error	Exact	|Estimate - Exact|
$\underline{N_f = 3}$			
424	212	448	24
5159	2270	___	___
$\underline{N_f = 4}$			
358	179	306	52
2607	636	___	___
$\underline{N_f = 5}$			
298	149	162	136
677	238	___	___

TABLE IX. Padé estimates for γ_4.

Estimate	Error	Exact	|Estimate - Exact|
$N_f = 3$			
636	318	762	126
8606	3146	——	——
$N_f = 4$			
517	259	503	14
4953	387	——	——
$N_f = 5$			
410	205	239	175
954	297	——	——

TABLE X. Padé estimates for γ_6.

Estimate	Error	Exact	|Estimate - Exact|
$\underline{N_f = 3}$			
744	372	946	202
10,676	4001	——	——
$\underline{N_f = 4}$			
596	298	621	25
5245	1266	——	——
$\underline{N_f = 5}$			
464	232	290	174
1201	348	——	——

TABLE XI. Padé estimates for γ_8.

Estimate	Error	Exact	|Estimate - Exact|
$\underline{N_f = 3}$			
833	417	1081	248
12,225	4629	——	——
$\underline{N_f = 4}$			
662	331	709	47
6018	2552	——	——
$\underline{N_f = 5}$			
510	255	330	180
1401	393	——	——

TABLE XII. Padé estimates for the low temperature ferromagnetic susceptibility coefficients in the Ising model (hc lattice).

Estimate	Error	Exact	|Estimate - Exact|
8749	818	8792	43
35,682	120	35,622	60
143,333	447	143,079	254
569,470	950	570,830	1360
2,264,740	631	2,264,649	91
8,942,853	2031	8,942,436	417
35,159,776	8724	35,169,616	9840
137,838,225	5787	137,839,308	1083
538,596,320	10,430	—	—

TABLE XIII. Padé estimates for the low temperature ferromagnetic susceptibility coefficients in the Ising model (sq lattice).

Estimate	Error	Exact	|Estimate - Exact|
449	138	416	33
2715	830	2791	76
18,699	1592	18,296	403
118,069	35,392	118,016	53
751,928	146	752,008	80
4747×10^3	1410×10^3	4746×10^3	$O(10^3)$
2973×10^4	721×10^4	2973×10^4	$O(10^3)$
$18{,}502 \times 10^4$	5494×10^4	——	——